UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 31 May 2001 1.30 to 4.30

PAPER 17

CATEGORY THEORY

*Attempt **SIX** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1

Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be an adjunction, $F \dashv G$.

(i) Define the *unit* η and the *counit* ε of the adjunction.

(ii) Prove the triangle identities, $(\varepsilon F)\circ(F\eta) = 1_F$ and $(G\varepsilon)\circ(\eta G) = 1_G$.

(iii) Prove that given functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ and natural transformations $1 \xrightarrow{\eta} GF$, $FG \xrightarrow{\varepsilon} 1$ satisfying the triangle identities, there is a unique adjunction between F and G with η as its unit and ε as its counit.

2 (i) Fix a nonempty topological space S, and let $\mathcal{O}(S)$ be the poset of open subsets of S, ordered by inclusion. Let

$$\Delta : \mathbf{Set} \longrightarrow [\mathcal{O}(S)^{\mathrm{op}}, \mathbf{Set}]$$

be the functor assigning to a set A the presheaf ΔA with constant value A. Exhibit a chain of adjoints

$$\Lambda \dashv \Pi \dashv \Delta \dashv \Gamma \dashv \nabla.$$

(ii) Let $O : \mathbf{Cat} \longrightarrow \mathbf{Set}$ be the functor taking a small category to its set of objects. Exhibit a chain of adjoints

$$C \dashv D \dashv O \dashv I.$$

(iii) Do either of these chains of adjoints extend further in either direction?

(*In parts (i) and (ii), when you define a functor you are only required to describe its effect on objects, and when you show adjointness you are not required to carry out any formal checks of naturality.*)

3

State and prove the Yoneda Lemma. Deduce:

(i) that the Yoneda embedding is full and faithful

(ii) that for objects A, B of a locally small category $\mathcal{C}$, $A \cong B$ if and only if $\mathcal{C}(C, A) \cong \mathcal{C}(C, B)$ naturally in $C \in \mathcal{C}$

(iii) that a functor $X : \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is representable if and only if there exist an object $A \in \mathcal{C}$ and an element $u \in X(A)$ which is 'universal' in a sense you should make precise.

4 (i) Show that if a category has a terminal object, all binary products and all equalizers, then it has all finite limits.

(ii) Let $\mathcal{C}$ be a category with all finite limits and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a functor which preserves finite products and equalizers. Show that F preserves all finite limits.

(iii) Deduce from (i) that if a category has a terminal object and all pullbacks then it has all finite limits.

5

Let $\mathbb{C}$ be a small category and $\mathcal{S}$ a complete category.

(i) Show that the functor category $[\mathbb{C}, \mathcal{S}]$ is complete, and that for each object C of $\mathbb{C}$ the evaluation functor

$$\begin{array}{rcl} \mathrm{ev}_C : [\mathbb{C}, \mathcal{S}] & \longrightarrow & \mathcal{S} \\ X & \longmapsto & X(C) \end{array}$$

preserves limits.

(ii) Characterize the epics in the category $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ of presheaves on $\mathbb{C}$, justifying your answer.

6

Let $\mathbb{C}$ be a small category.

(i) Prove that any presheaf $X \in [\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is a colimit of representable presheaves.

(ii) What does it mean for a category to be *cartesian closed*? Show that $[\mathbb{C}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed. (You may assume that products exist and are computed pointwise in a presheaf category.)

(iii) Now suppose that $\mathbb{C}$ is cartesian closed. Prove that the Yoneda embedding $H_\bullet$ preserves exponentials.

7

Consider the following three conditions on a functor U from a locally small category $\mathcal{C}$ to **Set**:

A. U has a left adjoint

R. U is representable

L. U preserves limits.

(i) Show that $\mathbf{A} \Rightarrow \mathbf{R} \Rightarrow \mathbf{L}$.

(ii) Show that if $\mathcal{C}$ has small coproducts then $\mathbf{R} \Rightarrow \mathbf{A}$.

(iii) Show that if $\mathcal{C}$ is complete, well-powered and has a coseparating set then the three conditions are equivalent.

8

State the General Adjoint Functor Theorem and the Special Adjoint Functor Theorem.

Either show that if a complete, locally small category has a weakly initial set of objects then it has an initial object, and explain in outline how this result leads to a proof of the General Adjoint Functor Theorem

Or assuming the General Adjoint Functor Theorem, prove the Special Adjoint Functor Theorem.

9 (i) What is a *monad* on a category? What is the *category of algebras* for a monad? What does it mean for a functor to be *monadic*?

(ii) Explain how an adjunction gives rise to a monad, and explain briefly why every monad arises in this way.

(iii) Prove that a monadic functor creates limits.

10

State the Monadicity Theorem, and sketch a proof. (You may assume all standard terminology.)