MATHEMATICAL TRIPOS **Part III**

Monday 11 June 2001 9 to 12

PAPER 16

THREE-DIMENSIONAL MANIFOLDS

Answer ***THREE*** *questions. The questions are of equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State without proof the Schönflies theorem for simple closed piecewise linear curves in S^2.

What is a piecewise linear n-ball? Suppose that B_1 and B_2 are piecewise linear 3-balls and $B_1 \cap B_2 = \partial B_1 \cap \partial B_2$, this intersection being a piecewise linear 2-ball. Prove that $B_1 \cup B_2$ is a piecewise linear 3-ball.

Explain what is meant by a *handle structure* for a closed piecewise linear n-dimensional manifold M and indicate briefly how such a structure can be obtained. A piecewise linear embedding $M \longrightarrow \mathbb{R}^{N-1} \times \mathbb{R}$ with $n < N$ is ambiently isotopic to a critical level embedding with respect to some collared handle decomposition of M. Explain what this means. State and prove the piecewise linear Schönflies theorem for embeddings of S^2 in $\mathbb{R}^3$.

What does it mean to say that a 3-manifold is *irreducible*? If a closed connected 3-manifold M has $\mathbb{R}^3$ as its universal cover, prove that M is irreducible.

2 Let $\star$ be a basepoint in ∂M, the boundary of a compact 3-manifold M. Suppose that the homomorphism of fundamental groups $\pi_1(\partial M, \star) \rightarrow \pi_1(M, \star)$ induced by the inclusion $\partial M \subset M$ is *not* an injection. Prove that there exists a piecewise linear pairwise *embedding* $e : (D, \partial D) \rightarrow (M, \partial M)$ of a disc D into M so that $e(\partial D)$ does not bound a disc in ∂M.

Let K be a knot in S^3. Show that if a piecewise linear map $f : (D, \partial D) \rightarrow (S^3, K)$ exists that is an embedding when restricted to the inverse image of a neighbourhood of K, then K is unknotted.

3 What does it mean to say that a surface F is *normal* with respect to a triangulation T of a closed connected 3-manifold M? Suppose that M has the property that any 2-sphere separates M into two components. Let Σ be a collection of k disjoint 2-spheres contained in M with the property that no component of M-cut-along-Σ is a ball with holes (that is, it is not homeomorphic to the closure of the complement of finitely many disjoint 3-balls in S^3). Prove there exists a collection of k such spheres having the same property that is *normal* with respect to T. Deduce that any closed connected 3-manifold X can be expressed as the connected sum of finitely many prime 3-manifolds.

When X is orientable, prove that this expression as a sum of primes is unique.

4 Let T be a triangulation of a compact 3-manifold M. Explain how a normal surface in M corresponds to an admissible solution to a system of linear equations with integer coefficients (the normal equations). What is a fundamental set of solutions to such a system of equations? Describe without proof how a fundamental set of solutions can in principle be found.

Let L be a 2-component link in S^3 that is split in the sense that there is a 2-sphere (a *splitting* 2-sphere) $\Sigma \subset S^3$ having one component of L in each component of $S^3 - \Sigma$. Let T be a triangulation of S^3 less the interior of a regular neighbourhood of L. Show that a splitting sphere for L can be chosen that corresponds to a fundamental solution of the normal equations defined with respect to T; deduce that there is an algorithm to determine if a link be a split link.

[Results about moving surfaces into normal position may be quoted without proof. It may help to notice that a surface $F \subset S^3 - L$ splits L if and only if $[F] \neq 0 \in H_2(S^3 - L; \mathbb{Z}/2)$.]

5 Give an example of a compact 3-manifold M, containing an incompressible surface F, with the property that M-cut-along-F is homeomorphic to M.

What is a hierarchy for a compact 3-manifold? Describe a hierarchy for $S^1 \times F$ where F is a closed connected surface not equal to S^2 or the projective plane. Show that any orientable, compact, connected, irreducible 3-manifold with non-empty boundary has a hierarchy.

[General results about the existence and the finite number of incompressible surfaces may be assumed without proof.]

Let $f : (M, \partial M) \longrightarrow (N, \partial N)$ be a map, respecting boundaries, between 3-manifolds which are compact, connected, orientable and irreducible. Suppose that

(i) $\partial M \neq \emptyset \neq \partial N$,

(ii) $f|\partial M$ is an injection,

(iii) $f_\star : \pi_1(M) \longrightarrow \pi_1(N)$ is an injection.

Prove that f is homotopic to a homeomorphism, keeping $f|\partial M$ fixed.

[Results concerning the possibility of changing a map between *surfaces* by a homotopy so that it becomes a homeomorphism may be quoted without proof.]