UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 11 June 2001 9 to 12

PAPER 15

COMPLEX MANIFOLDS

*Attempt any **THREE** questions. The questions are of equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe the hyperplane bundle $[H]$ on $\mathbf{P}^n(\mathbf{C})$, and its sheaf of holomorphic sections $\mathcal{O}_{\mathbf{P}^n}(1)$. If H is any hyperplane of $\mathbf{P}^n$, explain why $\mathcal{O}_{\mathbf{P}^n}(1)$ is isomorphic to the sheaf of meromorphic functions with at worst simple poles along H. Assuming the fact that $H^i(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}) = 0$ for all $i > 0$, calculate the dimensions $h^r(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(m)) = 0$ for all integers $r, m \geq 0$.

Let $\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n$ denote the standard morphism, and $X_0, \ldots, X_n$ coordinates on $\mathbf{C}^{n+1}$. Let $x_i = X_i/X_0$ denote affine coordinates on the relevant open affine subset U_0 of $\mathbf{P}^n$. If $P \in \mathbf{C}^{n+1} \setminus \{0\}$ has $\pi(P) \in U_0$, and we consider the tangent vector $\partial/\partial X_i$ at $P = (a_0, \ldots, a_n)$, show that its image in the holomorphic tangent space at $\pi(P)$ is

$$\pi_*(\partial/\partial X_i) = \begin{cases} a_0^{-1}\ \partial/\partial x_i & \text{for } i = 1, \ldots, n \\ \\ -\sum_1^n a_0^{-2} a_j\ \partial/\partial x_j & \text{for } i = 0. \end{cases}$$

If L is any linear homogeneous form in $X_0, \ldots, X_n$, show that $\pi_*(L\ \partial/\partial X_j)$ defines a holomorphic vector field on $\mathbf{P}^n$. What is the vector field $\pi_*(\sum_0^n X_i\ \partial/\partial X_i)$? Deduce the existence of a short exact sequence of locally free $\mathcal{O}_{\mathbf{P}^n}$-modules

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus(n+1)} \to \mathcal{O}_{\mathbf{P}^n}(T'_{\mathbf{P}^n}) \to 0,$$

explaining carefully the sheaf morphisms involved. Calculate $h^0(\mathbf{P}^n, \Omega^1_{\mathbf{P}^n}(1))$, where, as usual, $\Omega^1_{\mathbf{P}^n}(1)$ denotes the sheaf $\mathcal{O}_{\mathbf{P}^n}((T'_{\mathbf{P}^n})^* \otimes [H])$.

2 Write an essay giving an *overview* of integrable almost complex structures on a manifold M, including an account of the relationship between connections on the holomorphic tangent bundle T'_M which are compatible with the complex structure, and connections on the real tangent bundle which satisfy certain compatibility conditions, and an explanation of the role of Kähler metrics.

3 Define what is meant by a *connection* D on a complex vector bundle E on a C^∞ manifold M, and explain why connections on E always exist. Define what is meant by the *curvature* Θ associated with the connection. State and prove Cartan's equation for the curvature matrix (with respect to a given local frame for E). Explain how one obtains a globally defined 2-form $\mathrm{Tr}\,\Theta$ from the curvature.

If E has rank r and D a connection on E, show that there is a naturally defined connection D_r on $\Lambda^r E$, whose curvature 2-form is just $\mathrm{Tr}\,\Theta_D$. Show that this 2-form is closed. Show that the De Rham cohomology class determined by $\mathrm{Tr}\,\Theta$ is independent of the connection chosen. [*Hint: Consider the isomorphism* $H^2_{DR}(M, \mathbf{C}) \cong H^1(M, \mathcal{Z}^1_M)$, *where* $\mathcal{Z}^1_M$ *denotes the sheaf of closed 1-forms on* M.]

Suppose now that M is a complex manifold, and E is a holomorphic line bundle with a hermitian metric. If e is a local nowhere vanishing holomorphic section of E, consider the (locally defined) C^∞ function $h = \|e\|^2$; show that the metric determines a globally defined connection on E, with curvature form locally $\bar\partial\partial(\log h)$.

4 If M is a compact complex manifold equipped with a hermitian metric, explain briefly how this determines a hermitian inner-product on the space $A^{p,q}(M)$ of global (p,q)-forms. State carefully the Hodge theorem concerning the decomposition of $A^{p,q}(M)$ by means of the $\bar{\partial}$-Laplacian $\Delta_{\bar{\partial}}$, and deduce the standard orthogonal decomposition

$$A^{p,q}(M) = \mathcal{H}^{p,q}_{\bar{\partial}}(M) \oplus \bar{\partial} A^{p,q-1}(M) \oplus \bar{\partial}^* A^{p,q+1}(M),$$

with the $\bar{\partial}$-closed forms being the sum of the first two factors. [*Standard properties of* $\bar{\partial}^* = - * \bar{\partial} *$ *may be assumed.*]

Suppose now that M is also Kähler; define the Laplacians Δ_d, Δ_∂, and prove that (a) $\partial\bar{\partial}^* + \bar{\partial}^*\partial = 0$ (b) $\Delta_d = \Delta_\partial + \Delta_{\bar{\partial}}$ and (c) $\Delta_\partial = \Delta_{\bar{\partial}}$.
[*You may assume the Hodge identity* $[\Lambda, \partial] = i\bar{\partial}^*$, *provided you describe the operator* Λ.]

Let η be a $\bar{\partial}$-exact (p,q)-form on a compact Kähler manifold M; show that $\eta = \bar{\partial}\bar{\partial}^*\alpha$ for some (p,q)-form α. If η is also ∂-closed, prove that $\partial\bar{\partial}^*\alpha$ is harmonic; hence deduce that $\bar{\partial}^*\alpha$ is ∂-closed. Conclude that such an η can be expressed as $\eta = \bar{\partial}\partial\phi$, for some $\phi \in A^{p-1,q-1}(M)$.