UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 6 June 2001 9 to 12

PAPER 12

DIFFERENTIAL GEOMETRY

Answer ***THREE*** *questions, at most* ***ONE*** *of which should be from Section A.*
The questions are of equal weight.

All manifolds and related concepts should be assumed to be smooth.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

SECTION A

1 Give an account of the structure of the tangent bundle of a differential manifold, the various bundles associated to it and cross-sections of those bundles, noting any of particular relevance to differential geometry.

2 Let M be a submanifold of a manifold N. Explain what is meant by a vector field X along M and by a local extension $\tilde{X}$ of X. Show that such local extensions always exist and establish a simple criterion for $\tilde{X}$ to be a local extension of X. Prove that if $\tilde{Y}$ is a local extension of another vector field Y along M then $[\tilde{X}, \tilde{Y}]$ is an extension of $[X, Y]$.

If M^n is a hypersurface in $\mathbb{R}^{n+1}$ define carefully the shape operator, or Weingarten map, on M and prove that it is self-adjoint.

SECTION B

3 Define the curvature tensor of a Riemannian manifold. State and prove the symmetry relations that it satisfies.

Define sectional curvature, Ricci curvature and scalar curvature and state and prove expressions for the latter two in terms of the first.

4 Define what is meant by a normal neighbourhood of a point on a Riemannian manifold. Define the radial function and radial field on such a neighbourhood.

State and prove Gauss' Lemma. Hence identify the radial field in terms of geodesics and show that it is the gradient of the radial function.

5 Define what it means for a Riemannian manifold to be geodesically complete. State and prove the Hopf-Rinow Lemma and deduce the Hopf-Rinow Theorem.

6 For a geodesic γ in a Riemannian manifold define a geodesic variation of γ, a Jacobi field along γ and the index form for vector fields along γ. If p is a point on γ say what it means for a second point q to be conjugate to p along γ and to be a cut point of p along γ.

If p and q are conjugate points along the geodesic γ state how the index form characterizes the Jacobi fields among all piecewise smooth vector fields along γ that vanish at p and q.

Let γ be a geodesic from $p = \gamma(0)$ to $q = \gamma(1)$ such that no point $\gamma(t)$ for $t \in (0, 1]$ is conjugate to p. Let W be a piecewise smooth vector field along γ with $W(p) = 0$ and V be the Jacobi field along γ such that $V(p) = W(p) = 0$ and $V(q) = W(q) \neq 0$. Show that $I(VV) < I(WW)$, where I is the index form for vector fields along $\gamma|_{\{0,1]}$, unless $W = V$.

Deduce that the cut point along γ occurs at or before the first conjugate point. State without proof a characterization of the cut point along γ in terms of conjugate points and geodesics.