UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 5 June 2001 9 to 12

PAPER 11

PROBABILISTIC COMBINATORICS

*Answer **THREE** questions. The questions carry equal weight.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let $k \geq 2$ and $g \geq 3$ be integers. Prove that there exists a graph of chromatic number at least k and girth at least g.

(ii) Let $0 < p < 1$ be fixed. Prove that the chromatic number of a random graph $G(n,p)$ satisfies

$$\frac{n}{2\log_b n} \leq \chi(G(n,p)) \leq \frac{n}{\log_b n}(1+o(1))$$

almost surely, where $b = 1/(1-p)$.

2 State and prove the Harris-Kleitman correlation inequality.

State and prove Janson's concentration inequality.

Outline how Janson's inequality can be used to estimate accurately the chromatic number of a random graph.

3 Let $c > 0$ be fixed and let $p = p(n)$, $\mu = \mu(n)$ satisfy $\binom{n-1}{2}p^3 = \mu$ and $e^{-\mu} = c/n$. Let T be the number of vertices in a random graph $G(n,p)$ *not* lying in any triangle. Show that T is asymptotically Poisson with parameter c.

4 Let $A_1, \ldots, A_n$ be finite probability spaces and let $\Omega = \prod_{i=1}^n A_i$. Given $x \in \Omega$ and $A \subset \Omega$, the Talagrand distance $d_T(x,A)$ is defined as usual by

$$d_T(x,A) = \min\{\, t : \forall \alpha \in \mathbb{R}^n \text{ with } ||\alpha||_2 = 1,\ \exists y \in A \text{ with } \textstyle\sum_{x_i \neq y_i} \alpha_i \leq t\,\},$$

and the event $\overline{A_t}$ is defined by $\overline{A_t} = \{\, x \in \Omega : d_T(x,A) > t\}$.

Prove that $\Pr(A)\Pr(\overline{A_t}) \leq e^{-t^2/4}$ for all $t \geq 0$.

5 Let $\delta > 0$. Prove that there exists $\gamma = \gamma(\delta) > 0$ and $\Delta_0 = \Delta_0(\delta)$ with the following property: if H is a graph with maximum degree $\Delta \geq \Delta_0$, such that $e(H[\Gamma(v)]) \leq (1-\delta)\binom{\Delta}{2}$ for every vertex $v \in H$, then $\chi(H) \leq (1-\gamma)\Delta$.

Indicate briefly both how it might be proved that if H is triangle-free then $\chi(H) \leq O(\Delta/\log\Delta)$, and also why this result is best possible.