UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 1 June 2001 9 to 11

# PAPER 10

# RAMSEY THEORY

*Attempt any **THREE** questions. The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $m$ be a positive integer. Show that, whenever $\mathbb{N}^{(2)}$ is red-blue coloured, there exists

**either**

(i) an $m$-term arithmetic progression $M$ with $M^{(2)}$ blue

**or**

(ii) disjoint $m$-term arithmetic progressions $A$ and $B$ with every edge from $A$ to $B$ red.

[Hint: Suppose that every $m$-term arithmetic progression has at least one of its $\binom{m}{2}$ edges red. This gives a colouring of $\mathbb{N}^2$ with $\binom{m}{2}$ colours, by colouring $(a, d) \in \mathbb{N}^2$ according to which edge of the arithmetic progression with first term $a$ and common difference $d$ is red. Now apply Gallai's theorem.]

**2** State and prove the Hales-Jewett theorem, and deduce van der Waerden's theorem. Prove the strengthened van der Waerden's theorem.

State Gallai's theorem, and show how to deduce it from the Hales-Jewett theorem.

**3** State and prove Rado's theorem.

[You may assume that, for any $m, p, c$, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic $(m, p, c)$-set.]

Deduce that, for any $k$, whenever $\mathbb{N}$ is finitely coloured there exist $x_1, \ldots, x_k$ with $FS(x_1, \ldots, x_k)$ monochromatic.

**4** What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Prove that every $*$-Borel subset of $\mathbb{N}^{(\omega)}$ is Ramsey.