Topics in Set Theory

The following multiple choice self-test questions are intended to give an indication whether you have the first-level prerequisites for the Part III course *Topics in Set Theory.* All of them are basic and easy questions in set theory: you should get most of them right without effort or preparation.

Basics.

1. In the usual formalization of natural numbers and ordered pairs (i.e., $n = \{0, ..., n-1\}$ and $(x, y) := \{\{x\}, \{x, y\}\}$), one of the following statements is true. Which one?

   ☐ **A** $17 \in 4$.

   ☐ **B** $(0, 1) = 2$.

   ☐ **C** $2 \in (0, 1)$.

   ☐ **D** $4 \in (4, 17)$.

2. We often use informal mathematical notation using curly braces to denote sets. However, not every expression corresponds to a set; sometimes, we denote proper classes. Among the following expressions, one corresponds to a proper class. Which one?

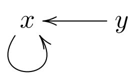
   ☐ **A** $\{x\ ;\ x$ is a nonempty subset of the natural numbers$\}$.

   ☐ **B** $\{x\ ;\ x$ is a finite set of real numbers$\}$.

   ☐ **C** $\{x\ ;\ x$ is a one-element set of rational numbers$\}$.

   ☐ **D** $\{x\ ;\ x$ is a two-element subset of a vector space$\}$.

3. Consider the following model $M = (\{x, y\}, \in)$ as a model of set theory (where $x \in x$ and $x \in y$, but not $y \in x$ or $y \in y$):

   One of the following axiom (scheme)s of set theory is true in M. Which one?

   ☐ **A** The Pairing Axiom.

   ☐ **B** The Axiom of Foundation.

   ☐ **C** The Union Axiom.

   ☐ **D** The Axiom Scheme of Separation.

4. The Zermelo numbers are defined by the following recursion: $z_0 := \varnothing$ and $z_{n+1} := \{z_n\}$. The von Neumann numbers are defined by: $v_0 := \varnothing$ and $v_{n+1} := v_n \cup \{v_n\}$ (i.e., $v_n = n$). One of the following statements is true. Which one?

   ☐ **A** $z_2 = v_2$.

   ☐ **B** $z_2 \in v_4$.

   ☐ **C** $z_2 \subseteq v_2$.

   ☐ **D** $z_2 \in z_4$.

5. One of the following statements about (X, R) implies that (X, R) is a wellorder. Which one?

☐ **A** R is a transitive relation.

☐ **B** R is a linear relation.

☐ **C** X has an R-minimal element.

☐ **D** There is a wellorder (Y, S) and an injective function $f : X \to Y$ such that for all $x_0, x_1 \in X$, we have $x_0 \, R \, x_1$ if and only if $f(x_0) \, S \, f(x_1)$.

6. Consider the integers $\mathbb{Z}$ with their natural order $<$ and their natural multiplication $\cdot$. One of the following sets is wellordered by $<$. Which one?

☐ **A** $\mathbb{Z} \backslash \{0\}$,

☐ **B** $\{z \in \mathbb{Z}\, ; \, \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,

☐ **C** $\{z \in \mathbb{Z}\, ; \, z < 0 \wedge \exists x \in \mathbb{Z}(z = 2 \cdot x)\}$,

☐ **D** $\{z \in \mathbb{Z}\, ; \, \exists x \in \mathbb{Z}(z = x \cdot x)\}$.

Ordinals.

7. One of the following is provable in ZF. Which one?

☐ **A** Every transitive set is an ordinal.

☐ **B** Every transitive set of ordinals is an ordinal.

☐ **C** Every set of ordinals is transitive.

☐ **D** None of the above.

8. Only one of the following statements is correct. Which one?

☐ **A** There are two different order isomorphisms between ω_1 and ω_1.

☐ **B** There are two different order isomorphisms between ω and ω.

☐ **C** There are two different order-preserving embeddings from ω to ω_1.

☐ **D** There are two different order-preserving embeddings from ω_1 to ω.

9. One of the following ordinal inequalities is true. Which one?

☐ **A** $5 \cdot \omega < \omega \cdot 5$.

☐ **B** $\omega \cdot 5 < 5 \cdot \omega + 5$.

☐ **C** $5 + \omega + \omega \cdot \omega \cdot \omega < \omega + 5 + \omega \cdot \omega \cdot \omega$.

☐ **D** $5 \cdot (20 + \omega_1) < 5 \cdot \omega_1$.

Cardinals.

10. The statement "there are no cardinal numbers between $\aleph_0$ and $\aleph_1$" is...

☐ **A** ...provable in ZF,

☐ **B** ...provable in ZFC, but not in ZF,

☐ **C** ...equivalent to the Continuum Hypothesis in the base theory ZFC.

☐ **D** None of the above.

11. In ZFC, one of the following statements is equivalent to the continuum hypothesis. Which one?

☐ **A** There is a bijection between the power set of $\mathbb{N}$ and the real numbers.

☐ **B** Every set of real numbers has an uncountable subset.

☐ **C** Every uncountable set of real numbers has a countable subset.

☐ **D** Every uncountable set of real numbers has a subset that is in bijection with the real numbers.

12. Work in ZF and consider $W := \{(A, R)\,;\, A \subseteq \mathbb{N} \text{ and } (A, R) \text{ is a wellorder}\}$. What is the cardinality of the set W?

☐ **A** ZF proves that it is $\aleph_0$.

☐ **B** ZF proves that it is $\aleph_1$.

☐ **C** ZF does not prove that it is $\aleph_1$, but ZFC proves that it is $\aleph_1$.

☐ **D** ZFC does not prove that it is $\aleph_1$, but ZFC + CH proves that it is $\aleph_1$.

13. Only one of the following statements is false. Which one?

☐ **A** There is an order preserving injection from ω to $\aleph_\omega$.

☐ **B** There is an order preserving injection from ω_1 to $\aleph_\omega$.

☐ **C** There is a cofinal order preserving injection from ω to $\aleph_\omega$.

☐ **D** There is a cofinal order preserving injection from ω_1 to $\aleph_\omega$.

14. Let κ be a cardinal with $\mathrm{cf}(\kappa) = \aleph_0$. Only one of the following statements is provable. Which one?

☐ **A** Every injective function from κ to κ has countable range.

☐ **B** The set κ is a countable union of sets of cardinality strictly smaller than κ.

☐ **C** The cardinal κ is countable.

☐ **D** If $\kappa = \aleph_\xi$, then ξ is a countable ordinal.