Tensors

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1 Relevant courses

The relevant Cambridge undergraduate course is IA Vector Calculus.

2 Books

- K. F. Riley, M. P. Hobson and S. J. Bence *Mathematical Methods for Physics and Engineering.* Cambridge University Press 2002.
- D. E. Bourne and P. C. Kendall *Vector Analysis and Cartesian Tensors.* 3rd edition, Nelson Thornes 1999

3 Notes

3.1 Definition and examples

Consider orthogonal right-handed bases $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ in $\mathbb{R}^3$ with corresponding Cartesian coordinates $\{x_i\}$ and $\{x'_i\}$. Then a vector $\boldsymbol{x} \in \mathbb{R}^3$ can be written as

$$\boldsymbol{x} = x_i \boldsymbol{e}_i = x'_i \boldsymbol{e}'_i$$

(using summation convention throughout these notes).

These two bases are related by a rotation:

$$\boldsymbol{e}'_i = R_{ip}\boldsymbol{e}_p \text{ and } x'_i = R_{ip}x_p$$

where R is a rotation matrix, so

- R is orthogonal: $R_{ip}R_{jp} = R_{qi}R_{qj} = \delta_{ij}$, and
- $\det R = 1$.

Tensors are geometrical objects which obey a generalised form of this transformation rule. By definition, a tensor T of rank n has components $T_{ij\ldots k}$ (with n indices) with respect to each basis $\{\boldsymbol{e}_i\}$ or coordinate system $\{x_i\}$, obeying the *tensor transformation rule*

$$T'_{ij\ldots k} = R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r}.$$

under a change of basis.

Examples Here are some examples of the tensor transformation rule for different ranks:

- Rank 0: $T' = T$. Rank 0 tensors are *scalars.*
- Rank 1: $T'_i = R_{ip}T_p$. As we have seen, these are *vectors.*
- Rank 2: $T'_{ij} = R_{ip}R_{jq}T_{pq}$. These are *matrices* which represent *linear maps* or *quadratic forms.*

Special cases The tensors δ_{ij} and ϵ_{ijk} are tensors of rank 2 and 3 respectively, with the special property that their components are unchanged under any change of basis:

$$R_{ip}R_{jq}\delta_{pq} = R_{ip}R_{jp} = \delta_{ij}$$

and

$$R_{ip}R_{jq}R_{kr}\epsilon_{pqr} = (\det R)\epsilon_{ijk} = \epsilon_{ijk}.$$

Symmetric and antisymmetric tensors A tensor of rank n obeying $T_{ijp\ldots q} = \pm T_{jip\ldots q}$ is said to be *symmetric/antisymmetric* in the indices i and j. A tensor is said to be *totally symmetric/antisymmetric* if it is symmetric/antisymmetric under any such swap of indices.

So, δ_{ij} is totally symmetric and ϵ_{ijk} is totally antisymmetric.

In $\mathbb{R}^3$, any totally antisymmetric tensor of rank 3 takes the form $T_{ijk} = \lambda\epsilon_{ijk}$ for some λ. There is no totally antisymmetric tensor of rank $n > 3$ (unless all components are zeros).

4 Exercises

4.1 Basic properties

Show each of the following:

- If T and S are both tensors of rank n, then $(T+S)_{ij\ldots k} = T_{ij\ldots k} + S_{ij\ldots k}$. (Hint: Use the transformation rule.)
- If α is a scalar, then $(\alpha T)_{ij\ldots k} = \alpha T_{ij\ldots k}$.
- Hence, any linear combination of rank n tensors is itself a rank n tensor.
- If T and S are tensors of rank n and m respectively, then the *tensor product*

  $$(T \otimes S)_{ij\ldots kpq\ldots r} = T_{ij\ldots k}S_{pq\ldots r}$$

  is a tensor of rank $n+m$.
- If $\boldsymbol{u}$, $\boldsymbol{v}$, $\ldots$, $\boldsymbol{w}$ are n vectors, then $T_{ij\ldots k} = u_i v_j \ldots w_k$ defines a tensor of rank n.
- If $T_{ijp\ldots q}$ is a tensor of rank n, then $S_{p\ldots q} = \delta_{ij}T_{ijp\ldots q}$ is a tensor of rank $n-2$. (Note that contracting on a different pair of indices in general results in a different tensor.)
- The trace of a matrix, T_{ii}, is a rank 0 tensor.

4.2 Further exercises

Let $u_i(\boldsymbol{x})$ be a vector field and let $\sigma_{ij}(\boldsymbol{x})$ be a second-rank tensor field. Show that:

- $\partial u_i/\partial x_j$ transforms as a rank 2 tensor.
- $\boldsymbol{\nabla}\cdot\boldsymbol{u} = \partial u_i/\partial x_i$ is a scalar.
- $\partial\sigma_{ij}/partial x_j$ transforms as a vector.

Let T be a tensor of rank 3, satisfying

$$T_{ijk} = T_{jik} \text{ and } T_{ijk} = -T_{ikj}.$$

Show that $T_{ijk} = 0$.

Let T be a tensor of rank 4, satisfying

$$T_{ijkl} = -T_{jikl} = -T_{ijlk} \text{ and } T_{ijij} = 0.$$

Show that

$$T_{ijkl} = \epsilon_{ijp}\epsilon_{klq}S_{pq}$$

where $S_{pq} = -T_{rqrp}$.