Scaling analysis

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1 Relevant courses

The relevant Cambridge undergraduate courses are IB Fluid Dynamics, II Fluid Dynamics and II Mathematical Biology.

2 Books

The principles of scaling analysis, and a number of examples, are given in *Elementary Fluid Dynamics* by D. J. Acheson, OUP 1990.

3 Notes

3.1 The heat equation

Consider the heat equation
$$\frac{\partial u}{\partial t} = \alpha \boldsymbol{\nabla}^2 u$$
where α is the thermal diffusivity (related to the thermal conductivity of a material k by $\alpha = k/(c_p\rho)$.

Specifically, consider this equation on the one-dimensional domain $x \in [0, L]$ for $t > 0$ (so that $\boldsymbol{\nabla}^2 = \frac{\mathrm{d}^2}{\mathrm{d}x^2}$). The initial and boundary conditions are

$$u(x, 0) = f(x) \text{ for some given } f$$

and
$$u(0, t) = u(L, t) \text{ for all t.}$$

What is the timescale for heat to decay?

There are two approaches for analysing this problem.

Approach 1: Exact solution We solve the equation exactly using separation of variables. Looking for $u(x, t)$ as a superposition of solutions of the form $T(t)X(x)$, we find that the solution takes the form

$$u(x, t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(\frac{-n^2\pi^2\alpha t}{L^2}\right).$$

Therefore, the decay happens over a timescale $T \propto L^2/\alpha$.

Approach 2: Dimensional/scaling analysis Let us write $u(x,t) = U\hat{u}(x,t)$, where U is a 'typical' temperature. Also, write $t = T\hat{t}$ and $x = L\hat{x}$. We are interested in finding the timescale T. The hatted quantities are all nondimensional.

Substituting these into the heat equation, and dropping hats, one gets

$$\frac{U}{T}u_t = \frac{\alpha U}{L^2}u_{xx}.$$

Since the dimensions must match up,

$$\frac{U}{T} \propto \frac{\alpha U}{L^2}$$

and so $T \propto L^2/\alpha$ as before.

3.2 The Reynolds number

Consider the Navier-Stokes equations

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} p + \mu\boldsymbol{\nabla}^2\boldsymbol{u} \text{ and } \boldsymbol{\nabla}\cdot\boldsymbol{u} = 0$$

supposing that there are no body forces.

Let $u = U\hat{u}$, $x = L\hat{x}$, $t = T\hat{t}$ and $p = P\hat{p}$, where T is the *advective* timescale, $T = L/U$ (so that the $\frac{\partial \boldsymbol{u}}{\partial t}$ and $\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}$ terms are scaled equally). Putting these into the Navier-Stokes equations, and dropping hats, we get

$$\frac{\rho U^2}{L}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{-P}{L}\boldsymbol{\nabla} P + \frac{\mu U}{L^2}\boldsymbol{\nabla}^2\boldsymbol{u}.$$

The Reynolds number is defined as the ratio of the inertia term to the viscous term:

$$\mathrm{Re} = \frac{\text{inertia}}{\text{viscosity}} = \frac{\rho U^2/L}{\mu U/L^2} = \frac{\rho L U}{\mu} = \frac{LU}{\nu}. \tag{1}$$

We can proceed in two different ways:

- Dividing by the viscous scale gives us

  $$\mathrm{Re}\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{-PL}{\mu U}\boldsymbol{\nabla} P + \boldsymbol{\nabla}^2\boldsymbol{u}.$$

  For *small* Re, we choose the viscous pressure scale $P \sim \frac{\mu U}{L}$, and we get the Stokes equations

  $$\boldsymbol{\nabla} p = \boldsymbol{\nabla}^2\boldsymbol{u}$$

  which hold at low Reynolds numbers (i.e. for viscous flows).

- Alternatively, dividing by the inertia scale gives us

  $$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{-P}{\rho U^2}\boldsymbol{\nabla} P + \frac{1}{\mathrm{Re}}\boldsymbol{\nabla}^2\boldsymbol{u}.$$

  For *large* Re, we choose the inviscid pressure scale $P \sim \rho U^2$, to get the inviscid Euler equations

  $$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\boldsymbol{\nabla} P.$$

4 Exercises

4.1 Exercise 1

A simple model of two competing species eating the same food takes the form

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = r_1 N_1\left(1 - \frac{N_1}{K_1} - b_{12}\frac{N_2}{K_2}\right),$$
$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = r_2 N_2\left(1 - \frac{N_2}{K_2} - b_{21}\frac{N_1}{K_1}\right),$$

where N_1 and N_2 are the population sizes. Rescale the equations to simplify them, and show that the solutions depend only on $\rho = r_2/r_1$, b_{12} and b_{21}.

4.2 Exercise 2

The concentration of a chemical $C(x,t)$ satisfies the nonlinear diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(C)\,\frac{\partial C}{\partial x}\right)$$

and the condition $\int_{-\infty}^{\infty} C(x,t)\,\mathrm{d}x = M$. The diffusivity is given by $D(C) = kC^p$, and M, k and p are positive constants.

Use dimensional analysis to find a suitable space-like scale ξ and a space-independent η for the similarity solution of the form

$$C(x,t) = \eta F(\xi).$$

Use this form to seek the solution initially localised to the origin, and show that F is of the form

$$F(\xi) = \begin{cases} \left(A - \frac{p}{2(2+p)}\xi^2\right)^{1/p} & \text{for } |\xi| < \xi_0 \\ 0 & \text{otherwise} \end{cases}$$

for some A and ξ_0. For the case when $p = 2$, find A and ξ_0.

4.3 Exercise 3: Flow in a 2D thin layer

Consider a flow in a 2D domain where $x \sim L$ and $y \sim \delta L$, where $\delta \ll 1$ so the domain is thin. How do $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ scale?

If $u \sim U$, explain why $v \sim \delta U$, and explain why the advective timescale T is proportional to L/U.

Rescale the Navier-Stokes equations, taking an advective timescale and choosing the pressure scale P so that it always balances the x-momentum equation. Show that three regimes are possible, depending on how large Re is compared to δ:

- If $\delta^2 \mathrm{Re} \ll 1$ then $P \sim \frac{\mu U}{\delta^2 L}$, and

  $$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \text{ and } 0 = -\frac{\partial p}{\partial y}.$$

  This is the *lubrication regime.*

- If $\delta^2 \mathrm{Re} \sim 1$ then again $P \sim \frac{\mu U}{\delta^2 L}$, but now

  $$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} \text{ and } 0 = -\frac{\partial p}{\partial y}.$$

  These are the *unsteady boundary layer equations.* They represent the flow of a low-viscosity fluid in a thin layer near a no-slip boundary; the thickness of the boundary layer is controlled by the viscosity of the fluid.

- For $\delta^2 \mathrm{Re} \gg 1$, one has $P \sim \rho U^2$, and

  $$\frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial p}{\partial x} \text{ and } 0 = -\frac{\partial p}{\partial y}$$

  This is the *shallow water regime.* This regime can be used to model the flow of low-viscosity fluids in chutes, rivers or even oceans (provided that horizontal lengths are far greater than the depth).

4.4 Exercise 4: Decay of vorticity

Writing $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ for the vorticity, and using the identities

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = (\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u} + \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{u}|^2\right)$$

and

$$\boldsymbol{\nabla}^2 \boldsymbol{u} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{u}),$$

show that

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \nu \boldsymbol{\nabla}^2 \boldsymbol{\omega}$$

provided that body forces are conservative. This is the *vorticity equation.*

Why does the $(\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u}$ term vanish in the 2D case?

Show that vorticity decays over a lengthscale $L \propto \frac{\nu}{U}$.