REPRESENTATION THEORY WORKSHOP

1. Introduction

Our aim is to cover the basic representation theory of finite groups. This leads to a first obvious question: what is a representation of a group?

Definition. Let G be a group. A *representation* of G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ where V is some vector space over a field k. We write it as the pair (V, ρ).

Remark. In practice, all of our representations will be over $k = \mathbb{C}$ and will be finite dimensional, i.e V will be a finite dimensional vector space over $\mathbb{C}$. Moreover all groups considered will be finite.

Example.
(1) For G any group, there is always a trivial representation given by $V = \mathbb{C}$ and $\rho(g) = \mathrm{id}$ for all $g \in G$.
(2) Let $G = C_n$ be the cyclic group of order n and $V = \mathbb{C}$. Say $g \in G$ generates the group. Then we have a representation

$$\begin{aligned} \rho : & G \to \mathrm{GL}(V) = \mathbb{C}^\times \\ & g \mapsto e^{\frac{2\pi i}{n}} \end{aligned}$$

(3) Let $G = S_n$ be the symmetric group. Then the signature of a permutation gives a representation $\varepsilon : G \to \{\pm 1\} \leq \mathbb{C}^\times$.
(4) Let $G = D_6 = \langle r, s | r^3 = s^2 = 1, srs = r^{-1} \rangle$. This is the group of symmetries of an equilateral triangle, where r corresponds to the rotation by $\frac{2\pi}{3}$ and s corresponds to the reflection through one of the axes. Then we have a representation

$$\begin{aligned} \rho : & G \to \mathrm{GL}_2(\mathbb{C}) \\ & r \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix} \\ & s \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

where $\omega = e^{\frac{2\pi i}{3}}$. After changing basis, the above matrices become equivalent to

$$\rho'(r) = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}, \quad \rho'(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

which gives what you expect given the description of G as the group of symmetries of a triangle.

In the last example above we saw two different representations which were really the same representation up to changing bases. This can be formalised as follows:

Definition. A *G-homomorphism*, or *intertwining map*, between two representations (V, ρ) and (W, ρ') is a linear map $\varphi : V \to W$ such that for all $v \in V$ and all $g \in G$, we have $\varphi(\rho(g)v) = \rho'(g)\varphi(v)$. A shorter way of writing this is $\varphi \circ \rho = \rho' \circ \varphi$, or $\varphi(gv) = g\varphi(v)$ if we drop the ρ's.

Remark. We will often drop the ρ's and just write gv to mean $\rho(g)v$.

In particular, if the homomorphism φ is a linear isomorphism, then we have $\rho' = \varphi \circ \rho \circ \varphi^{-1}$. In other words, once we identify V and W as vector spaces and once we pick a basis for V, the map φ then becomes a change of basis map and the two representations are the same up to changing bases. So we can talk about representations being isomorphic.

Notation. We will write $\mathrm{Hom}_G(V, W)$ to denote the set of all G-homomorphisms between two representations V and W of G.

Definition. Let V be a representation of G and $0 \neq W \leq V$ a vector subspace. We say that W is a *subrepresentation*, or a *G-invariant* subspace, if for all $g \in G$, $gW \subseteq W$. If V has no proper subrepresentation, then we say that V is *irreducible*.

Example. All the representations in the previous example are irreducible. Moreover 1-dimensional representations are always irreducible. But, say, the representation of $C_3 = \langle g \rangle$ given by

$$\rho(g) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$

is not irreducible since

$$W = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle$$

is a subrepresentation.

In general, over an algebraically closed field, commuting matrices always have a common eigenvector. So the only irreducible complex representations of an abelian group are the 1-dimensional representations (this is not necessarily true over non-algebraically closed fields).

2. Some results

Here are a few basic but very important results on representations of finite groups. In this section we return to the general case where representations are over some arbitrary field k.

Theorem. (Schur's lemma) *Let V, W be irreducible representations of G over k.*

(i) *Every G-homomorphism $\varphi : V \to W$ is either zero or an isomorphism.*
(ii) *If $k = \bar{k}$ then*

$$\dim_k \mathrm{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

In particular if $V \cong W$ then $\mathrm{Hom}_G(V, W) = \{\lambda id : \lambda \in k\}$.

Schur's lemma is a basic but crucial result that can be applied to many different areas of representation theory. You might encounter it in the course on Lie algebras for example.

Now, we saw that either representations contain smaller representations, or they are irreducible. It turns out these irreducible representations are the "building blocks" of all representations.

Theorem. (Maschke) *Suppose that* char$k$ *does not divide* $|G|$. *Then every finite dimensional representation* V *of* G *decomposes as a direct sum of subrepresentations*

$$V = \bigoplus_{i=1}^{n} W_i$$

where $W_i \leq V$ *is irreducible for all* i. *More generally, given a subrepresentation* $W \leq V$, *there exists a* G*-invariant* $W' \leq V$ *such that* $V = W \oplus W'$.

We will see later that the irreducible summands are unique up to isomorphism. We sometimes say that representations of G are *semisimple* or *completely reducible* to say that they decompose into irreducibles in that way. You should think of Maschke's theorem as a type of Jordan Normal Form decomposition for representations. Again there are many analogues of this result in other areas of representation theory, for example Weyl's theorem in the theory of Lie algebras.

Example. The representation of $C_3 = \langle g \rangle$ given by

$$\rho(g) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}$$

decomposes as

$$V = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle .$$

We will see a less trivial example of such decompositions in the next section.

3. The group algebra and permutation representations

We're going to see a different approach of what representations are.

Definition. Let G be a finite group. The *group algebra* $\mathbb{C}G$ is the $\mathbb{C}$-vector space with basis given by the abstract symbols $\{e_g : g \in G\}$. It is a ring with multiplication given by $e_g \cdot e_h = e_{gh}$ (extended $\mathbb{C}$-linearly) and unit $1 = e_1$.

Remark. By abuse of notation, we often write g instead of e_g and think of the group elements as the basis of $\mathbb{C}G$.

The reason for introducing the group algebra is that we have the following

Fact. Representations of G and modules over the ring $\mathbb{C}G$ are the same things, i.e V is a representation of G if and only if it is a $\mathbb{C}G$-module.

This other way of thinking about representations in terms of modules over rings is not going to play a big part for us, but is really important in modern representation theory. The group algebra $\mathbb{C}G$ is itself a representation of G, given by $g \cdot e_h = e_{gh}$ (extended linearly). This is called the *regular representation.*

Example. Let $G = C_3 = \{1, g, g^2\}$. Then with respect to the basis $\{e_1, e_g, e_{g^2}\}$ for $\mathbb{C}G$, we see that the regular representation is given by

$$\rho_{\text{reg}}(g) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Definition. More generally if X is a set on which G acts then we can form the *permutation representation* $\mathbb{C}X$, which is the $\mathbb{C}$-vector space with basis $\{e_x : x \in X\}$ and G-action given by $g \cdot e_x = e_{g \cdot x}$ (extended $\mathbb{C}$-linearly) for all $g \in G$ and $x \in X$.

Example. Remember we may think of D_6 as the symmetries of a triangle, where r is the rotation by $\frac{2\pi}{3}$ and s is the reflection through one of the axes. Let $G = D_6$ act on the set $X = \{\text{vertices of an equilateral triangle}\} = \{v_1, v_2, v_3\}$, where say

$$rv_1 = v_2, rv_2 = v_3, rv_3 = v_1$$

and

$$sv_1 = v_1, sv_2 = v_3, sv_3 = v_2.$$

So the action of G on $\mathbb{C}X$ is given by

$$\rho(r) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then the representation decomposes as $\mathbb{C}X = V_1 \oplus V_2$ where $V_1 = \langle v_1 + v_2 + v_3 \rangle$ and $V_2 = \langle v_1 + \omega v_2 + \omega^2 v_3, v_1 + \omega^2 v_2 + \omega v_3 \rangle$ (remember $\omega = e^{\frac{2\pi i}{3}}$).

We will see later that there are, up to isomorphism, only finitely many finite dimensional irreducible complex representations of a finite group. Using this, we finish this section with a nice

Fact. Every irreducible complex representation V of G is a direct summand of $\mathbb{C}G$, occuring with multiplicity $\dim V$. In other words, if $V_1, \ldots, V_n$ are the irreducible representations of G, then

$$\mathbb{C}G \cong \bigoplus_{i=1}^{n} (\dim V_i) V_i.$$

As $\dim \mathbb{C}G = |G|$, we deduce the following formula:

$$|G| = \sum_{i=1}^{n} (\dim V_i)^2.$$

4. Character Theory

Definition. The *character* of a representation $\rho : G \to \mathrm{GL}(V)$ is the function $\chi_\rho : G \to \mathbb{C}$ (sometimes also written χ_V) defined by

$$\chi_\rho(g) = \mathrm{tr}\rho(g).$$

We often drop the ρ and just write χ when no confusion can arise. Characters afforded by irreducible representations are called *irreducible characters*.

We can deduce a couple of things from the above definition:

- Since the trace is conjugate invariant, the character χ is constant on conjugacy classes. Hence we may think of it as a *class function*, i.e a function $\mathscr{C}(G) \to \mathbb{C}$, where $\mathscr{C}(G)$ denotes the set of conjugacy classes in G.
- Suppose $\mathscr{C}(G) = \{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$. Then the class functions form a vector space with basis given by

  $$\delta_{\mathcal{C}_i} : \mathcal{C}_j \mapsto \delta_{ij}.$$

  In particular this vector space has dimension the number of conjugacy classes, i.e $|\mathscr{C}(G)|$.

Theorem. *The irreducible characters form a basis of the space of class functions. Hence the number of irreducible characters is $|\mathscr{C}(G)|$.*

Example. (1) The cyclic C_n is abelian, so all of its irreducible complex representations are 1-dimensional. If $g \in C_n$ generates the group, then we can easily see that these representations are just given by $g \mapsto \omega^j$ where $\omega = e^{\frac{2\pi i}{n}}$ and $1 \leq j \leq n$. So there are indeed n of them, just as there are n conjugacy classes in C_n.

(2) The group D_6 has 3 conjugacy classes, namely

$$\{1\}, \{r, r^2\}, \{s, sr, sr^2\}.$$

It has two 1-dimensional representations, given by $r \mapsto 1$ and $s \mapsto \pm 1$ (exercise: check these are well-defined). It also has a 2-dimensional one that we saw earlier. By the theorem these are all the irreducible representations.

We saw above that the space of class functions is a vector space with basis the irreducible characters. Furthermore, it is actually an inner product space. Indeed, if we let χ and ψ be two class functions, then we define their inner product to be

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

We then have the following important result:

Theorem. (Orthogonality of characters) *The irreducible characters form an orthonormal basis of the space of class functions, i.e if we let χ and ψ be two irreducible characters afforded by the representations V and W respectively, then we have*

$$\langle \chi, \psi \rangle = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise} \end{cases}$$

This should remind you a bit of Schur's lemma, and indeed this is a consequence of it (hence it is very important that we work over $\mathbb{C}$ here).

Consequences. (1) *Let $V = \bigoplus n_i V_i$ where the V_i's are distinct irreducible representations and $n_i \geq 1$. Write χ for the character of V, and χ_i for the character of V_i. Then*

- $\chi = \sum_i n_i \chi_i$
- $n_i = \langle \chi_i, \chi \rangle$
- $\langle \chi, \chi \rangle = \sum_i n_i^2$

(2) *The decomposition into irreducible representations is unique.*

(3) *If V and W are two representations of G, then*

$$V \cong W \iff \chi_V = \chi_W.$$

(4) *Let χ be a character of G. Then χ is irreducible if and only if $\langle \chi, \chi \rangle = 1$*

Example. Let $G = D_6$ act on the set $X = \{v_1, v_2, v_3\}$ of vertices of a triangle. We already saw previously how the permutation representation decomposes. Let's do it again this time using only characters. Let χ be the corresponding permutation character. The conjugacy classes in D_6 have representatives 1, r and s. Since the character is a class function we only need to compute it at class representatives. We have already computed the action of G on $\mathbb{C}X$ before, from which we deduce that χ takes values

	1	r	s
χ	3	0	1

Now a quick computation gives

$$\langle \chi, \chi \rangle = 2 \text{ and } \langle 1, \chi \rangle = 1,$$

from which we conclude that $\chi = 1 + \psi$ where ψ is the irreducible character

	1	r	s
ψ	2	-1	0

In general we have this useful result:

Theorem. (Burnside's Lemma) *Let G act on a set X, and let $\mathbb{C}X$ be the permutation representation with character χ. Then*

$$\langle 1, \chi \rangle = |\{\text{orbits of } G \text{ in } X\}|.$$

Character tables.

Definition. List the conjugacy classes of G, $\mathcal{C}_1, \dots, \mathcal{C}_n$ say, and pick a representative $g_i \in \mathcal{C}_i$ (by convention $g_1 = 1$). Then list the irreducible characters $1 = \chi_1, \dots, \chi_n$ of G. The *character table* is the matrix

	1	g_2	$\cdots$	g_i	$\cdots$	g_n
1	1	1	$\cdots$	1	$\cdots$	1
$\vdots$				$\vdots$		
χ_j	$\cdots$	$\cdots$	$\cdots$	$\chi_j(g_i)$	$\cdots$	$\cdots$
$\vdots$				$\vdots$		
χ_n				$\vdots$		

Example.

- $G = C_3$

	1	g	g^2
1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

- $G = D_6$

	1	r	s
1	1	1	1
χ_2	1	1	-1
χ_3	2	$\omega + \omega^2$	0

We had orthogonality of characters before, which corresponds to orthogonality between the rows of the character table. But we also have

Theorem. (Column orthogonality) *Pick g_i, g_j as above. Then*

$$\sum_{\chi \in Irr(G)} \chi(g_i)\overline{\chi(g_j)} = \delta_{ij} |C_G(g_i)|.$$

In other words for $g, h \in G$, we have

$$\sum_{\chi \in Irr(G)} \chi(g)\overline{\chi(h)} = \begin{cases} 0 & \text{if } g, h \text{ not conjugate} \\ |C_G(g)| & \text{otherwise} \end{cases}$$

In particular, for $g = h = 1$, this gives $|G| = \sum (\dim V_i)^2$ which we saw before already.

All of these results are useful for computing character tables. These are then useful to understand the structure of the group. Indeed we have the following

Fact.
- For a representation ρ of G with character χ, we have
$$\ker \rho = \{g \in G : \chi(g) = \chi(1)\}.$$
- Let N be a subgroup of G. Then N is normal if and only if it is an intersection of kernels of irreducible representations.

From these two facts we can read off all normal subgroups of G from the character table and determine whether it is simple or not.

Example. We see from our above example that D_6 is not simple because $\ker \chi_2$ was non-trivial.

5. Induction and Lifting

Suppose that H is a subgroup of G. Then we can easily obtain representations of H from representations of G by restricting, i.e if ρ is a representation of G then $\rho|_H$ is a representation of H. How do we go the other way? More generally, how can we use representations of smaller groups to construct new representations of bigger groups?

Lifting. Suppose N is a normal subgroup, and that V is a representation of G/N. Then we get a representation of G by composing the homomorphisms
$$G \to G/N \to \mathrm{GL}(V)$$
where $G \to G/N$ is just the projection onto the quotient. This process is called *lifting.* All the representations of G with kernel containing N arise in this way.

Induction. Now we let H be a subgroup of G, and let V be a representation of H. Induction is process in which one constructs a representation $\mathrm{Ind}_H^G V$ of G from V.

Definition. The vector space $\mathrm{Ind}_H^G V$ is defined to be the space with basis given by the abstract symbols $t_j \otimes v_i$ where $t_1, \ldots, t_n$ is a left transversal, i.e a set of left coset representatives for H, and $v_1, \ldots, v_m$ is a basis of V. The G-action on $\mathrm{Ind}_H^G V$ is given by $g \cdot (t_j \otimes v_i) := t_s \otimes hv_i$ where t_s is the unique coset representative such that $gt_j \in t_s H$, and $h = t_s^{-1} g t_j$.

Secret. Really, as a $\mathbb{C}G$-module, $\mathrm{Ind}_H^G V = \mathbb{C}G \otimes_{\mathbb{C}H} V$ but this will only make sense after you see tensor products over arbitrary rings in the relevant Part III course.

But what's really useful to us is the formula for characters. Let ψ be a character of H and ket $g \in G$. Let $\mathcal{C}_G(g)$ denote the conjugacy class of g in G, and say we have
$$\mathcal{C}_G(g) \cap H = \bigcup_{i=1}^{m} \mathcal{C}_H(x_i)$$
where the x_i are the representatives of the m conjugacy classes of elements of H which are conjugate to g in G. Then we have
$$\mathrm{Ind}_H^G \psi(g) = \begin{cases} 0 \quad \text{if } m = 0 \\ |C_G(g)| \cdot \sum_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|} \quad \text{if } m \geq 1 \end{cases}$$

Example. $G = D_6$ and $H = C_3 = \langle r \rangle$. The character table of H is

	1	r	r^2
1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

In G:

	1	r	s
$\mathrm{Ind}_H^G 1$	2	2	0
$\mathrm{Ind}_H^G \chi_2$	2	$\omega + \omega^2$	0
$\mathrm{Ind}_H^G \chi_3$	2	$\omega + \omega^2$	0

Finally, to decompose induced characters into irreducibles, we have the following important result

Theorem. (Frobenius reciprocity) *Let χ be a character of G and ψ be a character of H. Then*

$$\langle Res_H^G \chi, \psi \rangle_H = \langle \chi, Ind_H^G \psi \rangle_G$$

where $Res_H^G \chi$ is the character of the same representation when viewed as a representation of H.

For the curious: if V is the representation of G with character χ, W is the representation of H with character ψ, then the above is equivalent to saying

$$\mathrm{Hom}_H(\mathrm{Res}_H^G V, W) \cong \mathrm{Hom}_G(V, \mathrm{Ind}_H^G W).$$