REPRESENTATION THEORY EXERCISES

In the following questions, you may assume that all representations are over $\mathbb{C}$ if you want, unless stated otherwise. Some of the questions might be a bit challenging.

(1) Check that the representation of $D_6 = \langle r, s | r^3 = s^2 = 1, srs = r^{-1}\rangle$ given by

$$\rho(r) = \begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ -\sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix}, \quad \rho(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is well-defined, i.e satisfies the relations for D_6: $\rho(s)\rho(r)\rho(s) = \rho(r)^{-1}$, etc. Is it irreducible?

(2) Find a representation of C_3 over $\mathbb{R}$ which is irreducible but not 1-dimensional.

(3) Suppose $\rho : G \to \mathrm{GL}(V)$ is a representation of G, and let ψ be a 1-dimensional representation. Show that the map $\psi \otimes \rho : G \to \mathrm{GL}(V)$ given by

$$\psi \otimes \rho(g) = \psi(g)\rho(g)$$

is a representation of G. Show that ρ is irreducible if and only if $\psi \otimes \rho$ is irreducible.

(4) The following questions are all fairly straightforward applications of Schur's lemma. They are not necessarily all inter-related.
  (a) Show that if V is irreducible, and W is any representation, then the multiplicity of V in W is $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, W)$.
  (b) Show that if $\rho : G \to \mathrm{GL}(V)$ is an irreducible representation over $\mathbb{C}$, then ρ maps the centre $Z(G)$ of G to scalar multiples of the identity. Deduce that the only complex irreducible representations of a finite abelian group are the 1-dimensional ones.
  (c) We say that a representation $\rho : G \to \mathrm{GL}(V)$ is *faithful* if $\ker \rho = 1$. Show that if G has a faithful irreducible representation over $\mathbb{C}$, then $Z(G)$ is cyclic.

(5) We say that a representation V of a group G is *indecomposable* if it cannot be decomposed into a direct sum $V = W_1 \oplus W_2$ of subrepresentations. Show that indecomposable complex representations of a finite group are always irreducible. Let p be a prime. We now consider the two maps $\rho_1 : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ and $\rho_2 : C_p \to \mathrm{GL}_2(\mathbb{F}_p)$ given by

$$\rho_1(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad \rho_2(g^n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Show that these define representations of $\mathbb{Z}$ (over $\mathbb{C}$) and C_p (over $\mathbb{F}_p$). Are they irreducible? Are they indecomposable?

(6) If $G = C_n$, describe the group algebra $\mathbb{C}G$ as a ring.

(7) Decompose explicitly the regular representation $\mathbb{C}G$ of the cyclic group $G = C_n$ into irreducible representations, i.e find explicitly, in terms of the basis elements, all the irreducible representations of G as subrepresentations of $\mathbb{C}G$ and check that $\mathbb{C}G$ does decompose into their direct sum.

(8) Show that the inner product

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}$$

between class functions does satisfy the definition of a complex inner product.

(9) For a representation V of G, let $V^G = \{v \in V : gv = v \text{ for all } g \in G\}$. Denote by χ the character of V, and write 1 for the trivial character, i.e the character of the trivial representation. Show that $\dim_{\mathbb{C}} V^G = \langle 1, \chi \rangle$.

(10) Let V and W be two representations of G over a field k, with characters χ and ψ respectively.
  (a) Show that the space $\mathrm{Hom}_k(V, W)$ of k-linear maps from V to W is a representation of G, with action given by $(gf)(v) = gf(g^{-1}v)$.
  (b) From now on assume $k = \mathbb{C}$. Show that $\mathrm{Hom}_k(V, W)$ has character $g \mapsto \overline{\chi(g)}\psi(g)$.
  (c) Using the previous exercise, deduce that $\dim_k \mathrm{Hom}_G(V, W) = \langle \chi, \psi \rangle$.
  (d) Conclude that the irreducible characters form an orthonormal set.

(11) Let S_n act on the set $X = \{1, \ldots, n\}$, and let χ be the corresponding permutation character. Show that for $\sigma \in S_n$, $\chi(\sigma) = |\{\text{fixed points of } \sigma\}|$. Now assume $n = 4$. Decompose χ into a sum of irreducible characters.

(12) Compute the character table of S_4 directly (hint: use question 3). Now compute the character table of D_8. Using induction from D_8 to S_4, obtain the character table of S_4 again.

(13) Compute the character table of S_5. Use it to compute the character table of A_5. Deduce from the table that A_5 is simple.

(14) (This question requires a bit of knowledge about algebraic integers)
  (a) Let χ be a character of a finite group G, and let $g \in G$. Show that there are roots of unity $\omega_1, \ldots, \omega_d$ such that

$$\chi(g^i) = \omega_1^i + \ldots, \omega_d^i$$

  for all integers i.
  (b) From now on assume that $G = S_n$ is the symmetric group. Deduce that $\chi(g) \in \mathbb{Z}$.
  (c) Is there an irreducible character χ of degree at least 2 with $\chi(g) \neq 0$ for all $g \in G$?

(hint: You may use the fact that characters always have values which are algebraic integers. Moreover you may use the fact that $\sum_{\gcd(i,n)=1} \omega^i \in \mathbb{Z}$ for all nth root of untiy ω.)