Introduction to nonlinear analysis (M24)

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This class in an introduction to the basic analytic tools needed for the mathematical study of nonlinear problems arising from mathematical physics. A particular emphasis will be made on the classical description of fundamental non linear waves discovered in the 19th century: solitons or solitary waves. The exact role of these bubbles of energy in many systems is still mysterious and the subject of an intense research activity. We will review the main classical methods at hand for the mathematical description of these objects:
(i) the Lyapounov-Schmidt theorem and the construction of nonlinear bifuration branches;
(ii) the nonlinear differential equations approach: local solutions, Lyapounov functional and phase portraits;
(iii) the variational approach: minimization of functionals in Hilbert and Banach spaces.

We will then discuss the dynamical properties of these objects and in particular the stability property for some canonical non linear models.

1. Basic tool box of modern analysis.

– Lebesgue's dominated convergence theorem, L^p spaces, Hölder and Young inequalities, the density of smooth functions in $L^p(\mathbb{R}^d)$.
– Hilbert spaces: weak convergence and compact operators.
– A crash course on distributions: definition, examples and basis properties. Derivative in the sense of distributions.
– Continuous Fourier analysis: Fourier in L^1, L^2 and tempered distributions.
– Sobolev spaces $H^s(\mathbb{R}^d)$: definition, Hilbertian structure.
– The Sobolev embedding Theorem and compactness.

2. A canonical model: the harmonic oscillator.

– The Cauchy Lipschitz theorem for ode's: local existence, global existence for linear equations and the blow up criterion.
– Inverting the Laplace operator: ode vs pde approach.
– Inverting the harmonic oscillator: the ode approach in dimension one.
– Inverting the harmonic oscillator: the pde approach, Lax Milgram theorem and variational methods in Hilbert spaces.
– Compactness of the resolvent, the spectral theorem and Legendre polynomials.

3. The Lyapounov-Schmidt approach.

– Computing the spectrum for a perturbed model: formal expansion.
– The Lyapounov-Schmidt approach and the implicit function theorem.
– Starting the bifurcation branch for a non linear model.

3. Phase portrait: solitons for the nonlinear Schrödinger equation.

– Phase portrait in dimension 1.
– Lyapounov monotonicity formulas for higher order non linearities.

4. An introduction to nonlinear variational methods.

– Spherically symmetric Sobolev embedddings.
– The ground state solution to the nonlinear Schrödinger equation.
– Extremal interpolation inequalities and best constant problems.
– Vortices in fluid dynamics: the travelling ring of smoke.

5. The stability of solitons problem

– The linear Schrödinger equation in $\mathbb{R}^d$ and dispersion of linear waves.
– The nonlinear Schrödinger equation in $\mathbb{R}$: local existence and blow up criterion.
– The stability problem for the ground state solitary wave.

Pre-requisites

Basic notions of Hilbert spaces (Hilbertian basis, weak convergence, compact operators). Basic notion of integration (Lebesgue's dominated convergence). Continuous Fourier transform (in L^1).

Literature

1. J.-M. Bony : Intégration et analyse hilbertienne, *cours de l'École Polytechnique*, 2006.
2. J.-M. Bony : Cours d'analyse, théorie des distributions et analyse de Fourier, Éditions de l'École Polytechnique.
3. H. Brézis : *Analyse fonctionnelle. Théorie et applications*, Masson, 1984.
4. Danchin, R.; Raphaël, P., Analyse nonlinéaire, sur la stabilité des ondes solitaires, Ecole Polytechnique 2016, https://math.unice.fr/ praphael/Teaching.html

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.