Modular Representation Theory (L24)

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Modular representation theory, the study of representations over fields of characteristic other than zero, was initiated by L.E. Dickson (1902). Such representations arise naturally in number theory, the theory of error-correcting codes, combinatorics and in topology.

The first really major developments came with Richard Brauer, who from 1935 onwards exploited this rich and virtually untapped area of mathematics. Brauer's methods were mainly character-theoretic. What is now called the theory of Brauer characters determines the composition factors of the representation but, unlike the situation over $\mathbf{C}$, not the equivalence type. One of Brauer's main goals was finding numerical constraints on the orders and internal structure of finite simple groups, and his methods were later used by Glauberman and others in the Classification program.

The next revolution in the theory came with Sandy Green, who considered the modules themselves. His techniques were completely different to those of Brauer, and his main goals lay in understanding the modules, rather than Brauer, who worked mostly with characters and blocks (a set of equivalence classes of characters). The aim of this course is to give a flavour of both classical techniques of Brauer and the more modern techniques of Green. We'll discuss Brauer characters, defect groups, blocks, decomposition numbers as well as projective and injective modules. The course culminates in a proof of Brauer's first main theorem, which establishes a correspondence between blocks with a certain defect group D and blocks of the normaliser $N_G(D)$ with defect group D.

- Review of assumed and basic material: group algebras, representations and modules, reducibility and decomposability, Maschke's theorem, tenors and homs, Frobenius reciprocity, Mackey's theorem, maximal and primitive ideals, Jacobson radical, complete reducibility, semisimple rings, Artin-Wedderburn structure theorem.
- Modular character theory: p-singular and p-regular elements, Brauer characters, Grothendieck groups.
- Irreducible, projective and injective modules: DVRs, (splitting) p-modular systems, forms, decomposition numbers and the decomposition matrix, counting modular irreducible modules.
- Projective indecomposable modules: the Higman criterion, primitive orthogonal idempotents, idempotent refinement.
- Block theory I: the socle and the head, Cartan matrix, lifting projectives, central idempotents.
- Central characters: the central character of an ordinary irreducible mod p determines the block.
- Block theory II: defect groups. relative projectivity and blocks of defect 0.
- The Brauer morphism and the Brauer correspondence; Brauer's first main theorem. If time, (statements of) the Alperin conjecture, comments on local–global conjectures, cyclic defect theory, representation type.

One or two sheets of examples will be provided backed up by classes.

Desirable Previous Knowledge

Basic group theory; ordinary representations/character theory from the Part II course; Sylow theory for finite groups; commutative algebra from Part III courses (rings, ideals, completions, local rings, primality, Artin-Wedderburn theorem); Lie algebras from the Part III course (weight spaces, root systems, Verma and Weyl modules); some categorical nonsense.

Introductory Reading

1. C.W. Curtis, Pioneers of representation theory: Frobenius, Burnside, Schur and Brauer (AMS 1999)
2. I.M. Isaacs, Character theory of finite groups (Dover reprint 1994)
3. G.D. James & M.W. Liebeck, Representations and characters of groups (CUP, 2nd edn 2001)
4. J.P. Serre, Linear Representations of Finite Groups (Springer GTM 42)
5. R.P. Langlands, Representation theory: its rise and its role in number theory. Proceedings of the Gibbs Symposium (New Haven, 1989).

Reading to complement course material

1. J.L. Alperin, Local representation theory (CUP 1986)
2. D.J. Benson, Representations and cohomology, Vol 1 (CUP 1991)
3. K. Erdmann & T. Holm, Algebras and representation theory (Springer 2018)
4. P. Landrock, Finite group algebras and their modules (CUP 1984)
5. P. Schneider, Modular representation theory of finite groups, (Springer 2013)
6. P. Webb, A course in finite group representation theory (CUP 2016). Chapters 6–12.