Mathematical Tripos
Part III Lecture Courses in 2018-2019

Department of Pure Mathematics & Mathematical Statistics

Department of Applied Mathematics & Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*, and are indicated as such after the title. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2017-18. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures and associated course materials as offered in this academic year define the syllabus. Each course lecturer has discretion to vary the material covered.
- Some courses have no writeup available at this time, in which case you will see "No description available" in place of a description. Course descriptions will be added to the online version of the Guide to Courses as soon as they are provided by the lecturer. Until then, the descriptions for the previous year (available at http://www.maths.cam.ac.uk/postgrad/mathiii/courseguide.html) may be helpful in giving a rough idea of course content, but beware of the comments in the preceding item on what defines the syllabus.

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Algebra

Algebra(M24)

Christopher Brookes

The primary aim of the course is to give an introduction to the theory of commutative Noetherian algebras and modules, a theory that is an essential ingredient in algebraic geometry, algebraic number theory and representation theory. In particular we shall learn about (commutative) polynomial and power series algebras.

I shall also include some introductory material about non-commutative algebras as a help to those attending the Lent term courses in Representation Theory and in Iwasawa Algebras.

Topics I hope to fit in will be

Examples, tensor products. Ideal theory for commutative Noetherian algebras, localisations. Artinian algebras (commutative and non-commutative), Artin-Wedderburn theorem. Integral dependence. Dimension theory. Filtratiohs and associated graded algebras. Injective and Projective modules; Ext and Tor. Derivations and differential operators. Hochschild (co-)homology.

Pre-requisites

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields will be helpful but not necessary.

Literature

1. M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra, Addison-Wesley, 1969.
2. N. Bourbaki, Commutative algebra, Elements of Mathematics, Springer, 1989.
3. I. Kaplansky, Commutative rings, University of Chicago Press, 1974.
4. H. Matsumura, Commutative ring theory, Cambridge Studies 8, Cambridge University Press, 1989.
5. M.Reid, Undergraduate Commutative Algebra, LMS student texts 29, Cambridge University Press, 1995.
6. R.Y. Sharp, Steps in commutative algebra, LMS Student Texts 19, Cambridge University Press, 1990.

   The basic introductory text for commutative algebra is Atiyah and Macdonald but it doesn't go into much detail and many results are left to the exercises. Sharp fills in some of the detail but neither book goes far enough. Both Kaplansky and Matsumura cover additional material though Matsumura is a bit tough as an introduction. Reid's book is a companion to one on algebraic geometry and that influences his choice of topics and examples. Bourbaki is encyclopaedic.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Lie Algebras and their representations (M24)

Beth Romano

This course is an introduction to the properties and representations of semisimple complex Lie algebras. The structure and representation theory of semisimple Lie algebras is one of the most beautiful and wide-reaching subjects in mathematics. It has applications to number theory, topology, algebraic geometry, and theoretical physics, to name a few examples, as well as to the representation theory of real and p-adic groups.

Lie algebras arise as tangent spaces to certain differential manifolds called Lie groups, yet they can be defined purely algebraically. The representation theory of a complex simple Lie algebra, which can be understood in terms of combinatorial data, completely determines that of a corresponding group. Understanding this data (e.g. roots, weights, and Weyl groups) will be at the heart of this course.

The following is an outline of the topics that will be covered in the course:

1. The basic structure theory of semisimple Lie algebras.
2. Root systems, Weyl groups, and the classification of simple complex Lie algebras.
3. The classification of finite-dimensional representations, Verma modules, and the Weyl character formula.
4. (If time permits) Chevalley bases and a brief introduction to Chevalley groups.

Desirable Previous Knowledge

The only real prerequisite is linear algebra, though students should be familiar with the language of group actions and of modules. It will be beneficial, but not necessary, to have some familiarity with basic representation theory, as well as tensor products, symmetric powers, and exterior powers of vector spaces.

Reading to complement course material

1. Humphreys, James E. *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, 9, Springer-Verlag, New York-Berlin, 1978.
2. Fulton, William and Harris, Joe. *Representation theory*, Graduate Texts in Mathematics, 129, Springer-Verlag, New York, 1991.
3. Carter, Roger W. *Simple groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1989.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Iwasawa algebras (L24)

Simon Wadsley

This will be a first-course in the p-adic representation theory of p-adic Lie groups via the study of Iwasawa algebras. Iwasawa algebras are completed group rings of p-adic Lie groups. The name comes from Iwasawa's work on infinite towers of Galois extensions with Galois group the p-adic integers. They also arise naturally elsewhere in arithmetic geometry. From a more algebraic perspective they form an interesting family of non-commutative Noetherian algebras.

The flavour of the course will be ring theoretic and representation theoretic but it will be of interest to those whose primary interest is in Number Theory.

At the start of the course p-adic Lie groups and Iwasawa algebras will be introduced and basic properties of them established. This part will be heavily based on Lazard's monograph and Schneider's exposition of it although with more time devoted to examples. What direction the course takes after this will depend on remaining time available and the interests of the audience. Possibilities include some discussion of the arithmetic applications of Iwasawa algebras, more detailed study of the structure of Iwasawa algebras as rings, dimension theory for finitely generated modules over Iwasawa algebras and the relationship between representations of Iwasawa algebras and associated Lie algebras.

Pre-requisites

It will be assumed that students have attended the Part III Algebra course lectured in the Michaelmas Term. The course Lie Algebras and their representations also lectured in the Michaelmas Term will provide useful background. Some familiarity with group theory, representation theory and basic topology (e.g. notions such as compact and Hausdorff) will also be assumed.

Literature

1. Dixon, J. D. and du Sautoy, M. P. F. and Mann, A. and Segal, D., *Analytic pro-p groups*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge 1999
2. Lazard, M., *Groupes analytiques p-adiques.* (French) Inst. Hautes tudes Sci. Publ. Math. No. 26 1965 389603.
3. Schneider, P., *p-Adic Lie Groups.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Heidelberg 2011

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

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Conformal Field Theory, Kac–Moody and Virasoro Algebras (M16)

Non-Examinable (Graduate Level)

Anthony Wasserman

This preamble concerns three theoretical physicists, two of them theologians and the third a self–declared atheist. Graduating in the war time, Freeman Dyson's early interest in number theory and mathematical physics has been described in his Gibbs lectures, "Missed opportunities." Starting from the fundamental work of Ramanujan and Hecke on number theory and complex function theory, Dyson discovered special modular functions: these are generating functions that count states. These later became part of Macdonald's celebrated identities underlying the Kac–Weyl character and denominator formulas. In axiomatic Quantum Field Theory, before the great breakthrough of the standard model, strong interactions were studied using Polkinghorne's analytic S–matrix. The axiomatic technique was later applied in 1+1 dimensions, where Wightman functions become operator–valued distributions on the circle (via Fourier series) and scattering of particles could be interpreted through operator product expansions. The theory of general relativity and black holes has so far had limited success at reconciling QFT with gravity. The research group of Hawking, Strominger, Perry and Haco aims to explain "Black Hole Entropy and Soft

Hair," using the language of conformal field theory: their work predicts that the boundary of a black hole has a rich algebraic structure consistent with CFT.

This graduate course will survey the representation theory of affine Kac–Moody algebras and the Virasoro algebra. These infinite–dimensional Lie algebras play an important rôle in string theory and conformal field theory; they are the Lie algebras of loop groups and the diffeomorphism group of the circle. Taking a unitary viewpoint, supersymmetry is used as one of the main techniques, following the supersymmetric coset constructions of Goddard–Kent–Olive and Kazama–Suzuki. Even in the case of finite–dimensional simple Lie algebras, this approach is fruitful. In particular it will be used to prove the classical Weyl character formula for $U(n)$, following unpublished notes of Peter Goddard. The character formula of Kac–Weyl for the affine KM algebra of $sl(2)$ at level one will be proved directly, using the boson–fermion correspondence and vertex operators. After introducing operator product expansions and vertex algebras, the general Kac–Weyl character formula for the affine KM algebra of $sl(2)$ for any level $\ell \geq 1$ will be proved using supersymmetry. The method generalises to all classical and exceptional cases, including E_8.

The course will also discuss braiding and fusion of primary fields between positive energy representations at a fixed level. Governed by the ODEs of Knizhnik and Zamolodchikov, gauge invariance of the compact group gives rise to so–called "symmetric Fuchsian systems." These special hypergeometric differential equations can be studied independently, without recourse to vertex operators or vertex algebras. In conformal mapping and complex function theory, H.A. Schwarz showed how hyperbolic equilateral triangles of angle π/n or zero (i.e. ideal triangles) could be tesselated by hyperbolic reflections. The monodromy of Gauss' hypergeometric equation, viewed as a rank 2 matrix–valued first–order complex ODE, relates B_3, the braid group on 3 strings, with the modular group $SL(2, Z)$ and the Hecke groups. Solutions of the Schwarzian yield automorphic functions and uniformising parameters; in particular, as n tends to ∞, one obtains the elliptic modular functions j and λ. This hypergeometric theory can be extended to rank N systems for certain remarkable classes of examples indexed by Dynkin–type diagrams or "quivers." These can be described by algebraic combinatorics or geometry: multiplicity–free products of characters of $U(n)$; or multiple flag varieties of finite type. A special case of Katz's rigidity theorem, involving only elementary linear algebra, shows how to compute the braiding–fusion coefficient matrices up to conjugation.

Literature

- 1. F.J. Dyson, *Missed Opportunities,* Gibbs Lectures, Bulletin of AMS, 1972
- 2. P. Goddard and D. Olive, "Kac–Moody and Virasoro Algebras," World Scientific, reprint volume, 1986.
- 3. T. Kohno, "New developments in the theory of knots," World Scientific, reprint volume, 1990.
- 4. F. Beukers, *Hypergeometric Functions, How Special Are They?,* Notices of AMS, 2014.
- 5. A. Ogg, "Modular Forms and Dirchlet Series," Benjamin, 1969.
- 6. M. Brion and C. Woodward, *Spherical Varieties, Moment Maps and Geometric Invariant Theorem,* CIRM courses, 2010
- 7. A. Szowronski and G. Zwara, "Trends in the representation theory of finite-dimensional algebras," Contemporary Mathematics, 1998.
- 8. W.W. Crawley–Boevey, *Geometry of the Moment Map for Representations of Quivers,* Compositio Mathematica, 2001.

Algebraic Geometry

Algebraic Geometry (M24)

Mark Gross

This will be a basic course introducing the tools of modern algebraic geometry. The most relevant reference for the course is the book of Hartshorne and the notes of Vakil.

The course will begin with a quick review of the theory of varieties as presented in the Part II algebraic geometry course (see e.g., the book of Reid for this background) and then proceeding to sheaves and the notion of an abstract variety. We then turn to an introduction to scheme theory, explaining why we want schemes and what they will do for us. We define schemes and introduce projective schemes. From there, we will pass to coherent sheaves, and introduce a number of tools, such as sheaf cohomology, necessary for any practicing algebraic geometer, with applications to problems in projective geometry.

Pre-requisites

Basic theory on rings and modules will be assumed. It is strongly recommended that students either have had a previous course on Commutative Algebra or had a quick read of the book on Commutative Algebra by Atiyah and MacDonald, and/or the elementary text by Reid on Algebraic Geometry.

Literature

Introductory Reading

1. M. Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press (1988) (preliminary reading).
2. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison–Wesley (1969) (basic text also for the commutative algebra we'll need).

Reading to complement course material

1. U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner, 2010.
2. R. Hartshorne, *Algebraic Geometry*, Springer (1977) (more advanced text).
3. R. Vakil, *The rising sea. Foundations of Algebraic Geometry*, available at http://math.stanford.edu/∼vakil/216blog/index.html

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Topics on moduli spaces in algebraic geometry (L24)

Non-Examinable (Graduate Level)

Caucher Birkar

In mathematics it is often the case that in order to understand a collection of objects one tries to find a space to parametrise them. If the parametrising space itself carries interesting structures, then methods of the subject can be applied to the parametrising space to get information about the collection. This philosophy is particularly fruitful in algebraic geometry because spaces parametrising certain varieties or sheaves, called *moduli spaces*, are often finite unions of algebraic varieties with a rich geometry. Such spaces not only help to understand families of varieties and sheaves but also provide interesting and non-trivial examples of varieties exhibiting certain behaviour.

A random collection of varieties or sheaves usually does not have a nice parametrising space. It is important to impose special properties ensuring that members of the collection fit together nicely to form a moduli space. Such properties are often some kind of *stability*.

This course provides an introduction to aspects of moduli theory of varieties and sheaves. Time permitting we hope to discuss topics such as the Grassmannian and Hilbert schemes, moduli of stable varieties, moduli of stable sheaves, derived categories and stability conditions, etc. In addition to the general theory we hope to look at special cases and specific examples.

Pre-requisites

Part III algebraic geometry. Students are expected to look up basic algebraic geometry material not covered by the part III course.

Literature

This is a list of general and expository references. More specific references will be mentioned as the course progresses.

1. Birkar, C, *Topics in algebraic geometry*,

   `https://www.dpmms.cam.ac.uk/~cb496/topics-ag-2010-final.pdf`

2. Huizenga, J, *Birational geometry of moduli spaces of sheaves and Bridgeland stability condition*,

   `https://arxiv.org/pdf/1606.02775.pdf`

3. Kollár, J, *Families of varieties of general type*,

   `http://web.math.princeton.edu/~kollar/book/modbook20170720.pdf`

4. Macrì, E and Schmidt, B, *Lectures on Bridgeland stability*,

   `https://arxiv.org/pdf/1607.01262.pdf`

Analysis

Analysis of Partial Differential Equations (M24)

Dr Warnick

This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on developing the theory and methods of the modern approach to PDE theory. Emphasis will be given to functional analytic techniques, relying on a priori estimates rather than explicit solutions. The course will primarily focus on approaches to linear elliptic and evolutionary problems through energy estimates, with the prototypical examples being Laplace's equation and the heat, wave and Schrödinger equations.

The following concepts will be studied: well-posedness; the Cauchy problem for general (nonlinear) PDE; Sobolev spaces; elliptic boundary value problems: solvability and regularity; evolutionary problems: hyperbolic, parabolic and dispersive PDE.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Preliminary Reading

The following article gives an overview of the field of PDEs:

1. Klainerman, S., *Partial Differential Equations*, Princeton Companion to Mathematics (editor T. Gowers), Princeton University Press, 2008.

Literature

1. Some lecture notes from a previous lecturer of the course are available online at:
   `http://cmouhot.wordpress.com/teachings/`.

   The following textbooks are excellent references:

2. Evans, L. C., *Partial Differential Equations*, Springer, 2010.
3. Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.
4. John, F., *Partial Differential Equations*, Springer, 1991.

Additional Information

This course is also intended for doctoral students of the Centre for Analysis (CCA), who will also be involved in additional assignments, presentations and group work. Part III students do not do these, and they will be assessed in the usual way by exam at the end of the academic year. Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be one office hour a week.

Topics in Ergodic Theory (M24)

Péter Varjú

Ergodic theory studies dynamical systems that are endowed with an invariant measure. There are many examples of such systems that originate from other branches of mathematics. This led to a fruitful interplay between ergodic theory and other fields, especially number theory.

I will explain some basic elements of ergodic theory, such as recurrence, ergodic theorems, mixing properties and entropy. I will also talk about some applications of the theory, such as Furstenberg's proof of Szemerédi's theorem, and Weyl's equidistribution theorem for polynomials.

I aim to cover the following topics:

- Furstenberg's correspondence principle,
- Poincaré recurrence, ergodicity,
- ergodic theorems,
- unique ergodicity,
- Weyl's equidistribution theorem for polynomials,
- mixing and weak mixing,
- the multiple recurrence theorem for weak mixing systems,
- entropy and its relation to mixing,
- Rudolph's theorem on $\times 2$, $\times 3$ invariant measures.

Pre-requisites

Measure theory, basic functional analysis, conditional expectation, Fourier transform.

Literature

Notes will be available on the lecturer's webpage.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Fractal Geometry (L24)

Non-Examinable (Graduate Level)

Ariel Rapaport

I aim to cover the following topics,

- self-similar sets and measures;
- fractal dimensions - Hausdorff, Minkowski, and entropy dimensions;
- projections of fractals to linear subspaces;

- intersections of fractals with affine subspaces;
- dimension of self-similar sets with overlaps;
- Bernoulli convolutions;
- self-affine sets and measures (as time permits).

Pre-requisites

Measure theory, basic ergodic theory, basic Fourier analysis.

Literature

The following sources are relevant for the first few topics mentioned above.

1. P. Mattila, Geometry of sets and measures in Euclidean spaces. Cambridge University Press, Cambridge, 1995.
2. K. Falconer, Techniques in fractal geometry, John Wiley & Sons, 1997.

Elliptic Partial Differential Equations (L24)

Iván Moyano

This course is intended as an introduction to the theory of linear second order elliptic partial differential equations. Second order elliptic equations play a fundamental role in many areas of mathematics including geometric analysis and mathematical physics. A strong background in the linear theory provides a foundation for studying a number of non-linear problems including minimal submanifolds, harmonic maps, geometric flows and general relativity. We will discuss both classical and weak solutions to linear elliptic equations focusing on the question of existence and uniqueness of solutions to the Dirichlet problem and the question of regularity of solutions. This involves establishing maximum principles, Schauder estimates and other a priori estimates for the solutions. As time permits, we will discuss other topics including the De GiorgiNashMoser theory (which provides the Harnack inequality and establishes Holder continuity for weak solutions), applications of the linear theory to quasilinear elliptic equations, regularity for hypoelliptic operators and spectral analysis of Schrödinger operators.

Pre-requisites

Lebesgue integration, Lebesgue spaces, Sobolev spaces and basic functional and Fourier analysis.

Literature

1. David Gilbarg and Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order. Springer-Verlag (1983).
2. Lawrence Evans, Partial Differential Equations. AMS (1998)
3. Qing Han and Fanghua Lin, Elliptic partial differential equations. Courant Lecture Notes, Vol. 1 (2011)
4. Lars Hörmander, The Analysis of Linear Partial Differential Operators, vols. I and II. Springer-Verlag (1983).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in analysis of many-particle systems (E12)

Non-Examinable (Graduate Level)

Clément Mouhot

This non-examinable course will present some mathematical tools and concepts for the rigorous derivation and study of nonlinear partial differential equations arising from many-particle limits: Vlasov transport equations, Boltzmann collision equations, nonlinear diffusion, quantum Hartree equations... Depending on time and interest it will include part or all of the following items: Liouville and master equations of a many-particle system, the notion of empirical measures, the BBGKY hierarchy, the Hewitt-Savage theorem, the Braun-Hepp-Dobrushin theorem, the coupling method, the concepts of chaos and entropic chaos, the recent works making progress on the mean-field limit, the hydrodynamic limit of lattice systems.

Pre-requisites

Basics in measure theory, functional analysis, partial differential equations and probability.

Literature

1. H. Spohn, *Large Scale Dynamics of Interacting Particles.* Springer 1991.
2. C. Kipnis and C. Landim, *Scaling Limits of Interacting Particle Systems.* Springer 1999.
3. F. Golse, *The Mean-Field Limit for the Dynamics of Large Particle Systems*, Journaux derivees partielles Forges-les-Eaux, 2-6 juin 2003.
4. F. Bolley,*Optimal coupling for mean field limits*, available online
5. P.-E. Jabin, *A review of the mean field limits for Vlasov equations*, Kinetic and Related models 2014, vol 7, pp 661-711.
6. P.-E. Jabin & Z. Wang, Mean Field Limit and Propagation of Chaos for Vlasov Systems with Bounded Forces. J. Funct. Anal. 271 (2016), no. 12, 3588-3627.
7. P.-E. Jabin & Z. Wang, *Quantitative estimates of propagation of chaos for stochastic systems with* $W^{-1,infty} kernels, preprint.$
8. D. Lazarovici, Dustin & P. Pickl, *A mean field limit for the Vlasov-Poisson system.* Arch. Ration. Mech. Anal. 225 (2017), no. 3, 1201-1231.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Combinatorics

Combinatorics (M16)

Prof. I. Leader

The flavour of the course is similar to that of the Part II Graph Theory course, although we shall not rely on many of the results from that course.

We shall study collections of subsets of a finite set, with special emphasis on size, intersection and containment. There are many very natural and fundamental questions to ask about families of subsets; although many of these remain unsolved, several have been answered using a great variety of elegant techniques.

We shall cover a number of 'classical' extremal theorems, such as those of Erdős-Ko-Rado and Kruskal-Katona, together with more recent results concerning such topics as 'concentration of measure' and hereditary properties of hypergraphs. There will be several indications of open problems.

We hope to cover the following material.

Set Systems

Definitions. Antichains; Sperner's lemma and related results. Shadows. Compression operators and the Kruskal-Katona theorem. Intersecting families; the Erdős-Ko-Rado theorem.

Isoperimetric Inequalities

The vertex-isoperimetric inequality in the cube (Harper's theorem). Inequalities in the grid. The classical isoperimetric inequality on the sphere. The 'concentration of measure' phenomenon. Applications, including Katona's t-intersecting theorem.

Projections

The trace of a set system; the Sauer-Shelah lemma. Bounds on projections: the Bollobás-Thomason box theorem. Hereditary properties. Intersecting families of graphs.

Desirable Previous Knowledge

The only prerequisites are the very basic concepts of graph theory.

Introductory Reading

1. Bollobás, B., Combinatorics, C.U.P. 1986.

Introduction to discrete analysis (M16)

W. T. Gowers

The aim of this course is to explain various key results and techniques in combinatorics, with an emphasis on parts of the subject with an analytic flavour and parts that have connections to other branches of mathematics such as harmonic analysis, analytic number theory and theoretical computer science (though many of these connections will not be made explicit).

The following is an approximate guide to the content of the course. Figures in square brackets represent the rough number of lectures needed for each section.

Discrete Fourier analysis. Roth's theorem, Bogolyubov's method. [2]

Sumsets. Plünnecke's theorem, the Balog-Szemerédi theorem, Freiman-type theorems. [3]

Quasirandomness. Quasirandom graphs, bipartite graphs, and subsets of finite Abelian groups, Szemerédi's regularity lemma, the triangle removal lemma. [4]

The polynomial method. Dvir's theorem, the combinatorial Nullstellensatz, solution of the cap-set problem. [2]

Introduction to higher-order Fourier analysis. Box norms, uniformity norms, related inequalities, Szemerédi's theorem for progressions of length 4 (some parts stated without proof or and some parts sketched). [5]

Prerequisites

There are very few prerequisites. It would be useful to know the basic definitions of graph theory and (discrete) probability. The discrete Fourier analysis will start from first principles, though a prior acquaintance with Fourier analysis will help you put it in context.

This lack of prerequisites does not necessarily make the course easy: experience suggests that the examples sheets, which are important if you want to get the most out of the course, are found quite hard.

Literature

Probably the best source of literature is the internet, where many different accounts of the topics covered in this course can be found. The nearest there is to a standard textbook in the area (which, however, covers far more material than this course will, and in a rather more abstract style) is the following.

T. C. Tao and V. Vu, *Additive Combinatorics*, Cambridge Studies in Advanced Mathematics **105**, Cambridge University Press.

Additional support

There will be three problem classes, with an examples sheet for each one, and possibly an additional sheet after the course has finished to give further practice in techniques of the course. There will also be a revision session in the summer term.

Introduction to approximate groups (L16)

Matthew Tointon

Approximate groups have been the subject of much research in recent years, with applications to fields as diverse as geometry, number theory and theoretical computer science. The main aims of this course will be to motivate the definition of approximate groups, to describe their algebraic structure, and to present some fairly general techniques that can be used in their study. We will also look at some applications to geometric group theory.

The content of the course will be roughly as follows. The numbers in square brackets are the approximate number of lectures that are likely to be required for each section.

Introduction: small doubling of subsets of groups; proof that sets of very small doubling are close to subgroups; definition of approximate groups and link to small doubling; basic examples; Freiman homomorphisms. [2]

Deduction of the Freiman–Ruzsa theorem from results of the Michaelmas course 'Introduction to Discrete Analysis': Minkowski's second theorem; proof that Bohr sets are contained efficiently inside coset progressions; Ruzsa's and Chang's covering arguments. [2]

Nilpotent groups: basic commutators; commutator calculus; nilprogressions. [2]

A Freiman–Ruzsa-type theorem for nilpotent groups of bounded class. [4]

The algebraic structure of approximate subgroups of arbitrary nilpotent groups. [3]

Residually nilpotent groups: introduction to residual properties; residual nilpotence of free groups; algebraic structure of approximate subgroups of residually nilpotent groups. [1]

Growth in groups: introduction to volume growth in finitely generated groups; a Gromov-type theorem for residually nilpotent groups. [2]

Pre-requisites

It will be helpful to have attended the course 'Introduction to Discrete Analysis' in the Michaelmas Term, and we will quote some results from it. Nonetheless, the techniques of this course will be mainly algebraic, in contrast to the analytic techniques of that course, so not having attended 'Introduction to Discrete Analysis' should not prevent anyone from understanding the material of this course.

Literature

Most of the material of the second half of the course at present appears only in research papers, but the following surveys give a high-level overview of the field and some of its applications.

1. E. Breuillard, *Lectures on approximate groups and Hilbert's 5th problem*, Recent Trends in Combinatorics, The IMA Volumes in Mathematics and its Applications **159** (2016), 369-404. Also available at
   `http://arxiv.org/abs/1512.01369`
2. B. J. Green, *Approximate groups and their applications: work of Bourgain, Gamburd, Helfgott and Sarnak*, Current events bulletin of the AMS (2010). Also available at
   `http://arxiv.org/abs/0911.3354`

Most of the material on nilpotent groups and commutators will be based on the following book.

3. M. Hall. *The theory of groups*, Amer. Math. Soc./Chelsea, Providence, RI (1999).

Finally, most of the material of the first two sections is covered in the following book.

4. T. C. Tao and V. H. Vu. Additive combinatorics, *Cambridge studies in advanced mathematics* **105**, Cambridge Univ. Press (2006).

Additional support

There will be three examples sheets and three associated examples classes. There will be a one-hour revision class in the Easter Term.

Connections between model theory and combinatorics (L16)

Non-Examinable (Graduate Level)

Julia Wolf

This course serves as an introduction to recent developments at the interface between model theory and combinatorics. It will focus on results of a finitary and combinatorial nature, including Ramsey-type theorems in graphs under additional tameness assumptions, and applications; efficient arithmetic regularity lemmas in groups as a consequence of local stability theory; bounded quotients and the structure of approximate groups; and dividing lines in unstable theories. If time permits, we will explore connections with computational complexity.

Pre-requisites

This course is particularly suitable for those students who have attended the Michaelmas courses "Introduction to discrete analysis" and "Model theory", and those who are planning to take "Introduction to approximate groups" in the Lent term. Students who have not had a chance to attend one or more of these courses are welcome to contact the lecturer to discuss suitable background reading. Every attempt will be made to keep the course as self-contained as possible.

Preliminary Reading

1. Lecture notes for any of the Part III courses mentioned under Pre-requisites.
2. D. Marker. *Model Theory.* Vol. 217, Springer-Verlag, New York, 2002.
3. T. Tao and V. Vu. *Additive Combinatorics.* Vol. 105, Cambridge University Press, 2010.

Literature

Most of the course material will be extracted from fairly recent research papers. The following sources cover some aspects of the course in a more expository fashion.

1. B. Poizat. *Stable Groups.* Mathematical Surveys and Monographs Vol. 87, American Mathematical Society, 2001.
2. L. van den Dries. *Approximate Groups [According to Hrushovski and Breuillard, Green, Tao].* Astérisque, no. 367, 2015.

Additional support

Exercises will be provided throughout the course, and three examples classes will be arranged to allow participants to deepen their understanding of the material through informal discussion.

Geometry and Topology

Algebraic Topology (M24)

Ivan Smith

Algebraic Topology, which studies topological spaces by associating algebraic invariants, permeates modern pure mathematics and theoretical physics. This course will focus on cohomology, with an emphasis on applications to the topology of smooth manifolds. Topics will include: basic theory of (co)chain complexes; singular and cellular (co)homology; cup-products; vector bundles; Thom isomorphism and the Euler class; Poincaré duality.

The course will not strictly assume any knowledge of algebraic topology, but will go quite fast at the beginning in order to reach more interesting material, so some previous exposure to chain complexes and simplicial homology, and/or to the fundamental group, would certainly be helpful. Some material from the Differential Geometry course (for background on smooth manifolds) might also be helpful.

Pre-requisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. Some knowledge of the fundamental group would be helpful though not a requirement. The books by Bott and Tu and by Hatcher are especially recommended, but there are many other suitable texts and many online resources.

Literature

1. Bott, R. and Tu, L. *Differential forms in algebraic topology.* Springer, 1982.
2. Hatcher, A. *Algebraic Topology.* Cambridge Univ. Press, 2002.
3. May, P. *A concise course in algebraic topology.* Univ. of Chicago Press, 1999.
4. Sutherland, W. *Introduction to metric and topological spaces.* Oxford Univ. Press, 1999.

Additional support

Three or four examples sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Differential Geometry (M24)

A. Kovalev

This course is intended as an introduction to modern differential geometry. It can be taken with a view of further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are e.g. in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, matrix Lie groups. Tangent vectors, the tangent and cotangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Exterior algebra of differential forms. Orientability of manifolds. Partition of unity and integration on manifolds, Stokes' Theorem. De Rham cohomology.

- *Vector Bundles.* Structure group, principal bundles. The example of Hopf bundle. Bundle morphisms. Three views on connections: vertical and horizontal subspaces, Christoffel symbols, covariant derivative. The curvature form and second Bianchi identity.

- *Riemannian Geometry.* Connections on manifolds, torsion. Riemannian metrics, the Levi–Civita connection. Geodesics, the exponential map, Gauss' Lemma. Decomposition of the curvature of a Riemannian manifold, Ricci and scalar curvature, low-dimensional examples. The Hodge star and Laplace–Beltrami operator. Statement of the Hodge decomposition theorem (with a sketch-proof, time permitting).

Pre-requisites

An essential pre-requisite is a working knowledge of linear algebra (including dual vector spaces and bilinear forms) and of multivariate calculus (e.g. differentiation in several variables and the inverse function theorem). Exposure to some ideas of classical differential geometry would be useful.

Literature

[1] R.W.R. Darling, *Differential forms and connections.* CUP, 1994.

[2] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry.* Springer-Verlag, 1990.

[3] V. Guillemin, A. Pollack, *Differential topology.* Prentice-Hall Inc., 1974.

[4] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer, 1983.

Roughly, half of the course material is taken from [4]. The book [3] covers the required topology. On the other hand, [1] which has a chapter on vector bundles and on connections assumes no knowledge of topology. Both [1] and [2] have a lot of worked examples. There are many other good differential geometry texts, e.g. the five volume series by M. Spivak.

Additional support

The lectures will be supplemented by four example classes, the fourth class to take place at the beginning of Lent Term will also fulfill a revision function. Printed notes will be available from
`https://www.dpmms.cam.ac.uk/~agk22/teaching.html`

Introduction to Geometric Group Theory (M16)

Ana Khukhro

Groups are algebraic objects which are well-suited to capturing notions of symmetry. As well as their intrinsic algebraic structure, groups have the ability to act on other mathematical objects such as sets or spaces. This action is often useful for learning more about either the object that the group is acting on, or the group itself. The Orbit-Stabiliser Theorem, which students will have already met, is a basic example of this phenomenon.

When a group acts on a metric space in a sufficiently nice way, one can often use the geometric properties of the space to deduce algebraic or analytic information about the group. In this way, one can build up a dictionary between algebra and geometry, which features beautiful and sometimes surprising connections between these two subjects. These connections can often be exploited to solve deep problems in other fields such as topology or analysis.

One metric space on which a group acts in a particularly pleasing way can be created using data from the group itself. Namely, by fixing a generating set of the group, one can construct a graph with vertex set equal to the set of elements of the group, with edges defined using multiplication by elements of the generating set. This graph, called a Cayley graph of the group, is not only a neat visualisation of the group, but is also an invaluable tool in modern group theory, since the geometric properties of this graph are profoundly connected to the group-theoretic properties of the group. Geometric group theory is the study of groups and spaces via these connections.

In this course, we will concentrate on some of the following aspects of this rich theory (time permitting):

- free groups and the Nielsen–Schreier Theorem; group presentations; group constructions such as free products, HNN extensions, and wreath products;
- Cayley graphs; quasi-isometries; the Švarc–Milnor Lemma;
- a Smörgåsbord of geometric properties and invariants of groups, such as growth, ends, hyperbolicity, and connections to algorithmic group theory;
- analytic properties of groups, such as amenability, and connections to actions on Banach spaces.

Pre-requisites

A good knowledge of basic group theory is essential. Part II Algebraic Topology (or equivalent) is required, and some intuition in graph theory and geometry would be helpful. Some functional analysis (such as the Part II Linear Analysis course or the beginning of the Part III Functional Analysis course) will be useful for the last part of the course.

Literature

1. P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics, 2000.
2. C. Druţu and M. Kapovich, *Geometric Group Theory*, Colloquium Publications 63, 2018. Also available at

   `https://www.math.ucdavis.edu/~kapovich/EPR/ggt.pdf`
3. P. W. Nowak and G. Yu, *Large Scale Geometry*, EMS Textbooks in Mathematics, 2012.
4. M. Clay and D. Margalit (Editors), *Office Hours with a Geometric Group Theorist*, Princeton University Press, 2017.
5. M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften 319, 1999.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Categorified Knot Invariants (M24)

Non-Examinable (Graduate Level)

Jacob Rasmussen

This course gives an introduction to the study of categorified knot invariants. In their simplest form, such invariants take a polynomial invariant of knots, like the Jones polynomial, and upgrade it to a graded homology theory whose graded Euler characteristic is the original knot polynomial. The first example of such an invariant (categorifying the Jones polynomial) was constructed by Khovanov in [1]. The theory of such invariants has expanded tremendously in the 20 years since Khovanov's seminal paper, and is now a subject of current research in fields including topology, representation theory, algebraic geometry, and theoretical physics.

Some topics which I intend to cover include:

1. The Jones polynomial and Khovanov homology. Applications to crossing number and slice genus.
2. The Temperly-Lieb algebra. Bar-Natan's formulation of Khovanov homology for tangles.
3. The HOMFLY-PT polynomial, Hecke algebras of type A, and the Kazhdan-Lusztig basis. HOMFLY-PT homology.
4. Reshetikhin-Turaev invariants of tangles and their categorifications. The colored HOMFLY-PT polynomial and Λ^k-colored HOMFLY-PT homology.

If there is time, I may also discuss Jones-Wenzl projectors and their categorifications.

Pre-requisites

No prior knowledge of knot theory is needed, but I will assume familiarity with chain complexes and their homology. Some prior exposure to the representation theory of either finite groups or Lie groups would be helpful.

Literature

1. M. Khovanov, A categorification of the Jones polynomial *DMJ* 101 (2000), 359-426. arXiv:math/9908171.
2. D. Bar-Natan, Khovanov's homology for tangles and cobordisms, *Geom. Topol.* 9 (2005), 1443-1499. arXiv:math/0410495.
3. V. Jones, Hecke Algebra Representations of Braid Groups and Link Polynomials, *Annals of Math.* 16 (1987) 335-388.
4. M. Khovanov, Triply-graded link homology and Hochschild homology of Soergel bimodules, *Int. J. Math.* 18 (2007), 869-885. arXiv:math/0510265.

Additional support

This is a non-examinable course, but I will provide example sheets. I would also be willing to hold some examples classes if there is enough demand.

Discrete subgroups of Lie groups (M16)

Non-Examinable (Graduate Level)

Prof Emmanuel Breuillard

The course will provide an introduction to discrete subgroups of Lie groups and to the diverse techniques used to study them. These include hyperbolic geometry and symmetric spaces, unitary representations, algebraic number theory, ergodic theory and dynamics of homogeneous flows as well as random walks. Several of these aspects will be presented and one of the goal of the lectures will be the Mostow-Margulis rigidity theorems and the Margulis arithmeticity theorem for lattices in semisimple Lie groups.

Pre-requisites

This course assumes some familiarity with differential geometry, probability and ergodic theory.

Literature

1. Borel, Armand *Introduction aux groupes arithmtiques.* Publications de l'Institut de Mathmatique de l'Universit de Strasbourg, XV. Actualits Scientifiques et Industrielles, No. 1341 Hermann, Paris 1969 125 pp.

2. Raghunathan, M. S. *Discrete subgroups of Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg, 1972.

3. Zimmer, Robert J. *Ergodic theory and semisimple groups.* Monographs in Mathematics, 81. Birkhuser Verlag, Basel, 1984. x+209 pp.

4. Margulis, G. A. *Discrete subgroups of semisimple Lie groups.* Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 17. Springer-Verlag, Berlin, 1991. x+388 pp.

Additional support

One or two Part III essays will be offered in connection to some of the topics of the course.

Loop spaces (M16)

Non-Examinable (Graduate Level)

Nathalie Wahl

The goal of the course is to study loop spaces, the loop space of a space being the space of all maps from a circle into that space. Loop spaces were already studied by Morse long ago to obtain information about geodesics in Riemannian manifolds. The main focus of the course will be an algebraic model defined by Jones using Hochschild homology. Such an algebraic model allows for example to do computations. At this cross-road between topology and algebra, we will use simplicial and cosimplicial spaces, spectral sequences, and meet higher homotopies. The connections to Hochschild cohomology, play an important role in symplectic topology and mirror symmetry, though this will not be the focus of the course.

Pre-requisites

This course assumes you have some background in Algebraic Topology, say Chapter 1-3 of Hatcher. A prior acquaintance with spectral sequences would also be helpful.

Literature

1. R. L. Cohen, Ralph and A. A. Voronov, Notes on string topology. String topology and cyclic homology, 195, *Adv. Courses Math.* CRM Barcelona, Birkhuser, Basel, 2006.

2. J. D. S. Jones, Cyclic homology and equivariant homology. *Invent. Math.* 87 (1987), no. 2, 403423.

3. A. Hatcher, *Spectral sequences*, available at

   `http://pi.math.cornell.edu/~hatcher/#SSAT`

Symplectic Topology (L24)

Ailsa Keating

The study of symplectic manifolds originated as an extension of classical mechanics; it has since developed into a field of its own right, with connections to e.g. low-dimensional topology, algebraic geometry, and theoretical physics. The first half (or a little more) of the course will develop the basic framework of symplectic topology; the second half will introduce invariants associated to counting pseudo-holomorphic curves, which are at the heart of modern symplectic topology. Time allowing, topics are expected to include:

Part I – Symplectic linear algebra. Hamiltons equations, cotangent bundles. Lagrangian submanifolds. Mosers trick, Darboux and Weinstein neighbourhood theorems. Symplectic circle actions and moment maps, symplectic reduction. Surgery constructions.

Part II – Almost complex structures and compatible triples. Kaehler manifolds. The Cauchy-Riemann equation. Basic theory of pseudo-homolorphic curves. Monotone symplectic manifolds. The quantum cohomology ring: simple computations.

Pre-requisites

Essential: Algebraic Topology and Differential Geometry, at the level of the Part III Michaelmas courses. Basic concepts from Algebraic Geometry (at the level of the Part II course) will be useful.

Literature

1. D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd edition. Oxford University Press, 2017.
2. D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, 2nd edition. American Mathematical Society, 2012.
3. A. Cannas da Silva, *Lectures on symplectic geometry*, Springer-Verlag, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Complex Manifolds (L24)

Ruadhaí Dervan

The goal is to help students learn the basic theory of complex manifolds. A preliminary outline of the course is as follows.

- Basic concepts of complex manifolds, holomorphic vector bundles, holomorphic tangent and cotangent bundles (for which the corresponding concepts from the real smooth manifolds will be assumed). Canonical line bundles, normal bundle for a submanifold and the adjunction formula.
- Brief description of sheaf cohomology, with deduction of de Rham and Dolbeault cohomology for complex manifolds.
- Hermitian metrics, connections, curvature and Chern classes for complex vector bundles. Case of holomorphic vector bundles.

- Harmonic forms: the Hodge theorem and Serre duality (general results on elliptic operators will be assumed).
- Compact Kähler manifolds. Hodge and Lefschetz decompositions on cohomology, Kodaira-Nakano vanishing, Kodaira embedding theorem.

Pre-requisites

A knowledge of basic Differential Geometry from the Michaelmas term will be essential.

Literature

1. J. P. Demailly *Complex analytic and differential geometry.* Available as a pdf at

   `https://www-fourier.ujf-grenoble.fr/~demailly/documents.html`

2. P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley, 1978.
3. D. Huybrechts, *Complex Geometry - an introduction*, Springer, 2004.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

3-manifolds (L24)

Sarah Rasmussen

This course introduces a selection of topics foundational to research in 3-manifold topology and geometry, including the interplay of knot theory with 3-manifold topology:

- *Knots and links.* Invariants of knots and links, including knot groups and Alexander polynomials.
- *Survey of 3-manifold decompositions.* Basic notions from prime decomposition, Jaco-Shalen-Johannson (JSJ) decomposition, Nielsen-Thurston theory, and geometrization. This section necessarily resorts to asserting several deep theorems without proof, but is fundamental to understanding the landscape of 3-manifold topology.
- *3-manifold constructions.* Mapping tori, handle decompositions from the standpoint of Morse theory, Heegaard splittings, and Dehn surgeries along knots and links.
- *Foliations.* Singular, Reeb, and taut foliations, with connections to fundamental group actions, topology, and geometry.

Pre-requisites

Part III Algebraic Topology and Part III Differential Geometry.

Literature

1. D. Rolfsen, *Knots and Links*, AMS Chelsea (1976).
2. V. Turaev, *Introduction to Combinatorial Torsions*, Lectures in Mathematics ETH Zürich, Birkhäuser (2001).
3. J. Hempel, *3-Manifolds*, Princeton University Press (1976).
4. R. Benedetti, C. Petronio, *Lectures on Hyperbolic Geometry*, Springer (1992).
5. D. Calegari, *Foliations and the Geometry of 3-Manifolds*, Oxford University Press (2007). An uncorrected version is available at

   `http://math.uchicago.edu/~dannyc/OUPbook/toc.html`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The topology of graphs (L16)

Non-Examinable (Graduate Level)

Henry Wilton

A graph is a 1-dimensional cell complex: the London Underground map may be the most well known example. One might think that 1-dimensional topology is trivial, but in this course I'll argue that, on the contrary, it's an extremely rich subject. Although the group of *automorphisms* of a finite graph Γ is necessarily finite, the group of *homotopy self-equivalences* of Γ (even modulo the subgroup of maps homotopic to the identity) is huge. In fact, it is $\mathrm{Out}(F_n)$, the group of outer automorphisms of a free group F_n, and there is a beautiful interplay between the topology of graphs and the algebra of free groups.

This graduate course builds on the Michaelmas Part III course *Introduction to Geometric Group Theory.* Free groups provide fundamental examples which much of geometric group theory aims to generalise. We'll learn the necessary tools for the modern study of a free group F_n, and prove some very recent theorems, such as Calegari's rationality theorem for stable commutator length.

Studying the free groups F_n naturally leads to the study of their outer automorphism groups $\mathrm{Out}(F_n)$, which can be thought of as non-commutative analogues of the matrix groups $GL(n, \mathbb{Z})$. These groups are superficially quite mysterious, but we'll see how they can be studied by thinking about the topology of graphs. In particular, we'll define *Culler–Vogtmann outer space*, the fundamental spaces that are used for studying $\mathrm{Out}(F_n)$.

(i) Covering spaces of graphs, immersions, Stallings' folds, trees.

(ii) Stable commutator length. Calegari's rationality theorem. If there's time, quasimorphisms and Bavard duality.

(iii) Bases of free groups and Whitehead's algorithm.

(iv) Outer automorphisms of free groups. Culler–Vogtmann outer space.

Pre-requisites

Part II Algebraic topology and Part III Introduction to Geometric Group Theory are required. Some familiarity with hyperbolic geometry, such as Part Ib Geometry, is also useful.

Literature

Relevant theorems were proved from the combinatorial point of view in Lyndon and Schupp, but as far as I know, the definitive topological account of this material has yet to be written.

1. Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory.* Springer-Verlag, 1977.
2. Danny Calegari, *scl.* MSJ Memoirs 20, Tokyo, 2009.

Topics in Floer Theory (L16)

Non-Examinable (Graduate Level)

Cheuk Yu Mak

This is an introductory course to Lagrangian Floer theory and Fukaya category. Lagrangian Floer theory is on one hand a generalization of Morse theory for smooth manifolds, and on the other a generalization of the intersection pairing between cycles in a smooth manifold. It is also the basic ingredient needed to construct the Fukaya category, which plays a fundamental role in understanding the homological mirror symmetry between symplectic manifolds and complex manifolds. The emphasis of the course will be put on concrete examples with the ultimate goal being illustrating the homological mirror symmetry.

Topics will be covered include

- A quick tour of basic symplectic geometry,
- Lagrangian Floer theory in exact symplectic manifolds, including punctured Riemann surfaces and symplectic Lefschetz fibrations, and
- Various versions of Fukaya categories associated to exact symplectic manifolds

Pre-requisites

Knowledge of materials in Michaelmas term Differential Geometry and Algebraic Topology is highly desirable. Exposure to some basic symplectic geometry would also be helpful. Students are encouraged to study the Lent term Symplectic Topology simultaneously, where more than enough background will be covered. Knowledge in complex geometry and category theory is not assumed.

Literature

There are a lot of literature on Lagrangian Floer theory and Fukaya category. We will pick up parts of it during the course. Some helpful literature for familiarizing with basic notions in symplectic geometry include

1. A. Cannas da Silva, *Lectures on Symplectic Geomety*, Springer, 2nd edition (2008), also available at https://people.math.ethz.ch/∼acannas/
2. D. McDuff and D.Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 3rd edition (2017).

Logic

Category Theory (M24)

Prof. P.T. Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]

Locally small categories. The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]

Limits. Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

Filtered colimits. Finitary functors, finitely-presentable objects. Applications to universal algebra.

Regular categories. Embedding theorems. Categories of relations, introduction to allegories.

Abelian categories. Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.

Monoidal categories. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

Fibrations. Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Pre-requisites

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. S. Mac Lane *Categories for the Working Mathematician.* Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.
2. S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.
3. T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.
4. E. Riehl *Category Theory in Context.* Dover Publications 2016. A new account of the subject by someone who first encountered it as a Part III student a dozen years ago.
5. F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Model Theory (M16)

Silvia Barbina

Model theory is a branch of mathematical logic. Initially, the focus was on how far a set of sentences in a first-order language determines the class of structures it describes. Later, the subject evolved in several directions. For example, part of the attention shifted to contexts where the subsets of a structure that are *definable* are also mathematically meaningful – for instance, the definable sets in an algebraically closed field are the constructible sets. This is one of several ways in which model theory interacts with other areas of mathematics. Connections have emerged to real and algebraic geometry, number theory and, more recently, combinatorics. This course introduces some basic model-theoretic tools and ideas up to initial notions in stability theory.

Topics likely to be covered include:

- preliminaries: structures, theories, elementarity (including elementary substructures, Tarski-Vaught test, downward Löwenheim-Skolem theorem)
- examples of relational structures (dense linear orders, the random graph)
- universal homogeneous models
- the compactness theorem for theories and for types
- saturation and the monster model
- omitting types theorem, possibly including ω-categorical theories and small theories
- preservation theorems and quantifier elimination
- imaginaries and the eq-expansion
- formulas without the order property (externally definable sets, stability and the number of types, stationary types and canonical bases).

Pre-requisites

The Part II course *Set Theory and Logic*, or an equivalent course, is essential.

Literature

References and notes will be provided during the course. A general model theory textbook is

1. D. Marker. *Model Theory.* Vol. 217, Springer-Verlag, New York, 2002.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in Set Theory (L24)

Benedikt Löwe

This course covers advanced topics in set theory, focusing on meta-mathematical techniques such as inner models and forcing.

Set theory and logic are intrinsically intertwined since the most interesting results in set theory are independence results showing that natural questions in set theory are not solvable using the standard axiomatic system of Zermelo-Fraenkel set theory with choice ZFC.

The most famous of these natural questions is Cantor's continuum hypothesis CH, "every uncountable set of reals is equinumerous to the set of all real numbers" or, equivalently, $2^{\aleph_0} = \aleph_1$. This question was elevated to the status of the foremost mathematical problem for the 20th century by David Hilbert in his address to the *International Congress of Mathematicians* in Paris in the year 1900. In 1938, Kurt Gödel proved that CH cannot be disproved in ZFC (inventing and using the method of inner models); in 1963, Paul Cohen proved that CH cannot be proved in ZFC (inventing and using the method of forcing). Together, these results show that CH is independent from ZFC.

We shall treat several of the following topics:

Model theory of set theory. Models of set theory. Transitive models of set theory. Inaccessible cardinals. Absoluteness. Simple independence results. Reflection principles.

Inner models. Definability. Ordinal definability. Constructibility. Condensation. Gödel's proof of the consistency of CH.

Forcing. Generic extensions. The forcing theorems. Adding reals; collapsing cardinals. Cohen's proof of the consistency of $\neg$CH.

A course webpage will be available at

`https://www.math.uni-hamburg.de/home/loewe/Lent2019/TST_L19.html`.

Pre-requisites

The Part II course *Logic and Set Theory* or an equivalent course is essential.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Connections between model theory and combinatorics (L16)

Non-Examinable (Graduate Level)

Julia Wolf

This course serves as an introduction to recent developments at the interface between model theory and combinatorics. It will focus on results of a finitary and combinatorial nature, including Ramsey-type theorems in graphs under additional tameness assumptions, and applications; efficient arithmetic regularity lemmas in groups as a consequence of local stability theory; bounded quotients and the structure of approximate groups; and dividing lines in unstable theories. If time permits, we will explore connections with computational complexity.

Pre-requisites

This course is particularly suitable for those students who have attended the Michaelmas courses "Introduction to discrete analysis" and "Model theory", and those who are planning to take "Introduction to approximate groups" in the Lent term. Students who have not had a chance to attend one or more of these courses are welcome to contact the lecturer to discuss suitable background reading. Every attempt will be made to keep the course as self-contained as possible.

Preliminary Reading

1. Lecture notes for any of the Part III courses mentioned under Pre-requisites.
2. D. Marker. *Model Theory.* Vol. 217, Springer-Verlag, New York, 2002.
3. T. Tao and V. Vu. *Additive Combinatorics.* Vol. 105, Cambridge University Press, 2010.

Literature

Most of the course material will be extracted from fairly recent research papers. The following sources cover some aspects of the course in a more expository fashion.

1. B. Poizat. *Stable Groups.* Mathematical Surveys and Monographs Vol. 87, American Mathematical Society, 2001.
2. L. van den Dries. *Approximate Groups [According to Hrushovski and Breuillard, Green, Tao].* Astérisque, no. 367, 2015.

Additional support

Exercises will be provided throughout the course, and three examples classes will be arranged to allow participants to deepen their understanding of the material through informal discussion.

Number Theory

Algebraic Number Theory (M24)

Ian Grojnowski

This will be a basic course in algebraic number theory, explaining some local aspects of the theory and some global aspects.

Topics that may be included:

Dedekind domains, localization, and passage to completion. The p-adic numbers, Witt vectors. Galois theory of Dedekind domains and ramification theory.

Artin and abelian L-functions. Adeles, ideles, and applications to class groups and groups of units. The analytic class number formula. Chebotarev density theorem.

Class field theory - statements of results.

Pre-requisites

Part II Galois Theory and Part IB Groups, Rings and Modules (or equivalent) are essential pre-requisites.

Part II Number Fields will not be assumed, but it is a beautiful and deep course - you should audit it if you haven't already taken it.

Literature

The lectures will be self contained, but there are many many textbooks on algebraic number fields. Here are some:

1. J. Neukirch, Algebraic Number Theory. Springer, 1999.
2. J.-P. Serre, Local fields. Springer, 1979.
3. J. W. S. Cassels and A. Frohlich, Algebraic Number Theory. 2nd edition. London Mathematical Society, 2010.
4. Yu. I. Manin and A. A. Panchishkin. Introduction to Modern Number Theory. Springer, 2005.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Elliptic Curves (M24)

Tom Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points. The following topics will be covered, and possibly others if time is available.

Weierstrass equations and the group law. Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process.

Isogenies. The degree of an isogeny is a quadratic form. The invariant differential and separability. The torsion subgroup over an algebraically closed field.

Elliptic curves over finite fields. Hasse's theorem and zeta functions.

Elliptic curves over local fields. Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.

Elliptic curves over number fields. The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Pre-requisites

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. The first few lectures will include a review of the necessary geometric background, but some previous knowledge of algebraic curves (at the level of the Part II course *Algebraic Geometry* or the first two chapters of [3]) would be very helpful. Later in the course, some basic knowledge of the field of p-adic numbers will be assumed.

Preliminary Reading

1. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Literature

2. J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.
3. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

Additional support

There will be four example sheets and four associated examples classes.

Analytic Number Theory (L24)

Thomas Bloom

We will give an introduction to analytic number theory, with an emphasis on classical results and techniques. The course is divided into roughly four parts, each focusing on different techniques.

1. Arithmetic functions and elementary estimates
2. The Riemann zeta function and the prime number theorem
3. Sieve methods
4. Dirichlet characters and the prime number theorem in arithmetic progressions

Pre-requisites

Basic knowledge of complex analysis (e.g. Cauchy's theorem) will be assumed. A handout will be provided summarising all the analysis knowledge required.

Literature

1. H. Montgomery and R. C. Vaughan *Multiplicative number theory. I. Classical theory.* Cambridge University Press, 2007.

2. H. Iwaniec and E. Kowalski *Analytic number theory.* American Mathematical Society, 2004.

3. M. Ram Murty *Problems in analytic number theory.* Springer, 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the ??? Term.

Probability and Finance

Advanced Probability (M24)

James Norris and Wei Qian

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

Review of measure and integration: sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini's theorem.
Conditional expectation: Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.
Martingales: Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.
Stochastic processes in continuous time: Kolmogorov's criterion, regularization of paths; martingales in continuous time.
Weak convergence: Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.
Sums of independent random variables: Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.
Brownian motion: Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.
Poisson random measures: Construction and properties; integrals.
Lévy processes: Lévy-Khinchin theorem.

Pre-requisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Literature

- Lecture notes online: `www.statslab.cam.ac.uk/~james/Lectures/ap.pdf`
- D. Applebaum, Lévy processes (2nd ed.), Cambridge University Press 2009.
- R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
- O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.
- D. Williams, Probability with martingales, CUP 1991.

Additional support

Four example sheets will be provided along with supervisions or examples classes. There will be a revision class in Easter term.

Percolation and Random Walks on Graphs (M16)

Perla Sousi

A phase transition means that a system undergoes a radical change when a continuous parameter passes through a critical value. We encounter such a transition every day when we boil water. The simplest mathematical model for phase transition is percolation. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for a solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm-Loewner evolutions (SLE), and to other models from statistical physics. The basic theory of percolation will be described in this course with some emphasis on areas for future development.

Our other major topic includes random walks on graphs and their intimate connection to electrical networks; the resulting discrete potential theory has strong connections with classical potential theory. We will develop tools to determine transience and recurrence of random walks on infinite graphs. Other topics include the study of spanning trees of connected graphs. We will present two remarkable algorithms to generate a uniform spanning tree (UST) in a finite graph G via random walks, one due to Aldous-Broder and another due to Wilson. These algorithms can be used to prove an important property of uniform spanning trees discovered by Kirchhoff in the 19th century: the probability that an edge is contained in the UST of G, equals the effective resistance between the endpoints of that edge.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

1. Bollobás, B. and Riordan, O., *Percolation*, Cambridge University Press, 2006
2. Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2010
   available at `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`
3. Grimmett, G. R., *Percolation*, Springer-Verlag, Berlin, second edition, 1999.
4. Lyon, R. and Peres, Y., *Probability on Trees and Networks*
   available at `http://mypage.iu.edu/~rdlyons/prbtree/book_online.pdf`

Advanced Financial Models (M24)

M.R. Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability and Stochastic Calculus & Applications.

- *Discrete time models.* Filtrations and martingales. Arbitrage, martingale deflators and equivalent martingale measures. Attainable claims and market completeness. European and American claims. Optimal stopping.
- *Brownian motion and stochastic calculus.* Brief survey of stochastic integration. Girsanov's theorem. Itô's formula. Martingale representation theorem.
- *Continuous time models.* Admissible strategies. Black–Scholes model. The implied volatility surface. Pricing and hedging via partial differential equations. Dupire's formula. Stochastic volatility models.

- *Interest rate models.* Short rates, forward rates and bond prices. Markovian short rate models. The Heath–Jarrow–Morton drift condition.

Pre-requisites

Familiarity with measure-theoretic probability will be assumed.

Literature

1. M. Baxter & A. Rennie. *Financial calculus: an introduction to derivative pricing.* Cambridge University Press, 1996
2. M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling.* Springer, 2006
3. D. Kennedy.*Stochasic Financial models.* Chapman & Hall, 2010
4. D. Lamberton & B. Lapeyre.*Introduction to stochastic calculus applied to finance.* Chapman & Hall, 1996
5. S. Shreve. *Stochastic Calculus for Finance: Vol. 1 and 2.* Springer-Finance, 2005

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Mixing Times of Markov Chains (L16)

Dr Jonathan Hermon

Elementary theory tells us that, after a sufficiently large time, the distribution of an irreducible, aperiodic, finite Markov chain is close to the invariant distribution. But how long should one wait in practice? For instance, how many times should a deck of cards be shuffled before its distribution is approximately uniform? This type of questions is at the heart of the theory of mixing times of Markov chains. It is a surprisingly rich question, with ramifications in analysis, geometry, combinatorics, representation theory, etc.

We shall focus on the basic theory and expose some of the main techniques which have been used to tackle this question. Our main goal will be to discuss the *cutoff phenomenon*, which says that a Markov chain reaches its stationary distribution in an abrupt fashion, after a well-defined number of steps called the *mixing time*. Surprisingly this phenomenon seems to be widespread.

A rough plan of the course is as follows:

Coupling method: convergence in total variation distance.

Spectral methods: eigenvalue decomposition and relaxation time.

Geometric methods: canonical paths, Cheeger's inequality, expanders.

Analytic methods: comparison theorem of Diaconis and Saloff-Coste.

Other notions of stationarity: strong stationary times, cover times, Lovász–Winkler theory.

Pre-requisites

This course assumes almost no background, except for prior exposure to Markov chains at an elementary level.

Literature

1. D. Levin, Y. Peres and E. Wilmer. *Markov Chains and Mixing Times.* American Mathematical Society, 2008.

2. N. Berestycki. *Mixing Times of Markov Chains: Techniques and Examples.* Available on the webpage of the author.

3. R. Montenegro and P. Tetali. *Mathematical aspects of mixing times in Markov chains.* Foundations and Trends in Theoretical Computer Science: Vol. 1: No. 3, pp 237–354, 2006.

Additional support

Examples sheets will be provided and associated examples classes will be given.

Stochastic Calculus and Applications (L24)

R. Bauerschmidt

This course provides an introduction to Itô calculus.

- *Stochastic calculus for continuous processes.* Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita-Watanabe theorem, and Itô's formula.
- *Applications to Brownian motion and martingales.* Lévy characterization of Brownian motion, Dubins-Schwarz theorem, martingale representation, Girsanov theorem, and Dirichlet problems.
- *Stochastic differential equations.* Strong and weak solutions, notions of existence and uniqueness, strong Markov property, and relation to second order partial differential equations.
- *Applications and examples.*

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with continuous-time martingales and Brownian motion.

Literature

1. J.-F. Le Gall, *Brownian Motion, Martingales, and Stochastic Calculus.* Springer. 2016

2. D. Revuz and M. Yor, *Continuous martingales and Brownian motion.* Springer. 1999

3. I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus.* Springer. 1998

4. L.C. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales.* Cambridge. 2000

Schramm-Loewner Evolutions (L16)

J. P. Miller

Schramm-Loewner Evolution (SLE) is a family of random curves in the plane, indexed by a parameter $\kappa \geq 0$. These non-crossing curves are the fundamental tool used to describe the scaling limits of a host of natural probabilistic processes in two dimensions, such as critical percolation interfaces and random spanning trees. Their introduction by Oded Schramm in 1999 was a milestone of modern probability theory.

The course will focus on the definition and basic properties of SLE. The key ideas are conformal invariance and a certain spatial Markov property, which make it possible to use Itô calculus for the analysis. In particular we will show that, almost surely, for $\kappa \leq 4$ the curves are simple, for $4 \leq \kappa < 8$ they have double points but are non-crossing, and for $\kappa \geq 8$ they are space-filling. We will then explore the properties of the curves for a number of special values of κ (locality, restriction properties) which will allow us to relate the curves to other conformally invariant structures.

The fundamentals of conformal mapping will be needed, though most of this will be developed as required. A basic familiarity with Brownian motion and Itô calculus will be assumed but recalled.

Literature

1. Nathanaël Berestycki and James Norris. Lecture notes on SLE.
   `http://www.statslab.cam.ac.uk/~james/Lectures`

2. Wendelin Werner. *Random planar curves and Schramm-Loewner evolutions*, arXiv:math.PR/0303354, 2003.

3. Gregory F. Lawler. *Conformally Invariant Processes in the Plane*, AMS, 2005.

Additional support

Two examples sheets will be provided and examples classes given. There will be a revision class in Easter Term.

Statistics and Operational Research

The courses in statistics form a coherent Masters-level course in statistics, covering statistical methodology, theory and applications. You may take all of them, or a subset of them. Core courses are Modern Statistical Methods and Applied Statistics in the Michaelmas Term.

All statistics courses for examination in Part III assume that you have taken an introductory course in statistics and one in probability, with syllabuses that cover the topics in the Cambridge undergraduate courses Probability in the first year and Statistics in the second year. It is helpful if you have taken more advanced courses, although not essential. For Applied Statistics and other applications courses, it is helpful, but not essential, if you have already had experience of using a software package, such as R or Matlab, to analyse data. The statistics courses assume some mathematical maturity in terms of knowledge of basic linear algebra and analysis. However, they are designed to be taken without a background in measure theory, although some knowledge of measure theory is helpful for Topics in Statistical Theory.

The desirable previous knowledge for tackling the statistics courses in Part III is covered by the following Cambridge undergraduate courses. The syllabuses are available online at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
First	*Essential*	Probability
Second	*Essential*	Statistics
	Helpful for some courses	Markov Chains
Third	*Helpful*	Principles of Statistics
	Helpful for applied statistics courses	Statistical Modelling
	For additional background	Probability and Measure

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation. If you have more time, then it would be helpful to review other courses as indicated.

Modern Statistical Methods (M24)

Rajen Shah

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data where we have huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to study such datasets. We aim to cover the following topics.

- Kernel machines: Ridge regression, the kernel trick, kernel ridge regression, the support vector machine, the hashing trick.
- Penalised regression: Model selection, the Lasso, variants of the Lasso.

- Graphical modelling and causal inference: Conditional independence graphs, Neighbourhood selection, the graphical Lasso, structural equation models, the PC algorithm.
- Multiple testing and high-dimensional inference: the closed testing procedure and the Benjamini–Hochberg procedure, the debiased Lasso.

Pre-requisites

Basic knowledge of statistics, probability, linear algebra and real analysis. Some background in optimisation would be helpful but is not essential.

Literature

1. T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning.* 2nd edition. Springer, 2001.
2. P. Bühlmann, S. van de Geer, *Statistics for High-Dimensional Data.* Springer, 2011.
3. T. Hastie, R. Tibshirani and M. Wainwright, *Statistical learning with sparsity: the lasso and generalizations.* CRC Press, 2015.
4. C. Giraud, *Introduction to High-Dimensional Statistics.* CRC Press, 2014.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Bayesian Modelling and Computation (M24)

Sergio Bacallado

The course will cover a range of algorithms for sampling and numerical integration which are useful in Bayesian inference. A third of the lectures will deal with applications in specific statistical models.

- **Fundamentals:** Monte Carlo integration; variance reduction techniques; rejection sampling and adaptive rejection sampling; importance sampling. Exponential families and conjugate priors.
- **Graphical models:** Belief networks, Markov random fields, and factor graphs; Hammersley–Clifford theorem; computational reduction between marginalisation, computing partition function, and sampling; belief propagation in trees.
- **Algorithms:** Markov chain Monte Carlo, conditions for convergence. Metropolis–Hastings and pseudo-marginal variants. Hamiltonian Monte Carlo; symplectic reversible integrators. Gibbs sampling. Simulated annealing and parallel tempering. Sequential Monte Carlo.
- **Approximate inference:** Expectation maximisation. Variational inference; mean-field methods. Stochastic variational inference. The parametric Bootstrap and Bayesian methods.
- **Modelling examples:** Hidden Markov Models for time series. Bayesian generalised linear models; mixed-effects models; variable selection priors. Mixture models; block and collapsed Gibbs sampler; hierarchical mixtures. Feature models. Nonparametric priors and slice sampling. Gaussian process priors in spatio-temporal models.

Pre-requisites

This course assumes familiarity with probability and basic Markov chain theory. Knowledge of statistical modelling and Bayesian analysis is helpful.

Literature

1. Murphy, K., *Machine Learning: a Probabilistic Perspective,* MIT Press, 2012.
2. Brooks, S., Gelman, A., Jones, G., Meng, X-L. *Handbook of Markov Chain Monte Carlo,* 1st edition. Chapman and Hall, 2011.
3. Owen, A.B., *Monte Carlo theory, methods, and examples*, 2013. Available at

   `http://statweb.stanford.edu/~owen/mc/`

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Statistics in Medicine (3 units)

Statistics in Medical Practice (M12)

Statistics in Medicine (3 units)

Lecturers from the MRC Biostatistics Unit

This part of the course includes three modules covering a range of statistical methods and their application in three areas of biostatistics.

A. Stochastic Models for Chronic and Infectious Diseases [4 Lectures] **(C. Jackson, A. Presanis & P. Birrell)**

Continuous-time multi-state and Markov models: properties and quantities of interest, and fitting models to individual disease history data. Applications to modelling the onset and progression of chronic diseases. Multi-state modelling to estimate incidence of infectious diseases from population-level prevalence data. Backcalculation methods for the estimation of incidence of disease with long incubation periods. Dynamic modelling of infectious disease transmission.

B. Design and Analysis of Randomised Trials [4 Lectures] **(S. Villar, D. Robertson & S. Seaman)**

Sample size estimation for clinical trials; group-sequential designs and treatment effect estimation following a group-sequential trial. Adaptive and multi-stage designs: the multi-armed bandit problem and the optimal sequential allocation of patients. Types of randomisation procedures. Non-parametric and parametric response-adaptive procedures: asymptotic behaviour of allocation ratios and likelihood-based inference. Handling missing data: classification of missingness mechanisms, maximum likelihood, and multiple imputation.

C. Causal Inference [4 Lectures] **(S. Burgess)**

It is well known that "correlation is not causation". But how then do you assess causal claims? Is it possible to show that X is a cause of Y? What does it even mean to say that X is a cause of Y? In this module, we introduce definitions of causal concepts, starting with the work of Rubin, Pearl, and Robins, and discuss practical approaches for assessing causal claims from observational data.

Analysis of Survival Data (L12)

Statistics in Medicine (3 units)

P. Treasure

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

D. Time-to-Event Analysis [4 Lectures]

‘Survival analysis’ is generalised to *time-to-event* analysis. The implications of event times which are unknown or in the future (*censored* data) are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (*Kaplan-Meier*) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (*log-rank* test).

E. Modelling Hazard [4 Lectures]

The *hazard* function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size ...) on the time-to-event distribution (*proportional hazards* modelling). Model checking procedures are introduced with an emphasis on excess event (*Martingale*) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (*competing risk* analysis) and (ii) the cause of death is unknown (*net survival*). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar time (e.g. 2017) are addressed: *period* survival analysis.

Statistics in Medicine (3 units)

Additional Information

Pre-requisites

Undergraduate-level statistics and probability, including analysis and interpretation of data, maximum likelihood estimation, hypothesis testing, basic stochastic processes.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. A few books to complement the course material are listed below.

1. Armitage P, Berry G, Matthews JNS, *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001. [A good introductory companion to the whole course]

2. van den Hout, A, *Multi-State Survival Models for Interval-Censored Data.* Chapman and Hall, 2016 [Module A]

3. Keeling, M. J., & Rohani, P. *Modeling Infectious Diseases in Humans and Animals.* Princeton University Press, 2008 [Module A]

4. Senn, S. *Statistical Issues in Drug Development.* Wiley, 2007. [Module B]

5. Jennison C, Turnbull B, *Group Sequential Methods with Applications to Clinical Trials.* Chapman and Hall, 2000. [Module B]

6. Burgess S, Thompson SG, *Mendelian Randomization: Methods for Using Genetic Variants in Causal Estimation* Chapman and Hall, 2015 [Module C]

7. Cox DR, Oakes D, *Analysis of Survival Data.* Chapman and Hall, 1984 [Modules D, E, F: the classic text]

8. Collett D, *Modelling Survival Data in Medical Research.* CRC Press, 2015 [Modules D, E, F: modern, applied, supports and extends lectures.]

9. Aalen OO, Borgen Ø, Gjessing HK, *Survival and Event History Analysis: A Process Point of View.* Springer, 2008 [Modules D, E, F: excellent modern approach]

Additional support

Example and revision classes will be given, with question sheets and solutions.

Topics in Statistical Theory (L16)

Richard Nickl

This course will present a selection of ideas, mathematical techniques and rigorous results relevant in modern statistics, focusing on high- and infinite-dimensional ('nonparametric') statistical models that are frequently used in modern data science and machine learning.

I. High-dimensional and nonparametric regression models, basic approximation theory, Gaussian process techniques (chaining and concentration), upper and lower bounds for the minimax risk, procedures based on complexity penalties and thresholding, adaptation

II. Bayesian Non-parametrics: prior measures in function spaces, frequentist contraction rates for posterior distributions, Gaussian approximations of posterior distributions and Bernstein-von Mises theorems, consistency of Bayes solutions for statistical inverse problems

Pre-requisites

Background in probability & measure theory as well as real analysis is useful, as is any preliminary course on mathematical statistics.

The references [1,2] cover most materials in the course.

Literature

1. R.M. Dudley, *Real analysis and probability*, Cambridge University Press, Cambridge 2002

2. E. Giné & R. Nickl, *Mathematical foundations of infinite-dimensional statistical models*, Cambridge University Press, Cambridge 2016

Additional support

Examples sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Statistical Learning in Practice (L24)

Alberto J. Coca

Statistical learning is the process of using data to guide the construction and selection of models, which are then used to predict future outcomes. In this course, consisting of 12 lectures and 12 practical classes, we will examine some of the most successful and widely used statistical methodologies in modern applications. The practical classes will deal with an introduction to R, exploratory data analysis and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models for regression and classification
- Model selection and regularisation
- Mixed effects models and quasi-likelihood methods
- Linear discriminant analysis and support vector machines
- Introduction to neural networks
- Time series and spatial statistics

Pre-requisites

Elementary probability theory. Maximum likelihood estimation, hypothesis tests and confidence intervals. Linear models.

Previous experience with R is helpful but not essential.

Literature

1. Dobson, A.J. and Barnett A. (2008) *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC.
2. Faraway, J. J. (2005) *Extending the linear model with R: generalized linear, mixed effects and non-parametric regression models.* CRC press.
3. Gaetan, C. and Guyon, X. (2010). *Spatial Statistics and Modeling.* Springer.
4. Hastie, T., Tibshirani, R. and Friedman, J. (2009) *The Elements of Statistical Learning.* Second Edition. Springer.
5. Shumway, R. H., and Stoffer, D. S. (2010) *Time Series Analysis and Its Applications: with R Examples.* Springer Science & Business Media.

Additional support

This course includes practical classes, in which statistical methods are introduced in a practical context and students carry out analysis of datasets using R. In the practical classes, the students have the opportunity to discuss statistical questions with the lecturer. Four examples sheets will be provided and there will be four associated examples classes. There will be a revision class in the Easter Term.

Astrostatistics (L24)

Kaisey Mandel

This course will cover statistical methods necessary to properly interpret today's increasingly complex datasets in astronomy. Particular emphasis will be placed on principled statistical modeling of astrophysical data and statistical computation of inferences of scientific interest. Statistical problems and techniques, such as Bayesian modeling, nonparametric methods, density estimation, regression, classification, time series analysis, sampling methods, and machine learning, will be examined in the context of applications to modern astronomical data analysis. Topics and examples will be motivated by case studies across astrophysics and cosmology.

Pre-requisites

Students of astrophysics, physics, statistics or mathematics are welcome. Astronomical context will be provided when necessary. Students without a previous statistics background should familiarise themselves with the material in Feigelson & Babu, Chapters 1-4, and Ivezić, Chapters 1, 3-5, by the beginning of the course. (Note that the two textbooks cover many of the same topics). These texts are freely available online to Cambridge students via the library website.

Literature

1. E. Feigelson and G. Babu. *Modern statistical methods for astronomy: with R applications.* Cambridge University Press, 2012.

2. Z. Ivezić, A. Connolly, J. VanderPlas & A Gray. *Statistics, Data Mining, and Machine Learning in Astronomy.* Princeton University Press, 2014.

3. C. Schafer. *A Framwork for Statistical Inference in Astrophysics.* 2015, Annual Review of Statistics and Its Application, 2: 141-162

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Particle Physics, Quantum Fields and Strings

The courses on Symmetries, Fields and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of the course Symmetries, Fields and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
Second	*Essential*:	Quantum Mechanics, Methods, Complex Methods.
	Helpful:	Electromagnetism.
Third	*Essential*:	Principles of Quantum Mechanics, Classical Dynamics.
	Very helpful:	Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Statistical Field Theory (M16)

David Tong

This course is an introduction to the renormalization group, the basis for a modern understanding of field theory. The course is primarily focussed on statistical systems such as spin models, but the connection to quantum field theory is never far from the surface.

The simplest spin system – known as the Ising model – is a constant companion throughout the course. After a description of this model, Landau's mean field theory is introduced and used as a framework to discuss the phenomology and classification of phase transitions. The extension to Landau-Ginzburg theory provides for a more complete understanding of fluctuations, and makes the connection to quantum field theory manifest.

These approaches struggle near a second order phase transition, also known as a "critical point". Here the tools of the renormalisation group become essential. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems.

Pre-requisites

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Literature

1. J M Yeomans, *Statistical Mechanics of Phase Transitions*, Clarendon Press (1992).
2. M Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press (1991).
3. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992).
4. M Kardar, *Statistical Physics of Fields*, Cambridge University Press (2007).
5. D Amit and V Martín-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005).
6. L D Landau and E M Lifshitz, Statistical Physics, Pergamon Press (1996).

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Quantum Field Theory (M24)

B.C. Allanach

Quantum Field Theory is the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics.

Interactions betweeen fields are examined next, using the interaction picture, Dyson's formula and Wick's theorem. A 'short version' of these techniques is introduced: Feynman diagrams. Decay rates and interaction cross-sections are developed.

Fermions and the Dirac equation are explored in detail, along with parity and γ^5. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions.

Finally, quantum electrodynamics (QED) is developed. A connection between the field strength tensor and Maxwell's equations is carefully made, before gauge symmetry is introduced. Lorentz gauge is used as an example, before quantisation of the electromagnetic field and the Gupta-Bleuler condition. The interactions between photons and charged matter is governed by the principal of minimal coupling, which is covered next. A few examples of quantum amplitudes (squared) in QED are given.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Literature

1. David Tong, *Lectures on Quantum Field Theory*
   `http://www.damtp.cam.ac.uk/user/tong/qft.html` videos of lectures and printed lecture notes
2. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996).
3. A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2010)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. One revision lecture will be given in Easter term.

Symmetries, Fields and Particles (M24)

N. Dorey

This course introduces the theory of Lie groups and Lie algebras and their applications to high energy physics. The course begins with a brief overview of the role of symmetry in physics. After reviewing basic notions of group theory we define a Lie group as a manifold with a compatible group structure. We give the abstract definition of a Lie algebra and show that every Lie group has an associated Lie algebra corresponding to the tangent space at the identity element. Examples arising from groups of orthogonal and unitary matrices are discussed. The case of $SU(2)$, the group of rotations in three dimensions is studied in detail. We then study the representations of Lie groups and Lie algebras. We discuss reducibility and classify the finite dimensional, irreducible representations of $SU(2)$ and introduce the tensor product of representations. The next part of the course develops the the theory of complex simple Lie algebras. We define the Killing form on a Lie algebra. We introduce the Cartan-Weyl basis and discuss the properties of roots and weights of a Lie algebra. We cover the Cartan classification of simple Lie algebras in detail. We describe the finite dimensional, irreducible representations of simple Lie algebras, illustrating the general theory for the Lie algebra of $SU(3)$. The last part of the course discusses some physical applications. After a general discussion of symmetry in quantum mechanical systems, we review the approximate $SU(3)$ global symmetry of the strong interactions and its consequences for the observed spectrum of hadrons. We introduce gauge symmetry and construct a gauge-invariant Lagrangian for Yang-Mills theory coupled to matter. The course ends with a brief introduction to the Standard Model of particle physics.

Pre-requisites

Linear algebra including direct sums and tensor products of vector spaces. Basic finite group theory, including subgroups and orbits. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

Literature

1. J. Fuchs and C. Schweigert, *Lie Algebras and Representations.* Cambridge University Press, 2003.
2. H.F. Jones, *Representations and Physics.* 2nd edition. Taylor and Francis, 1998.
3. H. Georgi, *Lie Algebras in Particle Physics.* Westview Press, 1999.

Additional support

A set of course notes will provided as handouts in the lectures. Printed notes of previous version of the course are also available on the Part III Examples and Lecture Notes webpage. Four examples sheets will be provided and four associated examples classes in moderate-sized groups will be given by graduate students.

Advanced Quantum Field Theory (L24)

MB Wingate

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalization of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantizing a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study renormalization. Wilson's picture of renormalization is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the renormalization group (RG) flow. The course explores renormalization systematically, from the use of dimensional regularization in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as "asymptotic freedom," this phenomenon revolutionized our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrize possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

Preliminary Reading

1. Zee, A., *Quantum Field Theory in a Nutshell*, 2^{nd} edition, PUP (2010).

Literature

1. Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).
2. Schwarz, M., *Quantum Field Theory and the Standard Model*, CUP (2014).
3. Srednicki, M., *Quantum Field Theory*, CUP (2007).
4. Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).
5. Deligne, P. *et al.*, *Quantum Fields and Strings*, vol. 1, AMS (1999).

Additional support

There will be four problem sheets handed out during the course. Classes for each of these sheets will be arranged during Lent term (the 4th class may be scheduled for Easter term). There will also be a general revision class during Easter term.

The Standard Model (L24)

C.E. Thomas

The Standard Model of particle physics is, by far, the most successful application of quantum field theory (QFT). At the time of writing, it accurately describes all experimental measurements involving strong, weak, and electromagnetic interactions. The course aims to demonstrate how this model, a QFT with gauge group $SU(3) \times SU(2) \times U(1)$ and fermion fields for the leptons and quarks, is realised in nature. It is intended to complement the more general Advanced QFT course.

We begin by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content (spin-half leptons and quarks, spin-one gauge bosons and spin-zero Higgs boson). The parity P, charge-conjugation C and time-reversal T transformation properties of the theory are investigated. These need not be symmetries manifest in nature; e.g. only left-handed particles feel the weak force and so it violates parity symmetry.

Ideas of spontaneous symmetry breaking are applied to discuss Goldstone's theorem and the Higgs mechanism. We then describe how the weak and electromagnetic interactions arise from the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry. We show how CP violation becomes possible in the electroweak sector when there are three generations of particles and describe its consequences. The topic of neutrino masses and oscillations is touched upon, an important window to physics beyond the Standard Model.

We show how to obtain cross sections and decay rates, quantities which can be measured in experiments, from the matrix element of a process. Because the couplings are small, these can be computed for various scattering and decay processes in the electroweak sector using perturbation theory.

The strong interaction is described by quantum chromodynamics (QCD), the non-abelian gauge theory of the (unbroken) $SU(3)$ gauge symmetry. At low energies quarks are confined and form bound states called hadrons. The coupling constant decreases as the energy scale increases, to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. Time permitting, we may discuss nonperturbative approaches to QCD.

Both very high-energy experiments and very precise experiments are currently striving to observe effects that cannot be described by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model.

Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or to be familiar with the material covered in them. It would be advantageous to attend the Advanced QFT course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

Literature

1. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1995).
2. F. Halzen and A.D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics*, Wiley (1984).
3. I.J.R. Aitchison and A.J.G. Hey, *Gauge Theories in Particle Physics*, CRC Press (two volumes or earlier 1989 edition in one volume).
4. J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model*, Cambridge University Press (2014).
5. H. Georgi, *Weak Interactions and Modern Particle Theory*, Benjamin/Cummings (1984).
6. T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press (1984).
7. M. Thomson, *Modern Particle Physics*, Cambridge University Press (2013).

Additional support

Four example sheets will be provided and four associated examples classes will be given. There will also be a revision class in Easter Term.

String Theory (L24)

R A Reid-Edwards

String theory is the quantum theory of interacting one-dimensional extended objects (strings). What makes the theory so apealing is that it is a quantum theory that contains gravitational interactions and therefore provides the first tentative steps towards a full quantum theory of gravity. It has become clear that string theory is also much more than this. It has become a framework in which to understand problems in quantum field theory, to ask meaningful questions about what we expect from a quantum theory of gravity and as a test bed for new ideas in mathematics.

This course provides an introduction to String Theory. We begin by generalising the worldline of a particle to the two-dimensonal surface swept out by a string. The quantum theory of the embedding of these surfaces in spacetime is goverened by a two-dimensional quantum field theory and we shall study the simplest example - the bosonic string - in detail.

An introduction to relevant ideas in Conformal Field Theory (CFT) will be given. The quantisation of the string will be studied, its specturm obtained, and the relationship between states on the two dimensional CFT and fields in spacetime will be discussed. We will see the necessity of the critical dimension of spacetime. Time permitting we will discuss boundary conditions and D-branes.

The path integral approach to the theory will be discused in detail. Fadeev-Popov and BRST methods will be introduced to deal with the redundanceies that appear in the theory. Vertex operators will be introduced and scattering amplitudes will be computed at tree level. Perturbation theory at higher loops and the role played by moduli space of Riemann surfaces will be sketched.

Time permitting, RNS supersymmetry will be introduced on the worldsheet and the corresponding superstring theories sketched. There may also be some discusion of compactification and duality.

Pre-requisites

Knowledge of the Quantum Field Theory course in Michaelmas term is assumed. Advanced Quantum Field Theory will complement this course but will not be assumed.

Literature

1. Polchinski, *String Theory: Vol. 1: An Introduction to the Bosonic String*, CUP 1998
2. Green, Schwarz and Witten, *Superstring Theory: Vol. 1:Introduction* CUP 1987.
3. Lust and Theisen, *Lecture Notes in Physics: Superstring Theory*, Springer 1989. (Note there is also a more recent expanded edition witten with Blumenhagen).
4. David Tong, *String Theory*, arXiv:0908.0333

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Supersymmetry (L16)

David Skinner

This course provides an introduction to the role of supersymmetry in quantum field theory, with the emphasis on mathematics rather than phenomenology. We begin with supersymmetric quantum mechanics, covering the Witten index, supersymmetric localization, SQM on a Riemannian manifold, localization of the path integral and the Atiyah–Singer index theorem. The course then moves to higher dimensions studying representations of the super Poincaré algebra in $d = 2$ and $d = 4$. We study superfields and construct supersymmetric actions for gauge and matter theories. Further topics may include $\mathcal{N} = 2$ theories in $d = 2$, their chiral rings and the associated A and B models, Seiberg's non-renormalization theorems in $d = 4$, and Seiberg–Witten theory and its topological applications.

Pre-requisites

You will need to familiar with the material in both the QFT and General Relativity course from Michaelmas. In particular, we will assume knowledge of differential geometry to the level of Prof. Reall's notes, available at

`http://www.damtp.cam.ac.uk/user/hsr1000/lecturenotes_2012.pdf`

It is also strongly recommended that you attend the Lent AQFT course in parallel with this one; the material on path integrals introduced in that course will be needed for this one.

Literature

1. K. Hori, S. Katz, C. Vafa *et al. Mirror Symmetry* Clay Math Monographs, AMS (2003).
2. P. Deligne, E. Witten *et al.*, *Quantum Fields and Strings: A Course for Mathematicians* vols. 1&2, AMS (1999).
3. J. Terning, *Modern Supersymmetry*, International Series of Monographs on Physics, OUP (2009).
4. E. Witten, *Supersymmetry and Morse Theory*, J. Diff. Geom. **17**, (1982) no. 4, 661-692. Also available at

https://projecteuclid.org/euclid.jdg/1214437492

5. E. Witten, *Phases of $\mathcal{N} = 2$ theories in two dimensions*, Nucl. Phys. **B403** (1993) 159-222. Also available at

   https://www.sciencedirect.com/science/article/pii/055032139390033L

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Classical and Quantum Solitons (E16)

N.S. Manton

Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantisation, solitons give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, vortices of the abelian Higgs model in two dimensions, and Skyrmions in three dimensions. Quantised Skyrmions give us a model for protons and neutrons and larger nuclei like the alpha particle, where the topological charge is the conserved baryon number.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Literature

1. N. Manton and P. Sutcliffe, *Topological Solitons.* C.U.P., 2004 (Chapters 1,3,4,5,7,9).
2. E.J. Weinberg, *Classical Solutions in Quantum Field Theory.* C.U.P., 2012 (Chapters 1,2,3,4,8).
3. R. Rajaraman, *Solitons and Instantons.* North-Holland, 1987.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Introduction to the AdS/CFT Duality (E16)

Non-Examinable (Part III Level)

Aron Wall

AdS/CFT is an amazing duality that relates certain theories of quantum gravity (with a negative cosmological constant) to ordinary quantum field theories living in a smaller dimensional spacetime. This is the most precise known realization of the holographic principle, the idea that all information in the universe is encoded somehow at the boundary of the universe. These lectures will describe in detail the “dictionary” used to relate observables on the bulk side to observables on the boundary side. Special attention will be given to the holographic entanglement entropy formula of Ryu and Takayanagi, and the information puzzle for black holes.

Pre-requisites

You should be familiar with quantum field theory, general relativity, and black holes. Basic exposure to quantum information theory and string theory may also be useful.

Although most known forms of AdS/CFT rely on supersymmetry, this aspect will not be heavily emphasized in these lectures.

Literature

TBD

Relativity and Gravitation

These courses provide a thorough introduction to General Relativity and Cosmology. The Michaelmas term courses introduce these subjects, which are then developed in more detail in the Lent term courses on Black Holes and Advanced Cosmology. A non-examinable course explores the application of spinor techniques in General Relativity.

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/undergrad/course`

Year		*Courses*
First	*Essential*:	Vectors & Matrices, Diff. Eq., Vector Calculus, Dynamics & Relativity.
Second	*Essential*:	Methods, Quantum Mechanics, Variational Principles.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Essential*:	Classical Dynamics.
	Very helpful:	General Relativity, Stat. Phys., Electrodynamics, Cosmology.
	Helpful:	Further Complex Methods, Asymptotic methods.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

General Relativity (M24)

Malcolm J. Perry

General Relativity is a classical theory of spacetime and gravitation. It is successfully used to describe small effects in the solar system that Newtonian physics cannot explain, it is used to describe the Earth's gravitational field for GPS, it is used to describe compact stellar objects and black holes, it describes the recently discovered gravitaional waves and it describes the entire Universe from the big-bang to the present era. One of the greatest mysteries and one of the most exciting areas of research in fundamental physics today is a quantum theory of gravity which must underlie general relativity. A thorough knowledge of general relativity is required to proceed with investigations into the quantum theory.

This course is designed to be taken by those who have no knowledge of general relativity. Accordingly, the start will be a discussion of the tools of differential geometry needed to tackle the physics of spacetime and gravitation. We will then use these tools to examine some of the issues raised in the first paragraph.

Pre-requisites

Special Relativity (essential) Relativistic Electrodynamics (highly desirable)

Literature

1. S. Carroll, "Spacetime and Geometry: An Introduction to General Relativity," Pearson International Edition.
2. C.W. Misner, K.S. Thorne, and J.A. Wheeler, "Gravitation," Princeton University Press.
3. Robert M. Wald, "General Relativity," University of Chicago Press.
4. "Einstein Gravity in a Nutshell," A. Zee, Princeton University Press.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Cosmology (M24)

Dr Enrico Pajer and Dr Blake Sherw

Invitation to Cosmology

Cosmology is one of the most exciting and fast developing area of research in Physics. Cosmology stands at the intersection between the deepest problems in theoretical physics, such as the quantum mechanical nature of gravity, the unification of forces and the origin of the observable universe, and the furthest reaching and most precise measurements of our cosmos, such as the Cosmic Microwave background and the systematic survey of most of the galaxies in our observable universe. While in the last 20 years we have reached a well-tested and self-consistent picture of the constituents of our universe, coronated by two Nobel prizes (2006 and 2011), the bluefuture greenlooks red even yellowbrighter.

- What is Dark Matter, which seems to be 6 times more abundant in the universe than atoms and light?
- How is the current expansion of the universe accelerating, how do we know it and why do we call it Dark Energy?
- How did the universe (see map) get so homogeneous and isotropic at large distances? What is cosmological inflation?
- How might Dark Energy and Inflation hold the key to the quantum regime of gravity?

If any of these questions excites your interest, if you want to see general relativity at work, if you wonder about the laws of physics at energies higher than those reached at colliders (at the LHC it is about 10 TeV), if you would like to *use* what you have learnt in statistical and quantum field theory, as well as relativity and hydrodynamics to describe well-tested physical systems (e.g. our whole universe), if you are just curious, then this class is for you.

Cosmology in a broader context

Cosmology connects to many other disciplines in theoretical physics and fits well in any master curriculum in Theoretical Physics.

- Cosmology offers an exciting playground to apply *Quantum Field theory* techniques, as well as to test the *standard model* of particle physics and its extensions. Examples include baryogenesis, nucleosynthesis, cosmic neutrino background and (the weak) Dark matter interactions
- Cosmology is a natural continuation of the study of *general relativity*, which it extends in a pedagogic way to the treatment of *quantization in curved spacetime*. Examples include the quantum generation of primordial fluctuations during inflation, primordial gravitational waves and eternal inflation in the landscape
- Cosmology connects to *hydrodynamics* and *soft matter theory*. Examples include the stochastic dynamics of the large scale structure of the universe ($\hbar = c^{-1} = 0$) and the background of multifield inflation
- Cosmology makes systematic use of tools from *Statistical field theory* and stochastic processes. Examples include the (relativistic) Boltzmann equation, the dynamics of phase transitions in the early universe (QCD, electroweak, etc) and inflation
- Cosmology is related to *astrophysics* and the field of *big-data science*. Examples include observational techniques for galaxy and weak lensing survey, X-ray observations of clusters and the analysis of the largest dataset in the history of humankind (Euclid, LSST, ets)
- Cosmology puts many things into perspective. Examples include the pale blue dot

Course objectives

After the course, the students should be able to:

1. Discuss, using the formalism of general relativity, the evidence for an expanding universe, Hubble's law, cosmological redshift, the big bang theory and the LCDM concordance model. Discuss the the particle content of the universe, relating it to the standard model of particle physics.
2. Write down Friedmann and acceleration equation and use them to predict the evolution of the universe in the presence of any perfect fluid. Sketch the geodesic equation for photon propagation in a FLRW universe.
3. Discuss cosmological perturbation theory, gauge fixing and the qualitative dynamics of perturbations during different stages of our universe (radiation, matter and Dark energy domination).
4. Qualitatively describe the physics of the Cosmic Microwave Background (spectrum and angular power spectrum) and how it constraints our current cosmological model
5. Qualitatively describe the linear matter power spectrum and some features of large scale structures (redshift space, and angular power spectrum)
6. Explain the evidence in favor of primordial inflation and describe its kinematics and dynamics within general relativity and quantum field theory. Qualitatively discuss the primordial power spectrum of curvature perturbations.

Pre-requisites

This course assumes you know some rudiments of Classical Field Theory, General Relativity and Statistical Physics.

Literature

- main reference: Enrico Pajer's lecture notes

For other useful references, see

1. Scot Dodelson, *Modern Cosmology* , Academic Press 2003, ISBN-9780122191411. **Nota Bene:** unfortunately, depending on the version, this textbook has *many typos and errors*. The good news is that almost all of these errors have been found. You can find the list of errors and correction on this errata page. I recommend that you check the errata page every time you use the book and/or solve exercises.
2. Viatcheslav Mukhanov, *Physical foundations of cosmology*, Cambridge University Press 2005, ISBN-9780521563987. See also the very short errata page
3. Steven Weinberg, *Cosmology* , Oxford University Press 2008, ISBN-9780198526827. See also the errata page if something seems to be incorrect

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Black Holes (L24)

J. Santos

A black hole is a region of spacetime that is causally disconnected from the rest of the Universe. These objects appear to be pervasive in Nature, and their properties have direct implications for the recent advances in gravitational wave astronomy. Besides being astrophysically relevant, black holes also play a fundamental role in quantum theory and are a natural arena to study and test any consistent quantum theory of gravity.

The following topics will be discussed:

1. Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
2. The initial value problem, strong cosmic censorship.
3. Causal structure, null geodesic congruences, Penrose singularity theorem.
4. Penrose diagrams, asymptotic flatness, weak cosmic censorship.
5. Reissner-Nordström and Kerr black holes.
6. Energy, angular momentum and charge in curved spacetime.
7. Positivity of energy theorem.
8. The laws of black hole mechanics. The analogy with laws of thermodynamics.
9. Quantum field theory in curved spacetime. The Hawking effect and its implications.

Pre-requisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

1. R.M. Wald, *General relativity*, University of Chicago Press, 1984.
2. S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
3. V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.
4. N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
5. R.M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be distributed during the course. Four examples classes will be held to discuss these. A revision class will be held in the Easter term.

Advanced Cosmology (L24)

Anthony Challinor and Tobias Baldauf

This course will take forward at much greater depth some of the topics in modern cosmology covered in the Michaelmas Term *Cosmology* course. The prediction from fundamental theory for the statistical properties of the primordial perturbations remains the key area of confrontation with cosmological observations, both from large-scale structure and the cosmic microwave background (CMB). This course will develop the mathematical tools and physical understanding necessary for research in this very active area.

Cosmic microwave background

- Statistics of random fields
- Relativistic kinetic theory
- The Boltzmann equation
- The CMB temperature power spectrum
- Photon scattering and diffusion
- Primordial gravitational waves and the CMB
- CMB Polarization

Inflationary theory and Large-Scale Structure

- Primordial non-Gaussianities
- Effective field theory of inflation
- CMB bispectrum and optimal estimators
- Modelling late time non-linearities in large-scale structure
- Effective field theory of large-scale structure
- Tracers of large-scale structure and the peak formalism

Pre-requisites

Material from the Michaelmas term *Cosmology* is essential. Familiarity with introductory Quantum Field Theory and General Relativity is recommended.

Literature

Textbooks

1. Dodelson, S., *Modern Cosmology*, Elsevier (2003).
2. Mukhanov, V., *Physical Foundation of Cosmology*, Cambridge (2005).
3. Weinberg, S., *Cosmology*, Oxford University Press (2008).
4. Misner, C.W., Thorne, K.S., and Wheeler, J.A., *Gravitation*, Freeman (1973).
5. Durrer, R., *The Cosmic Microwave Background*, Cambridge (2008).

Useful references

1. Bardeen, J.M., *Cosmological Perturbations From Quantum Fluctuations To Large Scale Structure*, DOE/ER/40423-01-C8 Lectures given at 2nd Guo Shou-jing Summer School on Particle Physics and Cosmology, Nanjing, China, Jul 1988. (Available on request.)
2. Mukhanov, V.F., Feldman, H.A., and Brandenberger, R.H., *Theory of cosmological perturbations*, Physics Reports, 215, 203 (1992).
3. Ma, C., and Bertschinger, E., *Cosmological Perturbation Theory in Synchronous and Conformal Newtonian Gauges*, Astrophysical Journal, 455, 7 (1995) [astro-ph/9506072].
4. Hu, W. and White, M., *CMB anisotropies: Total angular momentum method*, Physical Review D, 56, 596 (1997) [astro-ph/9702170].
5. Hu, W. and White, M., *A CMB polarization primer*, New Astronomy, 2, 323 (1997) [astro-ph/97006147].
6. Maldacena, J., *Non-gaussian features of primordial fluctuations in single field inflationary models*, Journal of High Energy Physics, 5, 13 (2003).
7. Chen, X., *Primordial Non-Gaussianities from Inflation Models* [arxiv:1002.1416].
8. Wang, Yi., *Inflation, Cosmic Perturbations and Non-Gaussianities* , [arXiv:1303.1523] (Conference Lecture Notes).
9. Senatore, L., Smith, K., Zaldarriaga, M. *Non-Gaussianities in Single Field Inflation and their Optimal Limits from the WMAP 5-year Data*, Journal of Cosmology and Astroparticle Physics, 1, 28 (2010) [arXiv:0905.3746]
10. Ligouri, M., Sefusatti, E., Fergusson, J.R., and Shellard, E.P.S., *Primordial Non-Gaussianity and Bispectrum Measurements in the Cosmic Microwave Background and Large-Scale Structure*, Advances in Astronomy, 2010, 73 (2010) [arxiv:1001.4707]
11. Bernardeau, F., Colombi, S., Gaztanaga, E., Scoccimarro, R., *Large-scale structure of the Universe and cosmological perturbation theory*, Physics Reports, 367 (2002) [arXiv:astro-ph/0112551]
12. Hertzberg, M., *Effective field theory of dark matter and structure formation: Semianalytical results*, Physical Review D, 89, 4 (2014) [arXiv:1208.0839]

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Spinor Techniques in General Relativity (L24)

Non-Examinable (Graduate Level)

Irena Borzym (12 Lectures) and Peter O'Donnell (12 Lectures)

Spinor structures and techniques are an essential part of modern mathematical physics. This course provides a gentle introduction to spinor methods which are illustrated with reference to a simple 2-spinor formalism in four dimensions. Apart from their role in the description of fermions, spinors also often provide useful geometric insights and consequent algebraic simplifications of some calculations which are cumbersome in terms of spacetime tensors.

The first half of the course will include an introduction to spinors illustrated by 2-spinors. Topics covered will include the conformal group on Minkowski space and a discussion of conformal compactifications, geometry of scri, other simple simple geometric applications of spinor techniques, zero rest mass field equations, Petrov classification, the Plucker embedding and a comparison with Euclidean spacetime. More specific references will be provided during the course and there will be worked examples and handouts provided during the lectures.

The second half of the course will include: Newman-Penrose (NP) spin coefficient formalism, NP field equations, NP quantities under Lorentz transformations, Geroch-Held-Penrose (GHP) formalism, modified GHP formalism, Goldberg-Sachs theorem, Lanczos potential theory, Introduction to twistors. There will be no problem sets.

Pre-requisites

The Part 3 general relativity course is a prerequisite.

No prior knowledge of spinors will be assumed.

Literature

Introductory material.

1. L. P. Hughston and K. P. Tod, *Introduction to General Relativity.* Freeman, 1990.
2. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation.* Freeman, 1973.

Best Course Reference Text for Lectures 1 to 12.

J.M. Stewart, *Advanced General Relativity.* CUP, 1993.

Best Course Reference Text for Lectures 13 to 24.

P O'Donnell, *Introduction to 2-spinors in general relativity.* World Scientific, 2003.

Reading to complement course material.

1. Penrose and Rindler, *Spinors and Spacetime Volume 1.* Cambridge Monographs on Mathematical Physics, 1987.

2. S. Ward and Raymond O. Wells, *Twistor Geometry and Field theory.* Cambridge Monographs on Mathematical Physics, 1991 .
3. Robert J. Baston, Michael G. Eastwood, *The Penrose Transform.* Clarendon Press, 1989.
4. S. A Huggett and P. Tod, *Introduction to Twistor Theory.* World Scientific, 2003.
5. R.M. Wald, *General Relativity.* World Chicago UP, 1984.
6. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime.* CUP, 1973.

Astrophysics

Introduction to Astrophysics courses

These courses provide a broad introduction to research in theoretical astrophysics; they are taken by students of both Part III Mathematics and Part III Astrophysics. The courses are mostly self-contained, building on knowledge that is common to undergraduate programmes in theoretical physics and applied mathematics. For specific pre-requisites please see the individual course descriptions.

Astrophysical Fluid Dynamics (M24)

Roman Rafikov

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as their stability and their response to tidal forcing.

Provisional synopsis

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Stellar oscillations. Introduction to asteroseismology and astrophysical tides.
- Local dispersion relation. Internal waves and instabilities in stratified rotating astrophysical bodies.

Pre-requisites

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of vector calculus, fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas.* Cambridge University Press.
2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.
3. Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows.* Cambridge University Press. Available as an e-book from

   `http://ebooks.cambridge.org`

4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics.* University Science Books.
5. Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics.* Imperial College Press.
6. Ogilvie, G. I. (2016). *Lecture Notes: Astrophysical Fluid Dynamics.* J. Plasma Phys. **82**, 205820301.

Additional support

Four example sheets will be provided and four associated classes will be given. Extended notes supporting the lecture course are available from reference 6 in the list above. There will be a revision class in Easter Term.

Planetary System Dynamics (M24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of 1000s of extrasolar planets and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems.

Specific topics to be covered include:

1. Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation
2. Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits
3. Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift
4. Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits
5. Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide

6. Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces
7. Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments
8. Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems
9. Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap

Pre-requisites

This course is self-contained.

Literature

1. Murray C. D. and Dermott S. F.,*Solar System Dynamics.* Cambridge University Press,1999.
2. Armitage P. J., *Astrophysics of Planet Formation.* Cambridge University Press, 2010.
3. de Pater I. and Lissauer J. J., *Planetary Sciences.* Cambridge University Press, 2010.
4. Valtonen M. and Karttunen H., *The Three-Body Problem.* Cambridge University Press, 2006.
5. Seager S., *Exoplanets.* University of Arizona Press, 2011.
6. Perryman M., *The Exoplanet Handbook.* Cambridge University Press, 2011.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extrasolar Planets: Atmospheres and Interiors (M24)

Nikku Madhusudhan

The field of extrasolar planets (or 'exoplanets') is one of the most dynamic frontiers of modern astronomy. Exoplanets are planets orbiting stars beyond the solar system. Thousands of exoplanets are now known with a wide range of sizes, temperatures, and orbital parameters, covering all the categories of planets in the solar system (gas giants, ice giants, and rocky planets) and more. The field is now moving into a new era of Exoplanet Characterization, which involves understanding the atmospheres, interiors, and formation mechanisms of exoplanets, and ultimately finding potential biosignatures in the atmospheres of rocky exoplanets. These efforts are aided by both high-precision spectroscopic observations as well as detailed theoretical models of exoplanets.

The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and stratospheres, equilibrium/non-equilibrium chemistry, atmospheric dynamics, clouds/hazes, etc) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, as well as parametric models and retrieval techniques) (3) exoplanetary interiors (equations of state, mass-radius relations, and internal structures of giant planets, super-Earths, and rocky exoplanets), and (4) relating atmospheres and interiors to planet formation.

Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres of all kinds. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed. The course will also include a discussion on detecting biosignatures in rocky exoplanets, the relevant theoretical constructs and expected observational prospects with future facilities.

Pre-requisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self-contained and basic concepts will be introduced as required.

Literature

1. Chapters on exoplanetary atmospheres and interiors in the book Protostars and Planets VI, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning. Most of these chapters are available publicly on the astro-ph arXiv.
2. Seager, S., Exoplanet Atmospheres: Physical Processes, Princeton Series in Astrophysics (2010).
3. Exoplanets, University of Arizona Press (2011), ed. S. Seager.
4. de Pater, I. and Lissauer J., Planetary Sciences, Cambridge University Press (2010).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Structure and Evolution of Stars (M24)

A.N.Żytkow

Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a mathematical description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These basic equations have to be supplemented by a number of appropriately chosen equations describing the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed. Some familiarity with the principles of hydrodynamics, thermodynamics, quantum mechanics, atomic and nuclear physics will be assumed.

Approximate solutions of the equations will be shown; polytropic gas spheres, homology principles, the virial theorem will be presented. The evolution of a star will be discussed, starting from the main-sequence, following the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

The only way in which we may test stellar structure and evolution theory is through comparison of the theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

Pre-requisites

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics although a detailed knowledge of all of these is not expected.

Preliminary Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Literature

1. Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution, Second Edition*, Springer-Verlag, 2012.
2. Iben, I. *Stellar Evolution Physics, Vol. 1 and 2.* Cambridge University Press, 2013.
3. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
4. Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

Additional support

There will be four example sheets each of which will be discussed during an examples class. There will be a one-hour revision class in the Easter Term.

Optical and infrared astronomical telescopes and instruments (M16)

Ian Parry

Astronomy is an observational science. Our understanding of the universe beyond the Earth comes mostly from interpreting the electromagnetic radiation we see coming from the sky. This course is about the equipment and techniques that we use to collect and measure the optical and near infra-red component of this radiation (approximately 0.3 to 5 microns in wavelength).

The material presented will give the student a thorough understanding of how telescopes and their instruments actually work. An important aim of the course is to quantify how well they work leading to an understanding of what defines the state-of-the-art and what its limitations are.

Specific topics will be selected from the following list;

1. Introduction: Effects of the Earths atmosphere, transparency, seeing, refraction, dispersion, basic definition of magnitudes.
2. Positional astronomy and coordinate systems: Sidereal time, right ascension, declination, hour angle, aberration of starlight, spherical trigonometry, great circles, small circles, spherical triangles, cosine and sine rules, the analogue formula, tangent plane.
3. Optics: geometrical optics, lens-makers equation, principle planes, focal length, f-number, aberrations, paraxial approximation, ray-tracing, image planes, pupil planes, conjugate planes, simple lens design, achromatic lenses, the Petzval lens, methods of designing complex optical systems, étendue, physical optics, quantum optics, Fourier treatment of wave propagation.
4. Telescopes. Refractors, reflectors, parabolic reflectors, Ritchey-Chretien telescopes, 3-mirror anastigmats, the Schmidt telescope, ground-based telescopes, telescope mounts, space-based telescopes.

5. Detectors: photo-electrons, useful semiconductor types, readout architectures, image intensifiers, linearity, dynamic range, quantum efficiency, pixel-to-pixel variations, readout noise, dark-current, sensitivity to cosmic rays, defects, charge transfer, artefacts.
6. Adaptive Optics: wavefront sensors, deformable mirrors, control algorithms, Kolmogorov turbulence, Zernike polynomials.
7. Imagers: eyepieces, the eye, magnification, collimators, cameras, detector matching, image scale, field of view, filters, magnitude systems. Fabry Perot interferometers, polarimetry.
8. Coronagraphs: Fourier modelling, speckles, occulters, lyot-stop, apodization,
9. Spectrographs: Dispersive spectrometers (long-slit, multi-slit, multi-fibre, echelle, integral field), disperser types, grating equation, spectro-polarimetry, Fourier-transform spectrometers.
10. Interferometers:, Stellar interferometry: Fizeau-Stephan interferometer, Michelson stellar interferometer, closure-phase, non-redundant masks.
11. Signal-to-noise ratio: exposure time, Poisson noise, systematic errors, beam-switching, cryogenics.
12. Future projects: E-ELT, LSST, JWST, HDST.

Pre-requisites

This course is self-contained.

Literature

1. Roy, A.E., & Clarke, D., Astronomy Principles and Practice, 4th ed., Institute of Physics, 2003.
2. Kitchin, C.R.: Astrophysical Techniques, 4th ed., Institute of Physics, 2003.
3. Smart, W.M., Spherical Astronomy, 6th ed., Cambridge University Press, 1977.
4. Saha S. K., Diffraction-limited imaging with large and moderate telescopes, World Scientific, New Jersey, 2007.
5. Born & Wolf, Principles of Optics, 7th ed., Cambridge University Press, 2002
6. Optics : E.Hecht
7. Speckle Phenomenon in Optics : J.W.Goodman
8. Astronomical Optics : D.J.Shroeder
9. Astronomical Techniques : W.A.Hiltner
10. Optical Detectors for Astronomy : J.W.Beletic & P.DAmico

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Galaxy Formation (M16)

N Wyn Evans

This new course describes our current state of knowledge of galaxy formation and evolution in a cold dark matter cosmology. We will start with structure formation in the non-linear regime, the formation and evolution of dark matter haloes and Press-Schechter theory. We will cover physical processes (shock heating, radiative cooling and star formation) as well as dynamical transformations (dynamical friction, tidal shocking, accretion and mergers) that are responsible for the shapes and properties of the galaxies we see today. We will end with a study of the formation and current day attributes of disk galaxies (Sersic profiles, thin and thick disks, stellar haloes) and elliptical galaxies (fast/slow rotators, major/minor mergers, Faber-Jackson relation). Recent discoveries on the structure of the Local Group and the Milky Way galaxy will be used as illustrative examples of formation processes throughout the course.

There is a complementary course by V. Belokurov in Lent Term on the present-day life of the galaxies.

Pre-requisites

This Part III course assumes that you have taken undergraduate courses in cosmology, relativity and dynamics.

Literature

1. J. Binney and S. Tremaine *Galactic Dynamics* 2nd edition, Princeton University Press, 2008
2. J. Bland-Hawthorn, K. Freeman *The Origin of the Galaxy and the Local Group*, Springer, 2014
3. A. Loeb *How Did the First Stars and Galaxies Form*, Princeton, 2010 (Background reading)
4. M. Longair, *Galaxy Formation* 2nd edition, Springer, 2008
5. H. Mo, F. van den Bosch and S. White, *Galaxy Formation and Evolution*, Cambridge Universitry Press, 2010
6. S. Phillips, *The Structure and Evolution of Galaxies*, Wiley, 2005
7. L. Sparke, J. Gallagher, *Galaxies in the Universe*, 2nd edition, Cambridge University Press, 2007 (Background reading)

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Dynamics of Astrophysical Discs (L16)

Gordon Ogilvie

A disc of matter in orbital motion around a massive central body is found in numerous situations in astrophysics. For example, Saturn's rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a (non-Newtonian) fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary star systems and in the centres of active galaxies, where they can reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of exoplanets.

Provisional synopsis:

Occurrence of discs in various astronomical systems, basic physical and observational properties.

Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.

Evolution of an accretion disc.

Vertical disc structure, scaling relations and timescales, thin-disc approximations, thermal and viscous stability.

Shearing sheet, symmetries, shearing waves.

Incompressible dynamics: hydrodynamic stability, vortices and dust dynamics.

Compressible dynamics: density waves, gravitational instability and turbulence.

Satellite-disc interaction: tidal potential, resonant torques, migration and gap opening.

Magnetorotational instability and turbulence.

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some previous knowledge of basic magnetohydrodynamics is helpful for the magnetorotational instability, but self-contained notes on this topic will be available.

Literature

Much information on the astrophysical background is contained in [1]. Some of the basic theory of accretion discs is described in review articles [2,3].

1. Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.
2. Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.
3. Papaloizou, J. C. B. & Lin, D. N. C. (1995), Annu. Rev. Astron. Astrophys. 33, 505.

Additional support

Three example sheets will be provided and three associated example classes will be given. There will be a revision class in Easter Term.

Binary Stars (L16)

Christopher Tout

A binary star is a gravitationally bound system of two component stars. Such systems are common in our Galaxy and a substantial fraction interact in ways that can significantly alter the evolution of the individual stellar components. Many of the interaction processes lend themselves to useful mathematical modelling when coupled with an understanding of the evolution of single stars.

In this course we begin by exploring the observable properties of binary stars and recall the basic dynamical properties of orbits by way of introduction. This is followed by an analysis of tides, which represent the simplest way in which the two stars can interact. From there we consider the extreme case in which tides become strong enough that mass can flow from one star to the other. We investigate the stability of such mass transfer and its effects on the orbital elements and the evolution of the individual stars. As a prototypical example we examine Algol-like systems in some detail. Mass transfer leads to the concept of stellar rejuvenation and blue stragglers. As a second example we look at the Cataclysmic Variables in which the accreting component is a white dwarf. These introduce us to novae and dwarf novae as well as a need for angular momentum loss by gravitational radiation or magnetic braking. Their formation requires an understanding of significant orbital shrinkage in what is known as common envelope evolution. Finally we apply what we have learnt to a number of exotic binary stars, such as progenitors of type Ia supernovae, X-ray binaries and millisecond pulsars.

Pre-requisites

The Michaelmas term course on Structure and Evolution of Stars is very useful but not absolutely essential. Knowledge of elementary Dynamics and Fluids will be assumed.

Literature

1. Pringle J. E. and Wade R. A., *Interacting Binary Stars.* CUP.

Reading to complement course material

1. Eggleton P. P., *Evolutionary Processes in Binary and Multiple Stars.* CUP.

Additional support

Three examples sheets will be provided and three associated two-hour examples classes will be given. There will be a two-hour revision class in the Easter Term.

Astrophysical black holes (L16)

Debora Sijacki

Black holes are one of the most fascinating objects lying at the interface of mathematics, physics and astronomy. From the astrophysical stand point they give rise to extremely rich and complex phenomena occurring from sub-parsec to Mega-parsec scales and covering the full electromagnetic spectrum. A large body of state-of-the-art current and upcoming observational facilities and theoretical models is aimed at investigating black hole properties and their link with the larger scale environment, making this an exciting and fast paced research field. With the recent gravitational wave detections of merging black hole binaries the field has experienced further stimulus, as black holes have become unique multi-messengers to explore cosmology, gravity in the strong regime, high energy phenomena and complex (magneto)hydrodynamic flows.

This course will cover a range of concepts pertinent to astrophysical (supermassive) black holes highlighting both observational and theoretical advances in the field. The aim of the course is to give an overview of the possible formation and growth channels of these objects and to discuss various mechanisms through which black holes interact with their surroundings, with the provisional synopsis as follows:

- Basic concepts; observational evidence for dormant and non-dormant objects (SgrA*)
- AGN properties and classification
- Formation pathways for supermassive black holes
- Black hole growth overview
- Fuelling mechanisms from kpc to sub-pc scales
- Bondi-Hoyle solution and limitations
- Accretion disk models: thin, slim and thick discs
- Stellar capture (TDEs)
- Black hole binaries and hardening processes
- Black hole mergers and associated GW emission
- Black hole mass and spin evolution during inspiral and GW-driven recoils
- Outflows: basic concepts; collimated wind and jet phenomena
- Energy-, momentum- and radiation pressure-driven outflow solutions
- Impact of outflows on host properties

Pre-requisites

Newtonian mechanics, fluid dynamics and basic knowledge of magnetohydrodynamics. Some knowledge of (thermo)dynamics, electromagnetism and galaxy formation is advantageous but not strictly necessary.

Literature

1. Peterson, B. M., *An Introduction to Active Galactic Nuclei*, Cambridge University Press, 1997.
2. Netzer, H., *The Physics and Evolution of Active Galactic Nuclei*, Cambridge University Press, 2013.
3. Krolik, J. H., *Active Galactic Nuclei: from the Central Black Hole to the Galactic Environment*, Princeton University Press, 1999.
4. Frank, J., King, A. R., & Raine, D., *Accretion Power in Astrophysics*, Cambridge University Press, 2002.
5. Pringle, J. E., & King, A. R., *Astrophysical Flows*, Cambridge University Press, 2007.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The Life of Galaxies (L16)

Vasily Belokurov

This course will provide the observational perspective on the evolution of galaxies and will complement the theoretical Part III course "The Birth of Galaxies".

The course will cover the following topics. Elements of observational cosmology, structural properties of galaxies, galactic luminosity and mass functions, colors and spectra of galaxies, galaxy scaling relations, high redshift galaxies, gravitational lensing, interstellar medium, intergalactic medium, black holes, active galactic nuclei, quasars, clusters of galaxies, interacting galaxies, Local Group of galaxies, star clusters, elements of observational stellar evolution, star formation, dwarf galaxies, observable manifestations of Dark Matter, feedback, death of galaxies.

Pre-requisites

It is preferable (but not strictly required) that you have attended "The Birth of Galaxies" course in M18.

Literature

Please note that there exists no all-encompassing up-to-date book on the subject.

1. Binney, J., and Tremaine, S. *Galactic Dynamics* Second Edition. Princeton University Press, 2008
2. Mo, H., van den Bosch, F., and White, S. *Galaxy Formation and Evolution* Cambridge University Press, 2010
3. Sparke, L., and Gallagher, J. *Galaxies in the Universe* Second Edition. Cambridge University Press, 2007
4. Longair, M. *Galaxy Formation* Second Edition. Springer, 2008

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

High energy radiative processes in astrophysical plasma (L16)

Non-Examinable (Part III Level)

Giulio Del Zanna

The main aim of this non-examinable 16 lecture course for the Lent term is to provide the students with a broad overview of the physics underlying the main radiative processes occurring in astrophysical plasma. We will focus first on emission line diagnostics, which mostly covers the lower-energies, then discuss the continuum emission. We provide general background theory but devote a good fraction of the lectures to applications, to provide an insight on current research in astrophysics, based on high-resolution spectroscopy, which is becoming widely available.

We start with an introduction to radiation quantities and concepts such as radiation and matter in thermodynamical equilibrium. We then introduce the basics of radiative transfer in one dimension, linking Einstein's coefficients with opacity in a spectral line. We then review the basics of atomic structure for complex atoms, and then introduce the fundamental processes affecting the level populations within an ion and the ion charge states in a plasma. We summarise how rates are calculated and where they are available. We then illustrate how spectral line intensities are calculated assuming thermal equilibrium and used for plasma diagnostics, with emphasis on measurements of electron densities, temperatures and chemical composition. We provide several specific example applications for planetary nebulae, inter-galaxy hot plasma, stellar coronae. We then briefly review non-equilibrium effects. Finally, we introduce the theory of the X-ray satellite lines and how they can be used to measure the state of high-energy plasma. We provide examples based on solar flare spectra and high-energy Hitomi spectra of the Perseus cluster.

We then briefly describe continuum processes, starting with the basics of free-free and free-bound emission, cyclotron and synchrotron emission and absorption, Thomson scattering and Compton scattering. We provide examples of high-energy spectra of astrophysical sources. Finally, we introduce Active Galactic Nuclei: accretion disks, broad and narrow emission line spectra, X-ray emission and the 6 keV iron feature.

Pre-requisites

This course assumes you would have taken the Part II course on Principles of Quantum Mechanics (D Skinner).

It would be useful if students had taken one of the following courses. However, the same topics will be covered in this course.

1) Advanced Quantum Physics (R. Batley)

as this covers: A) Atomic structure: Hydrogen atom; fine structure: relativistic corrections; spin-orbit coupling; hyperfine structure. Multi-electron atoms: LS coupling; Hund's rules; Zeeman effect.

B) Elements of quantum field theory: Quantization of the electromagnetic field, photons; number states. Radiative transitions, dipole approximation, selection rules, spontaneous emission and absorption, stimulated emission, Einstein ¡92¿s A and B coefficients;

2) Part II: Advanced Quantum Mechanics (Ben Simons) as it covers several aspects, including an extended version of atomic structure, Radiative transitions, Scattering theory.

3) Lectures on Applications of Quantum Mechanics (David Tong)

as it covers: Hydrogen; Spin-Orbit coupling, Fine structure, Hyperfine structure; Helium, Exchange energy; Hartree method, Slater determinant, Hartree-Fock method; The Zeeman effect; Spontaneous emission, Selection rules

Literature

1. Del Zanna, G. and Mason, H.E., *Solar UV and X-Ray Spectral Diagnostics*, Living Reviews in Solar Physics, in press.
   `https://www.springer.com/br/livingreviews/solar-physics/lrsp-articles`

2. Landi Degl' Innocenti, E., *Atomic Spectroscopy and Radiative Processes*, 2014, Springer.

3. Rybicki, G. B. and Lightman, A. P., *Radiative processes in astrophysics.*, 1979, New York, Wiley-Interscience.

4. Ghisellini, G., *Radiative processes in high energy astrophysics*, 2013.
   `http://adsabs.harvard.edu/abs/2013LNP...873.....G`

Quantum Computation, Information and Foundations

Quantum Computation (M16)

Richard Jozsa

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

This course will be a 'second' course in the subject, following the Part II course Quantum Information and Computation (see below in prerequisites) that was introduced in the year 2017-2018.

In this course we will discuss a selection from the following topics (and possibly others too):
- Review of Shor's algorithm; The hidden subgroup problem.
- Quantum algorithmic primitives of amplitude amplification and phase estimation.
- Quantum simulation for local hamiltonians.
- The Harrow-Hassidim-Lloyd quantum algorithm for systems of linear equations.
- Clifford operations in quantum computation.

Pre-requisites

This course will assume a prior basic acquaintance with quantum computing, to the extent presented in the course notes for the Cambridge Part II course Quantum Information and Computation. These course notes are available to download from `http://www.qi.damtp.cam.ac.uk/node/261`. In particular you should be familiar with Dirac notation and basic principles of quantum mechanics, as presented in the course notes sections 2.1, 2.2 and 2.3. You should also have a basic acquaintance with quantum computation to the rough extent of the second half of the course notes, pages 47 to 86 (Chapters 6-11). It would be desirable for you to look through this material before the start of the course.

Literature

Further useful literature includes the following.

1. Nielsen, M. and Chuang, I., *Quantum Computation and Quantum Information.* CUP, 2000.
2. John Preskill *Lecture Notes on Quantum Information Theory* (especially Chapter 6) available at `http://www.theory.caltech.edu/people/preskill/ph219/`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Information Theory (L24)

Nilanjana Datta

Quantum Information Theory (QIT) is an exciting, young field which lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the mathematical theory of acquisition, storage, transmission and processing of information. QIT is the study of how these tasks can be accomplished, using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical analogues. These new features can be exploited, not only to improve the performance of certain information-processing tasks, but also to accomplish tasks which are impossible or intractable in the classical realm.

This is an introductory course on QIT, which should serve to pave the way for more advanced topics in this field. The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of QIT. Topics in this part of the course will include a brief discussion of data compression, transmission of data through noisy channels, Shannon's theorems, entropy and channel capacity.

The quantum part of the course will commence with a study of open systems and a discussion of how they necessitate a generalization of the basic postulates of quantum mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs, the Kraus Representation Theorem and the Choi-Jamilkowski isomorphism. Entanglement and some applications elucidating its usefulness as a resource in QIT will be discussed. This will be followed by a study of the von Neumann entropy, its properties and its interpretation as the data compression limit of a quantum information source. Schumacher's theorem on quantum data compression will be discussed in detail. The definitions of ensemble average fidelity and entanglement fidelity will be introduced in this context. Definitions and properties of the quantum conditional entropy, quantum mutual information, the quantum relative entropy and coherent information will be discussed. Various examples of quantum channels will be given and the different capacities of a quantum channel will be discussed. The Holevo bound on the accessible information and the Holevo-Schumacher-Westmoreland (HSW) Theorem will also be covered.

Pre-requisite Mathematics

Knowledge of basic quantum mechanics will be assumed. However, an additional lecture can be arranged for students who do not have the necessary background in quantum mechanics. Elementary knowledge of Probability Theory, Vector Spaces and Linear Algebra will be useful.

Literature

The following books and lecture notes provide interesting and relevant reading material.

1. M.A.Nielsen and I.L.Chuang, Quantum Computation and Quantum Information; Cambridge University Press, 2000.
2. M. M. Wilde, *From Classical to Quantum Shannon Theory*, http://arxiv.org/abs/1106.1445
3. J. Watrous, Lecture Notes, https://cs.uwaterloo.ca/ watrous/LectureNotes.html
3. J.Preskill, Chapter 5 of his lecture notes: Lecture notes on Quantum Information Theory, http://www.theory.caltech.edu

Philosophy of Physics

The courses in Philosophy of Physics are open to all students doing Part III, but are formally listed as graduate courses. This means there is no exam at the end of May for any such course; but a Part III student can get credit for them by doing their submitted Part III essay in association with one of the courses. More generally, the Philosophy of Physics courses are intended as a refreshing and reflective companion to the other Part III courses, especially the courses in theoretical physics.

Philosophical Aspects of Symmetry and Duality (M8)

Non-Examinable (Part III Level)

J. Butterfield

Symmetry principles have been at the centre of physics for over a hundred years. In recent decades, dualities have been increasingly important. In short: while a symmetry maps a state of the system into an appropriately related state (namely, one with the same values for a salient set of physical quantities): in a duality, an entire theory is mapped into another appropriately related theory. This course will introduce the philosophical literature about symmetries and dualities.

The content will be moulded by students' interests. But I propose four pairs of lectures, as follows. I will: first (i) review Noether's theorem in classical Lagrangian and Hamiltonian mechanics, using the modern geometric formulation; then (ii) review elements of quantum mechanics' use of group representation theory; then (iii) introduce the philosophical literature about symmetries, especially about the rationales for quotienting under a symmetry; then (iv) introduce dualities, and my preferred account of them. If we follow this plan, we will be setting aside various topics, such as identity in physics (i.e. representations of the symmetric group) and gauge structure. But we can decide what to emphasise, as we go along.

Pre-requisites

There are no formal prerequisites. Previous familiarity with classical and quantum mechanics, at undergraduate level, will be assumed. Of the Part III courses, the closest companion to this course is (needless to say): 'Symmetries, Particles and Fields'.

Preliminary Reading

From a vast literature, here is a selection, for each of the first three themes (i) to (iii), of one book that is appropriate for preliminary reading.

1. D. E. Neuenschwander, *Emmy Noether's Wonderful Theorem.* Johns Hopkins University Press ,2011.
2. M. Hamermesh, *Group Theory and its Application to Physical Problems.* Dover, 1989.
3. K. Brading and E. Castellani (eds.), *Symmetries in Physics.* Cambridge University Press, 2003.

Literature

The suggested literature is in the order of these four themes. For themes (i) (iii) and (iv), the suggestions are papers. For theme (ii), the two suggestions are books. The book by Landsman is Open Access, and freely downloadable.

1. J. Butterfield. *On Symmetries and Conserved Quantities in Classical Mechanics*, in W. Demopoulos and I. Pitowsky (eds.), *Physical Theory and its Interpretation*, Springer 2006; 43 - 99; Available at

http://arxiv.org/abs/physics/0507192

2. J. Butterfield. *On Symplectic Reduction in Classical Mechanics*, in J. Earman and J. Butterfield (eds.) *The Handbook of Philosophy of Physics*, North Holland 2006; 1 - 131. Available at http://arxiv.org/abs/physics/0507194
3. W Greiner and B Müller, *Quantum Mechanics: Symmetries.* Springer, 1994.
4. N. Landsman. *Foundations of Quantum Theory.* Springer 2017: especially Chapter 5. Open access: downloadable at: https://link.springer.com/book/10.1007/978-3-319-51777-3
5. G. Belot. Three articles: (i) *Notes on symmetries*, in K. Brading and E. Castellani (eds.), *Symmetries in Physics.* Cambridge University Press, 2003; (ii) *Symmetry and gauge freedom*, Studies in History and Philosophy of Modern Physics **34** 189-225 (2003); (iii) *Symmetry and equivalence*, in R. Batterman (ed.), *Oxford Handbook of Philosophy of Physics.* Oxford University Press, 2013. All available at https://sites.google.com/site/gordonbelot/home/papers-etc
6. A. Caulton. *The role of symmetry in the interpretation of physical theories.* Studies in the History and Philosophy of Modern Physics **52** 153-162 (2015). Available at https://www.sciencedirect.com/science/article/pii/S1355219815000635
7. N. Dewar. *Sophistication about symmetries*, British Journal for the Philosophy of Science 2017. Available at https://academic.oup.com/bjps/advance-article-abstract/doi/10.1093/bjps/axx021/4111183
8. S. De Haro and J. Butterfield. *A Schema for Duality, Illustrated by Bosonization*, in J. Kouneiher (ed.) *Foundations of Mathematics and Physics one century after Hilbert.* Collection Mathematical Physics, Springer 2017. Available at https://arxiv.org/abs/1707.0668
9. S. De Haro. *The heuristic function of duality*, Synthese 2018. Available at https://link.springer.com/article/10.1007/s11229-018-1708-9
10. J. Butterfield. *On Dualities and Equivalences Between Physical Theories*, in B. Le Bihan, N.Huggett and C. Wuthrich (eds.) *Philosophy Beyond Spacetime* Oxford University Press, 2018. Available at https://arxiv.org/abs/1806.01505

Additional support

One or two Part III essays will be offered in conjunction with this course.

Hamiltonian General Relativity (L8)

Non-Examinable (Part III Level)

J. B. Pitts

How does one set up a Hamiltonian formulation for GR, Maxwell or Yang-Mills, given that the Legendre transformation does not work? Does Hamiltonian General Relativity really lack change? Are observables nonlocal and constants of the motion? Such claims are part of the "problem of time" that has afflicted Hamiltonian General Relativity (and hence canonical quantum gravity!) since the 1950s. These lectures are a foundationally-oriented introduction to Rosenfeld-Dirac-Bergmann constrained Hamiltonian dynamics with special attention to real examples and the relation to the Lagrangian and 4-dimensional geometric formalisms. A mixed system, Einstein-Proca, with constraints related to gauge freedom and constraints unrelated to gauge freedom, will also appear, as will Einstein-Dirac and, time permitting, perhaps a bit of supergravity.

Pre-requisites

Familiarity with mechanics, electromagnetism, and Special and General Relativity is helpful.

Literature

1. J. Anderson and P. Bergmann, *Physical Review* **83** (1951), p. 1018.
2. P. Bergmann, *Reviews of Modern Physics* **33** (1961), p. 510.
3. L. Castellani, *Annals of Physics* **143** (1982), p. 357.
4. K. Kuchař, pp. 119-150 of *General Relativity and Gravitation 1992: Proceedings of the Thirteenth International Conference on General Relativity and Gravitation held at Cordoba, Argentina, 28 June–4 July 1992*, edited by R. J. Gleiser, C. N. Kozameh and O. M. Moreschi; arXiv:gr-qc/9304012.
5. C. Misner, K. Thorne, and J. Wheeler, *Gravitation*, chapter 21.
6. J. B. Pitts, *Classical and Quantum Gravity* **34** (2017), 055008, arXiv:1609.04812 [gr-qc].
7. J. Pons and D. Salisbury, *Physical Review D* **71** (2005), 124012, gr-qc/0503013.
8. J. Pons and D. Salisbury and K. Sundermeyer, *Physical Review D 80* (2009), 084015, arXiv:0905.4564v2 [gr-qc].
9. J. Pons, D. Salisbury and K. Sundermeyer, *Journal of Physics: Conference Series* **222** (2010), 012018, arXiv:1001.2726v2 [gr-qc].
10. L. Rosenfeld, *Annalen der Physik* **397** (1930), p. 113, translated by Donald Salisbury and Kurt Sundermeyer, *European Physical Journal H* **42** (2017), p. 63.
11. D. Salisbury and K. Sundermeyer, *European Physical Journal H* **42** (2007), p. 23.
12. L. Shepley, J. Pons, and D. Salisbury, *Turkish Journal of Physics* **24** (2000), p. 445.
13. K. Sundermeyer, *Constrained Dynamics: With Applications to Yang–Mills Theory, General Relativity, Classical Spin, Dual String Model* (Lecture Notes in Physics, volume 169). Springer, 1982.
14. K. Sundermeyer, *Symmetries in Fundamental Physics*, second edition (Fundamental Theories of Physics, volume 176). Springer, 2014.
15. R. Wald, *General Relativity*, appendix E.

Additional support

A Part III essay will be offered in conjunction with this course.

Applied and Computational Analysis

Topics in Convex Optimisation (M16)

Hamza Fawzi

Mathematical optimisation problems arise in many areas of science and engineering, including statistics, machine learning, robotics, signal/image processing, and others. This course will cover some techniques known as *convex relaxations*, to deal with optimisation problems involving polynomials, which are in general intractable. The emphasis of the course will be on semidefinite programming which is a far-reaching generalization of linear programming. A tentative list of topics that we will cover include:

- From linear programming to conic programming. Duality theory.
- Semidefinite optimisation and convex relaxations. Sums-of-squares and moment problems.
- Applications: binary quadratic optimisation and rounding methods (e.g., Goemans-Williamson rounding), stability of dynamical systems, matrix completion/low-rank matrix recovery, etc.

Pre-requisites

This course assumes basic knowledge in linear algebra and analysis. Some knowledge of convex analysis will be useful.

Literature

1. A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, 2001 (`http://dx.doi.org/10.1137/1.9780898718829`).
2. G. Blekherman, P. Parrilo, R. Thomas, *Semidefinite optimization and convex algebraic geometry*, SIAM 2013 (`http://dx.doi.org/10.1137/1.9781611972290`).
3. S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004 (`http://web.stanford.edu/~boyd/cvxbook/`).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Inverse Problems in Imaging (M16)

Yury Korolev

Inverse problems arise whenever there is a need to infer about quantities of interest from indirectly measured data. Inverse problems are ubiquitous in science; they arise in physics, biology, medicine, engineering, finance and computer science (e.g., in machine learning and computer vision). Many imaging problems, such as reconstruction of medical images (computer tomography, magnetic resonance imaging, positron-emission tomography) and deblurring or denoising of microscopy and astronomy images, are also instances of inverse problems. Inverse problems typically share a feature that makes them challenging to solve in practice: they lack continuous dependence on the data and, therefore, small errors in the measurements can lead to large errors in the reconstructions, making them practically useless. To deal with this issue, special *regularisation* techniques have been developed that overcome the instability by using additional a priori information about the solution, such as smoothness or sparsity in some basis. In this course we will present the mathematical theory of inverse problems and regularisation, from the classical foundations to modern variational regularisation methods, and apply it to some problems in imaging using state-of-the-art numerical algorithms.

Pre-requisites

This course assumes basic knowledge in linear algebra and analysis (e.g. linear analysis or analysis of functions). Additional knowledge in convex analysis is beneficial, but not mandatory.

Literature

1. H. W. Engl, M. Hanke and A. Neubauer. *Regularization of Inverse Problems.* Vol. 375, Springer Science & Business Media, 1996, ISBN: 9780792341574
2. O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier and F. Lenzen. *Variational Methods in Imaging.* Applied Mathematical Sciences, Springer New York, 2008, ISBN: 9780387309316
3. A. Chambolle, T. Pock, *An introduction to continuous optimization for imaging*, Acta Numerica, **25**, 161-319 (2016). Also available at

   `https://www.cambridge.org/core/journals/acta-numerica/article/an-introduction-to-continuous-optimization-for-imaging/1115AA7E36FC201E811040D11118F67F`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Distribution Theory and Applications (M16)

Dr. A. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will breifly look at Sobolev spaces in $\mathbf{R}^n$ and their description in terms of the Fourier transform of tempered distrbutions. The material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. The final section of the course will be concerned with Hörmander's oscillatory integrals.

Pre-requisites

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Methods or Complex Analysis).

Preliminary Reading

1. F.G. Friedlander & M.S. Joshi, *Introduction to the Theory of Distributions*, Cambridge Univ Pr, 1998.
2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge Univ Pr, 1958.
3. G.B. Folland, *Introdution to Partial Differential Equations*, Princeton Univ Pr, 1995.

Literature

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols* I-II, Springer Verlag, 1985.
2. M. Reed & B. Simon, *Methods of Modern Mathematical Physics: Vols* I-II, Academic Press, 1979.
3. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be made available. There will be a revision class in the Easter Term.

Boundary value problems for linear PDEs (M16)

Konstantinos Kalimeris

Recent developments in the area of the so-called *integrable nonlinear* Partial Differential Equations (PDEs) have led to the emergence of a new method for solving boundary value problems, which is usually referred to as the *Unified Transform* (UT) or the *Fokas Method*.

The UT will be implemented to:

(a) Linear evolution PDEs in one spatial variable formulated either on the half-line or on a finite interval. Examples include the heat equation and the Stokes equation (linearised version of the KdV).

(b) Linear elliptic PDEs in two spatial variables formulated in the interior of a convex polygon. Examples include the Laplace, the modified Helmholtz, and the Helmholtz equations.

For the above problems, in addition to presenting integral representations of the solution, simple numerical techniques for the effective computation of the solution will also be introduced.

Pre-requisites

The course only requires some elementary knowledge of complex analysis.

Literature

1. A.S. Fokas, *A unified method for boundary value problems.* 1st edition. SIAM, 2008.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Numerical Solution of Differential Equations (L24)

Arieh Iserles

The course will address modern algorithms for the solution of ordinary and partial differential equations, inclusive of finite difference and finite element methods, with an emphasis on broad mathematical principles underlying their construction and analysis.

Pre-requisites

Although prior knowledge of *some* numerical analysis and of abstract function spaces is advantageous, it will not be taken for granted. Reasonable understanding of basic concepts of analysis (complex analysis and analytic functions, basic existence and uniqueness theorems for ODEs and PDEs, elementary facts about PDEs) and of linear algebra is a prerequisite.

Literature

1. U. Ascher, *Numerical Methods for Evolutionary Differential Equations*, SIAM, 2008.
2. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations* (2nd edition), Cambridge University Press, 2006.

Additional support

An extensive printed handout, covering the entire material of the course, will be provided in the first week. There will be weekly examples' classes, starting from the third week, as well as a revision supervision in the Easter Term.

Introduction to Optimal Transport (L24)

Matthew Thorpe

Optimal transport crosses many branches of mathematics such as partial differential equations, probability, fluid mechanics and functional analysis. Applications of Optimal Transport are increasing as numerical developments have made computations ever more efficient. We now see applications of optimal transport in (i) image retrieval, registration and morphing, (ii) color and texture analysis, (iii) image denoising and restoration, (iv) morphometry, (v) super resolution, and (vi) machine learning. In this course I aim to give an overview of the theory of optimal transport. Whilst we will cover some of the numerical methods I will largely skip applications.

Course Content

We will cover the following topics.

Kantorovich Duality. Kantorovich duality forms the basis for many theoretical results regarding optimal transport, for example the equivalence of Monge and Kantorovich's formulation.

Existence and Characterisations of Optimal Transport Maps. We prove existence of optimal transport plans, and their characterisation as the subgradient of a convex function.

Wasserstein spaces, Geodesics, and Riemannian Structure. We define the Wasserstein distance and explore its topology, including proving existence of geodesics.

Gradient Flows for the Fokker-Planck Equation. We show how the Wasserstein distance arises naturally in the gradient flow approach for computing solutions to the Fokker-Planck equation.

Numerical Methods: The Entropy Regularised Approach. Entropy regularised optimal transport leads to a numerically efficient algorithm for the computation of optimal transport. Indeed, one can write entropy regularised optimal transport as a Kullback-Liebler divergence and use iterative optimisation methods such as Sinkhorns algorithm.

Pre-requisites

Familiarity with measure theory and functional analysis will be necessary. Prior exposure to convex analysis will be helpful but not essential.

Literature

1. C. Villani, *Topics in Optimal Transportation.* AMS, 2003.
2. C. Villani, *Optimal Transport: Old and New.* Springer, 2008. Also available at

   `http://cedricvillani.org/wp-content/uploads/2012/08/preprint-1.pdf`

3. F. Santambrogio, *Optimal Transport for Applied Mathematicians: Calculus of Variations, PDEs, and Modeling.* Birkhäuser, 2015. Also available at

   `https://www.math.u-psud.fr/~filippo/OTAM-cvgmt.pdf`

4. L. Ambrosio, N. Gigli and G. Saveré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Springer Science & Business Media, 2008. Also available at

   `http://www2.stat.duke.edu/~sayan/ambrosio.pdf`

5. I will also produce course notes which will be updated throughout term.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Bayesian Inverse Problems (L16)

Hanne Kekkonen

Inverse problems arise when only indirect measurement data are available and we would like to use them to gain information about an object of interest. Inverse problems have several applications in areas ranging from medical imaging and industrial process monitoring to ozone layer tomography and financial market modelling. The common feature of these problems is the need to understand indirect measurements and to overcome extreme sensitivity to noise and modelling inaccuracies. In this course, we employ a statistical approach to inverse problems. This approach enables us to find stable and meaningful solutions that quantify how uncertainty in data (or model) affects the obtained estimate.

Pre-requisites

This course assumes basic knowledge in analysis and probability theory (e.g. Linear Analysis, and Probability and Measure).

Literature

1. M. Dashti and A.M. Stuart, *The Bayesian approach to inverse problems, Handbook of Uncertainty Quantification.* Springer, 2017.
2. A.M. Stuart, *Inverse problems: a Bayesian perspective.* Acta Numerica, 2010.
3. T.J. Sullivan, *Introduction to Uncertainty Quantification.* Springer, 2015.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

An Introductory Course on Non-smooth Optimisation (L16)

Non-Examinable (Part III Level)

Jingwei Liang

Driven by the problems arising from fields such as signal/image processing, inverse problem, data science and machine learning, the past two decades have witnessed a tremendous developments of modern non-smooth optimisation. This is an introductory course to non-smooth optimisation, the topics will include:

- Classic first-order proximal splitting methods developed since 1950s.
- Acceleration techniques for developing fast numerical algorithms.
- Stochastic optimisation methods that are widely used in data science and machine learning.

Pre-requisites

Basic knowledge in linear algebra and analysis. Some knowledge of convex analysis will be useful.

Literature

1. S. Boyd and L. Vandenberghe *Convex Optimization.* Cambridge university press, 2004.
2. N. Parikh and S. Boyd, *Proximal Algorithms.* Foundations and Trends® in Optimization, **1.3**, 127-239 (2014).
3. A. Chambolle and T. Pock, *An Introduction to Continuous Optimization for Imaging.* Acta Numerica, **25**, 161-319 (2016).
4. Y. Nesterov, *Introductory Lectures on Convex Optimization: A Basic Course.* Vol. 87. Springer Science & Business Media, 2013.

Additional support

Two projects will be provided, middle and end of Lent term.

Mathematics of Image Reconstruction (L16)

Non-Examinable (Graduate Level)

Evren Yarman

Trying to infer about an unknown through observation and measurement appears in many disciplines. In applied sciences, observations are used to build measurement systems and construct mathematical models of the relationship between the unknown and the measurement. Given the mathematical model, image reconstruction deals with formation of an interpretable image from the measurement. A widely seen example is non-destructive visualization of the interior or remote sensing of exterior/interior of an object. In this course we will consider some of the mathematical modalities and associated image reconstruction methods appearing in biomedical, radar and seismic imaging.

Pre-requisites

This course assumes basic knowledge in linear algebra and analysis. Familiarity with programming is desirable.

Literature

1. F. Natterer and F. Wuebbeling. *Mathematical Methods in Image Reconstruction.* Mathematical Modeling and Computation. Society for Industrial and Applied Mathematics, 2001.
2. N. Bleistein, J.K. Cohen, and J.W.J. Stockwell. *Mathematics of Multidimensional Seismic Imaging, Migration, and Inversion.* Interdisciplinary Applied Mathematics. Springer New York, 2013.
3. M. Cheney and B. Borden. *Fundamentals of Radar Imaging.* CBMS-NSF Regional Conference Series in Applied Mathematics. Society for Industrial and Applied Mathematics, 2009.
4. H.H. Barrett and K.J. Myers. *Foundations of image science.* Wiley series in pure and applied optics. Wiley-Interscience, 2004.
5. H.P. Langtangen. *A Primer on Scientific Programming with Python.* Texts in Computational Science and Engineering. Springer Berlin Heidelberg, 2009.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Example sheets and classes will involve programming in Python. Anaconda distribution for scientific computing using Python can be downloaded at `https://www.anaconda.com/download/`.

Systems Biology (E10)

Non-Examinable (Graduate Level)

Johan Paulsson & Andreas Hilfinger

In this module we use applied mathematics tools to figure out design principles of biological systems. Our main focus is on stochastic processes and master equations to describe complex biochemical reaction networks in single cells. We will derive theorems that establish universal performance bounds and trade-offs due to low copy number noise that underlies all biochemical processes within cells. We show how these mathematical relations are powerful tools to understand non-genetic variability within clonal populations of cells.

Pre-requisites

Interest in biology and the challenges that come with modelling the real world. Basic familiarity with probability and differential equations.

Literature

1. Hilfinger, Andreas et al., *Constraints on Fluctuations in Sparsely Characterized Biological Systems*, Phys. Rev. Lett., **116**, (2016). Available online at

   `https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.116.058101`

2. Paulsson, Johan, *Summing up the noise in gene networks*, Nature, **427**, 415-418 (2004). Available online at

   `https://www.nature.com/articles/nature02257`

Additional support

This module will run during the course of one week in the Easter Term. 5×2 h of lectures will be supplemented with 3×2 h of tutorials.

Continuum Mechanics

The four courses in the Michaelmas Term are intended to provide a broad educational background for any student preparing to start a PhD in fluid dynamics. The courses in the Lent Term are more specialized and in some cases (see the course descriptions) build on the Michaelmas Term material.

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice, familiarity with the continuum assumption, the material derivative, the stress tensor and the Navier-Stokes equation will be assumed, as will basic ideas concerning incompressible, inviscid fluid mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable. Previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is desirable for some courses. No previous knowledge of solid mechanics, Earth Sciences, or biology is required.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses

Year	*Courses*
First	Differential Equations, Dynamics and Relativity, Vector Calculus, Vectors & Matrices.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves, Asymptotic Methods.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Pre-requisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Preliminary Reading

1. D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7
2. G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.
3. L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 & 5.

Literature

1. J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).
2. S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)
3. C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).
4. O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Hydrodynamic Stability (M24)

Colm-cille Caulfield & Richard R. Kerswell

Developing an understanding by which "small" perturbations grow, saturate and modify fluid flows is central to addressing many challenges of interest in fluid mechanics. Furthermore, many applied mathematical tools of much broader relevance have been developed to solve hydrodynamic stability problems, and hydrodynamic stability theory remains an exceptionally active area of research, with several exciting new developments being reported over the last few years.

In this course, an overview of some of these recent developments will be presented. After a brief introduction to the general concepts of flow instability, presenting a range of examples, the major content of this course will be focussed on the broad class of flow instabilities where velocity "shear" and fluid inertia play key dynamical roles. Such flows, typically characterised by sufficiently "high" Reynolds number Ud/ν, where U and d are characteristic velocity and length scales of the flow, and ν is the kinematic viscosity of the fluid, are central to modelling flows in the environment and industry. They typically demonstrate the key role played by the redistribution of vorticity within the flow, and such vortical flow instabilities often trigger the complex, yet hugely important process of "transition to turbulence".

A hierarchy of mathematical approaches will be discussed to address a range of "stability" problems, from more traditional concepts of "linear" infinitesimal normal mode perturbation energy growth on laminar parallel shear flows to transient, inherently nonlinear perturbation growth of general measures of perturbation magnitude over finite time horizons where flow geometry and/or fluid properties play a dominant role. The course will also discuss in detail physical interpretations of the various flow instabilities considered, as well as the industrial and environmental application of the results of the presented mathematical analyses.

Pre-requisites

Undergraduate fluid mechanics, linear algebra, complex analysis and asymptotic methods.

Literature

1. F. Charru *Hydrodynamic Instabilities* CUP 2011.
2. P. G. Drazin & W. H. Reid *Hydrodynamic Stability* 2nd edition. CUP 2004.
3. P. J. Schmid & D. S. Henningson, *Stability and transition in shear flows.* Springer, 2001.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Perturbation Methods (M16)

S.J. Cowley

This course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

A number of some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. The advantage of uniformly valid expansions for comparison with experiment and numerical solutions will be covered. [7]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. Further examples will be given of asymptotics beyond all orders. This section will include a brief introduction to the summation of [divergent] series, e.g. covering Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, and Domb-Sykes plots. [6]
- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKB[JLG]' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium). [3]

Pre-requisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward differential equations and partial differential equations and evaluate simple integrals.

Literature

Relevant Textbooks

1. Bender, C.M. & Orszag, S., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to 'Stokes' lines as 'anti-Stokes' lines, and vice versa. The course will use Stokes' convention.*
2. Hinch, E.J., *Perturbation Methods*, Cambridge University Press (1991). *This is the book of the course; some view it as somewhat terse.*
3. Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975). *This is the original book on perturbation methods; somewhat dated, but still a useful read.*

Reading to Complement Course Material

1. Berry, M.V., *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).
2. Boyd, J.P., *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, **56**, 1-98 (1999). Also available at

   `http://hdl.handle.net/2027.42/41670` and
   `http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf`.
3. Kevorkian, J. & Cole, J.D., *Perturbation Methods in Applied Mathematics*, Springer (1981).

Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.

Fluid Dynamics of the Environment (M24)

S.B. Dalziel

Understanding the environment and predicting the impact of human activity on it are critical challenges in our time. Whether we are concerned about climate change, pollution or thermal comfort within our buildings, the fluid dynamics of bodies of water (rivers, lakes and oceans) and the atmospheres play a vital role. This course introduces the basic fluid dynamics necessary to build mathematical models of the environment in which we live, and focuses on problems which occur over sufficiently small time and length scales to be largely unaffected by the earth's rotation.

The course begins by considering the fluid flow in the presence of (typically small) density variations. If the fluid is stably stratified, 'internal gravity waves' can occur since the stratification provides a restoring force when fluid parcels are displaced vertically. The course highlights some of the rich and surprising dynamics of these waves. For example, internal gravity waves radiate energy vertically as well as horizontally, and their interaction with boundaries can focus this energy and cause mixing far from where the energy was input.

Density variations within fluids can also drive the flow and the course will consider two important and related classes where the flow is either long and shallow or tall and thin. Both classes allow substantial simplification of the governing equations by integrating them over the smaller dimension. In the first, when there are lateral gradients in fluid density interacting with horizontal or sloping boundaries, turbulent 'density' or 'gravity' currents can develop. For the second, a relatively localised source can drive the rise of a turbulent 'plume' of buoyant fluid. Volcanic eruption clouds and accidental releases of pollution are just two examples of such plumes. Turbulence, entrainment and mixing play an important part here, and key aspect of stratified turbulence will be discussed. Additionally, for both classes, the buoyancy driving these flows may be due to differences in temperature or composition (e.g. salt or water vapour concentration), or due to the presence of a second phase such as particles or bubbles. Examples of particle-laden flows include snow avalanches, turbidity currents and pyroclastic flows, where particle deposition and/or resuspension can play a role.

Pre-requisites

Undergraduate fluid dynamics is desirable.

Literature

Reading to complement course material

1. B. R. Sutherland, Internal gravity waves, Cambridge University Press (2010).
2. J. S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).
3. J. Pedlosky, Geophysical Fluid Dynamics, Springer (1987).

Additional support

In addition to the lectures, four examples sheets will be provided and four associated examples classes will run in parallel to the course. There will be a revision class in the Easter Term.

Computation Methods in Fluid Mechanics (M16)

Non-Examinable (Graduate Level)

John Hinch

The aim of this Graduate course is to provide an overview of some of the computational methods used to solve the partial differential equations that arise in fluid dynamics and related fields. The idea is to provide a feel for the computational methods rather than study them in depth. Although the course is non-examinable, project-type essays will be set on some of the material.

The course will start with a four-lecture introduction to the numerical solution of the Navier-Stokes equations at moderate Reynolds number; the issues and difficulties will be highlighted.

Next some general issues will be covered in greater detail.

- Discretisations: finite difference, finite element and spectral.
- Time-stepping: explicit, implicit, multi-step, splitting, symplectic.
- Solution of Linear Systems: packages, LU and QR decompositions, sparse matrices, conjugate gradients, eigenproblems.

The remaining lectures will focus on specific issues selected from the following.

- Demonstration of the software package FreeFem++.
- Methods for hyperbolic equations.
- Representation of surfaces.
- Boundary Integral/Element Method.
- Fast Poisson Solvers: Multigrid, Fast Fourier, Domain Decomposition.
- Fast Multipole Method.
- Nonlinear considerations.
- Wavelets.
- Particle Methods.

Desirable Previous Knowledge

Attendance at an introductory course in Numerical Analysis that has covered (at an elementary level) the solution of ordinary differential equations and linear systems will be assumed. Some familiarity with the Navier-Stokes equations and basic fluid phenomena will be helpful (as covered by a first course in Fluid Dynamics).

Additional support

MatLab code will be provided for the first four lectures.

Fluid dynamics of the solid Earth (L24)

Jerome A. Neufeld & M. Grae Worster

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth's core is formed by the solidification of a mixture of molten iron and various lighter elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth's magnetic field. At very much longer time scales, radiogenic heating of the solid mantle drives solid-state convection resulting in plume-like features possibly responsible for features such as the Hawaiian sea mounts. Nearer the surface, convection drives the motion of brittle plates which are responsible for the Earth's topography as can be felt and imaged through the seismic record. Upwelling mantle material also drives partial melting of mantle rocks resulting in compaction, and ultimately in the propagation of viscous melt through the elastic crust. On the Earth's surface, and at very much faster rates, the same physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of sea ice to the flow of glacial ice.

This course will use the wealth of observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials.

Pre-requisites

A basic understanding of viscous fluid dynamics. Mathematical methods, particularly the solution of ordinary and partial differential equations.

Literature

1. M.G. Worster. *Solidification of Fluids.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
2. H.E. Huppert. *Geological fluid mechanics.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
3. D.L. Turcotte, G. Schubert. Geodynamics, second edition. CUP (2002)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Fluid Dynamics of Climate (L24)

P.H. Haynes & J.R. Taylor

Understanding the Earth's climate and predicting its future evolution is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the climate system, helping to make the planet hospitable for life. The dynamical complexity of this fluid motion and the wide range of space and time scales involved is one of the most difficult aspects of climate prediction.

This course, focusing on the large-scale behaviour of stratified and rotating flows, provides an introduction to the fluid dynamics necessary to build mathematical models of the climate system. The course begins by considering flows which evolve on a timescale which is long compared with a day, where the Earth's rotation plays an important role. The rotation is felt through the Coriolis force (a fictitious force arising from use of a frame of reference rotating with the Earth) which causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics, particularly in combination with stable density stratification. Canonical models are introduced and studied to illustrate phenomena such as adjustment to a state of geostrophic balance, where Coriolis force balances pressure gradient, new wave modes that can communicate dynamical information on both regional and global scales, and new hydrodynamic instabilities that lead to atmospheric weather systems and ocean eddies.

The course then moves on apply these basic ideas to important aspects of the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the structure and hence the effects of eddies and weather systems, the dynamics of ocean gyres and boundary currents like the Gulf Stream, the dynamics of the meridional (north/south) circulation in the ocean and atmosphere and the associated transport of heat and of chemical and biological tracers and special dynamics of tropical regions which give rise to phenomena such as El Nino.

Desirable Previous Knowledge

Undergraduate fluid dynamics

Reading to complement course material

1. Vallis, G.K. Atmospheric and Oceanic Fluid Dynamics (2nd edition). Cambridge University Press. (2017).
2. Gill, A.E., Atmosphere-Ocean Dynamics. Academic Press (1982).
3. Marshall, J. and R.A. Plumb. Atmosphere, Ocean, and Climate Dynamics. Academic Press. (2008).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Direct and Inverse Scattering of Waves (L16)

Orsola Rath Spivack

The study of wave scattering is concerned with how the propagation of waves is affected by objects, and has a variety of applications in many fields, from environmental science to seismology, medicine, telecommunications, materials science, military applications, and many others. If we know the nature of the objects and we want to find how an incident wave is scattered, we call this a 'direct scattering problem' and practical applications will include for example underwater sound propagation, light transmission through the atmosphere, or the effect of noise in built-up areas. If we measure and know the scattered field produced by an incident wave, but we do not know the nature of the objects that have scattered it, we call this an 'inverse scattering problem' and applications will include for example non-destructive testing of materials, remote sensing with radar or lidar, or medical imaging.

This course will provide the basic theory of wave propagation and scattering and an overview of the main mathematical methods and approximations, with particular emphasis on inhomogeneous and random media, and on the regularisation of inverse scattering problems. Only time-harmonic waves will be normally considered.

Topics covered will include:

1. Boundary value problems and the integral form of the wave equation.
2. The parabolic equation and Born and Rytov approximations for the scattering problem.
3. Scattering by randomly rough surfaces and propagation in inhomogeneous media.
4. Ill-posedness of the inverse scattering problem, and the Moore-Penrose generalised inverse.

5. Regularisation methods and methods for solving some inverse scattering problems.
6. Time reversal and focusing in inhomogeneous media.

Pre-requisites

This course assumes basic knowledge of PDEs, and of Fourier transforms. Some familiarity with linear algebra and with basic concepts in functional analysis is helpful, though by no means necessary.

The Part III course Inverse Problems in Imaging is a useful complement to part of this course, although not at all a prerequisite.

Preliminary Reading

1. C.W. Groetsch *Inverse Problems in the Mathematical Sciences.* Braunschweig 1993
2. L.D. Landau and E.M. Lifschitz *Fluid Dynamics.* Pergamon 1987 [Chapter 8]. Also available at

   `users-phys.au.dk/srf/hydro/Landau+Lifschitz.pdf`

Literature

1. D. Colton and R. Kress *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, 1992.
2. D.G. Crighton et al, *Modern Methods in Analytical Acoustics.* Springer, 1992.
3. H.W. Engl, M. Hanke and A. Neubauer, *Regularization of inverse problems.* Kluwer, 2000.
4. A. Ishimaru, *Wave Propagation and Scattering in Random Media.* Academic Press, 1978.
5. A. Kirsch, *An introduction to the mathematical theory of inverse problems.* Springer, 1996.
6. B. Uscinski, *The elements of wave propagation in random media.* McGraw-Hill, 1977.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Theoretical Physics of Soft Condensed Matter (L16)

Mike Cates

Soft Condensed Matter refers to liquid crystals, emulsions, molten polymers and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, generally with noise so that the equations of motion are stochastic PDEs. The form of these equations is helpfully constrained by the requirement that the Boltzmann distribution is regained in the steady state (when this indeed holds, i.e. for systems in contact with a heat bath but not subject to forcing). Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a 'smectic liquid crystal' (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field (and several steps short of the renormalization group, whose application to classical field theories is discussed in other courses but not this one).

Important models of soft matter include diffusive ϕ^4 field theory ('Model B'), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates 'Model H', a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth. We will also explore the continuum theory of nematic liquid crystals, which spontaneously break rotational but not translational symmetry, focusing on topological defects and their associated mathematical structure such as homotopy classes.

Finally, the course will cover some recent extensions of the same general approach to systems whose microscopic dynamics does not have time-reversal symmetry, such as self-propelled colloidal swimmers.

Pre-requisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements in part the following Michaelmas Term courses although none are prerequisites: Statistical Field Theory; Active Biological Fluids; Slow Viscous Flow; Quantum Field Theory.

In previous years the audience has included a mix of students whose main specialism is either fluid dynamics or field theory. People with these differing backgrounds may find different parts of the course easier or harder, but the intention is to create a roughly level playing field.

Preliminary Reading

1. D. Tong *Lectures on Statistical Physics*

   `http://www.damtp.cam.ac.uk/user/tong/statphys.html`

   Before embarking on this course you do need to understand the equation $F = -k_B T \ln Z$ and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Familiarity with the Landau theory of phase transitions is desirable. We will not need much abstract thermodynamics (e.g. Maxwell relations) but you do need to know the zeroth, first and second laws. The above lecture notes are an excellent resource for revising and reviewing the key material.

2. M. E. Cates and E. Tjhung *Theories of binary fluid mixtures: from phase-separation kinetics to active emulsions.* J. Fluid Mech. (2018), **836**, pp1-66.

   `https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/theories-of-binary-fluid-mixtures-from-phaseseparation-kinetics-to-active-emulsions/5BD133CB20D89F47E724D77C296FEF80/share/106fd30f307db12134745de39fd568fbbaa3f9d2`

   This JFM perspectives article has significant overlap (perhaps 50%) with the course but takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics when needed. It gives a good flavour of the types of problem we will address and some of the methodologies involved.

Literature

I am not aware of any books that treat this material at the right level. But it may be worth looking at:

1. P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics.* Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, topological defects all feature in the description of condensed matter systems at $\hbar = 0$. More for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will also be a two-hour revision class in the Easter Term.

Active Biological Fluids (L16)

Prof Eric Lauga

Fluid mechanics plays a crucial role in a number of biological processes, from the largest of animals to the smallest of cells. In this course, we will give an overview of the hydrodynamic phenomena associated with biological life at the cellular scale, by with a focus on the fluid mechanics of individual microorganisms and their appendages. We will combine physical description, scaling analysis, and detailed calculations in order to present a wide overview of the subject. Drawing examples from a variety of organisms, we will aim at providing a precise mathematical description of how cells actuate and exploit surrounding fluids in order to self-propel, how they interact with their mechanical environment, and how populations of cells dynamically influence each other. At the end of the course, students will be equipped to carry out independent research in biological physics and fluid dynamics relevant to the cellular world.

Pre-requisites

Undergraduate fluid dynamics, vector calculus and mathematical methods. Attendance to Part III “Slow Viscous Flows” is useful.

Literature

1. Lighthill (1975) Mathematical Biofluiddynamics, SIAM.
2. Purcell (1977) Life at low Reynolds number. *American Journal of Physics*, **45**, 3-11.
3. Childress (1981) Mechanics of Flying and Swimming, Cambridge University Press.
4. Yates (1986) How microorganisms move through water. *American Scientist* **74**, 358-365.
5. Vogel (1996) Life in Moving Fluids, Princeton University Press.
6. Berg (2000) Motile Behavior of Bacteria. *Physics Today*, **53**, 24.
7. Bray (2000) Cell Movements, Garland.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Demonstrations in Fluid Mechanics. (L8)

Non-Examinable (Part III Level)

Prof. S.B. Dalziel, Dr. J.A. Neufeld

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- porous media and carbon sequestration;
- fluid flow and elastic deformation;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

Pre-requisites

Undergraduate Fluid Dynamics.

Literature

1. M. Van Dyke. An Album of Fluid Motion. Parabolic Press.
2. G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.
3. M. Samimy, K. Breuer, P. Steen, & L. G. Leal. A Gallery of Fluid Motion. CUP.