Measure Theory Catch-up Lecture: Exercises and Solutions.

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1 What is a Measure Space

Here are some hopefully straightforward exercises:

1. Prove that if $(A_n, n \in \mathbb{N}) \subset \mathcal{E} \Rightarrow \bigcap_n A_n \in \mathcal{E}$.

- Its easiest to think about $A \cap B$ first. we can see that $\left(\bigcap_n A_n\right)^c = \bigcup_n A_n$.

2. Prove that if E is a countable set then $\mathcal{P}E$ is a σ-algebra.

-All the points are clearly satisfied. In fact we don't need that E is countable but that does mean the σ-algebra is generated by the point sets.

3. Is it always the case that if all the A_n are in $\mathcal{E}$ then $\mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n)$.

- Define a new sequence by $B_n = A_n - \left(\left(\bigcup_{k=1}^{n-1} A_k\right) \cap A_n\right)$. Then the B_n are pairwise disjoint and

$$\mu(A_n) = \mu(B_n) + \mu\left(\left(\bigcup_{k=1}^{n-1} A_k\right) \cap A_n\right) \Rightarrow \mu(B_n) \leq \mu(A_n).$$

Then we have

$$\mu\left(\bigcup_n A_n\right) = \mu\left(\bigcup_n B_n\right) = \sum_n \mu(B_n) \leq \sum_n \mu(A_n).$$

4. If A_n are all in $\mathcal{E}$ and $\mu\left(\bigcap_n A_n\right) = 0$ is it necessarilly the case that $\bigcap_n A_n = \emptyset$.

- No, take $A_n = (-1/n, 1/n)$.

5. If $\mathcal{E}$ is both a π-system and a d-system prove that it is a σ-algebra.

- The first two conditions are easy. Since $(A \cap B)^c = A^c \cup B^c$ we have that $A, B \in \mathcal{E} \Rightarrow A \cup B \in \mathcal{E}$. Then if we take a sequence A_n and define a new sequence $B_n = \bigcup_{k=1}^n A_n$ then we know from above that $B_n \in \mathcal{E}$ and it is an increasing sequence. We have that $\bigcup_n B_n = \bigcup_n A_n$ and this is in $\mathcal{E}$.

6. Speculate on how Lebesgue measure is defined in higher dimensions.

- $leb((a_1, b_1] \times (a_2, b_2] \times ... \times (a_n, b_n]) = (b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n)$. Then we extend this to the rest of the Borel σ-algebra.

2 Functions

Exercises on functions:

1. Prove that it is only necessary to check the measurablity criterion on a π-system generating the σ-algebra.

-Again its not really necessary that this is a π-system it just has to generate the σ-algebra. It almost always is a π-system in practice. We just need to note that $f^{-1}(A^c) = f^{-1}(A)^c$ and $f^{-1}(\bigcup_n A_n) = \bigcup_n f^{-1}(A_n)$.

2. Prove that a continuous function between two topological spaces with their Borel σ-algebras is measurable.

-This follows from the previous question as the open sets generate a Borel σ-algebra.

3. Prove that if f, g are measureable functions into $\mathbb{R}$ with its Borel σ-algebra then fg and $f+g$ are also measurable.

-want to look at something like $(fg)^{-1}((-\infty, a))$ depending on whether a is positive or negative we would have sets of the form (suppose negative)

$$\bigcup_{\alpha,\beta\in\mathbb{Q},\alpha<0,\beta>0} f^{-1}((-\infty,\alpha))\cap g^{-1}((\beta,\infty)).$$

This is a countable union of measurable sets so measurable. The case where a is positive is similar. For $f+g$ a similar strategy works but it is easier.

4. Prove that if f_n are all measurable functions into $\mathbb{R}$ with its Borel σ-algebra then $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$ and $\limsup_n f_n$ are all measurable.

-We can reduce to the case of just looking at inf by replacing f_n with $-f_n$. Then

$$(\inf_n f_n)^{-1}((a,\infty)) = \bigcup_n f_n^{-1}((a,\infty)), \quad (\liminf_n f_n^{-1})((a,\infty)) = \bigcap_n \bigcup_{m\geq n} f_m^{-1}((a,\infty)).$$

5. Try and come up with sequences that converge either in measure or a.s but not in both.

-For something that converges a.s. but not in measure take a gliding peak like $1_{n,n+1}$. For something that converges is measure but not almost everywhere take the function that is

$$\sum_{k=1}^{n} 1_{(k/n-1/n^2,k/n+1/n^2)}.$$

3 Integration

1. Find a sequence of measurable functions f_n such that f_n converges to some function f with $\mu(|f|)<\infty$, but $\mu(f_n)$ doesn't converge to $\mu(f)$.

-Again here can use a gliding peak.

2. The sequence

$$f_n(x) = \sum_{k=1}^{2^n} \sqrt{\frac{k}{2^n}} 1_{[(k-1)/2^n,k/2^n)}$$

converges to $f(x)$. Prove that $\mu(f_n)$ converges to $\mu(f)$ where μ is Lebesgue measure on $[0,1]$.

- It is fairly clear that $f_n(x)\to\sqrt{x}$ pointwise. Using this it is easy to bound the difference between $\mu(f_n)$ and $\mu(f)$. It is helpful to remember that

$$\sqrt{\frac{k+1}{2^n}} - \sqrt{\frac{k}{2^n}} \leq \frac{1}{2^{n/1}}.$$

3. Check that the π-system defined in the making of the product measure space is in fact a π-system.

-This should be straightforward.

4. Find a function which is in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$ and vice versa.

- $f(x)=\frac{1}{x}1_{x>1}$ is in L^2 but not L^1. Similarly $g(x)=\frac{1}{\sqrt{x}}1_{x\in(0,1)}$ is in L^1 but not L^2.

4 Inequalities

1. Show if X is a Normal random variable with mean 0 and variance 1. Show that

$$\mathbb{P}(X\geq\alpha)\leq\frac{1}{2\alpha^2}.$$

$$-\mathbb{P}(X \geq \alpha) = \frac{1}{2}\mathbb{P}(X^2 \geq \alpha^2) \leq \frac{\mathbb{E}(X^2)}{2\alpha^2} = \frac{1}{2\alpha^2}.$$

2. Show that if $f, g \in L^2(\mu)$ then

$$|\int f(x)g(x)\mu(\mathrm{d}x)| \leq \frac{1}{2}\int |f(x)|^2\mu(\mathrm{d}x) + \frac{1}{2}\int |g(x)|^2\mu(\mathrm{d}x).$$

-Use Cauchy-Shwartz and then Young's inequality.

3. Show that if $f \in L^1 \cap L^2$ then

$$\int_0^T f(x)^2\mathrm{d}x \leq \frac{1}{T}\left(\int_0^T f(x)\mathrm{d}x\right)^2.$$

- μ_T given by the average integral from 0 to T is a measure with mass 1 so then you can use this and Jensen's inequality.