Measure Theory Catch-up Lecture: Exercises.

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October 5, 2015

1 What is a Measure Space

Here are some hopefully straightforward exercises:

1. Prove that if $(A_n, n \in \mathbb{N}) \subset \mathcal{E} \Rightarrow \bigcap_n A_n \in \mathcal{E}$.
2. Prove that if E is a countable set then $\mathcal{P}E$ is a σ-algebra.
3. Is it always the case that if all the A_n are in $\mathcal{E}$ then $\mu\left(\bigcup_n A_n\right) \leq \sum_n \mu(A_n)$.
4. If A_n are all in $\mathcal{E}$ and $\mu\left(\bigcap_n A_n\right) = 0$ is it necessarilly the case that $\bigcap_n A_n = \emptyset$.
5. If $\mathcal{E}$ is both a π-system and a d-system prove that it is a σ-algebra.
6. Speculate on how Lebesgue measure is defined in higher dimensions.

2 Functions

Exercises on functions:

1. Prove that it is only necessary to check the measurablity criterion on a π-system generating the σ-algebra.
2. Prove that a continuous function between two topological spaces with their Borel σ-algebras is measurable.
3. Prove that if f, g are measureable functions into $\mathbb{R}$ with its Borel σ-algebra then fg and $f+g$ are also measurable.
4. Prove that if f_n are all measurable functions into $\mathbb{R}$ with its Borel σ-algebra then $\inf_n f_n$, $\sup_n f_n$, $\liminf_n f_n$ and $\limsup_n f_n$ are all measurable.
5. Try and come up with sequences that converge either in measure or probability but not in both.

3 Integration

1. Find a sequence of measurable functions f_n such that f_n converges to some function f with $\mu(|f|) < \infty$, but $\mu(f_n)$ doesn't converge to $\mu(f)$.
2. The sequence

$$f_n(x) = \sum_{k=1}^{2^n} \sqrt{\frac{k}{2^b}} 1_{[(k-1)/2^n, k/2^n)}$$

converges to $f(x)$. Prove that $\mu(f_n)$ converges to $\mu(f)$ where μ is Lebesgue measure on $[0, 1]$.

3. Check that the π-system defined in the making of the product measure space is in fact a π-system.
4. Find a function which is in $L^2(\mathbb{R})$ but not in $L^1(\mathbb{R})$ and vice versa.

4 Inequalities

1. Show if X is a Normal random variable with mean 0 and variance 1. Show that

$$\mathbb{P}(X \leq \alpha) \geq \frac{1}{2\alpha^2}.$$

2. Show that if $f, g \in L^2(\mu)$ then

$$|\int f(x)g(x)\mu(\mathrm{d}x)| \leq \frac{1}{2}\int |f(x)|^2\mu(\mathrm{d}x) + \frac{1}{2}\int |g(x)|^2\mu(\mathrm{d}x).$$

3. Show that if $f \in L^1 \cap L^2$ then

$$\int_0^T f(x)^2 \mathrm{d}x \leq \frac{1}{T}\left(\int_0^T f(x)\mathrm{d}x\right)^2.$$