Index notation

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1 Relevant courses

The relevant Cambridge undergraduate courses are IA Vectors and Matrices and IA Vector Calculus.

2 Books

- K. F. Riley, M. P. Hobson and S. J. Bence *Mathematical Methods for Physics and Engineering.* Cambridge University Press 2002.

3 Notes

Index notation and the summation convention are very useful shorthands for writing otherwise long vector equations. Whenever a quantity is summed over an index which appears exactly twice in each term in the sum, we leave out the summation sign.

Simple example: The vector $\boldsymbol{x} = (x_1, x_2, x_3)$ can be written as

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 = \sum_{i=1}^{3} x_i\boldsymbol{e}_i.$$

Under the summation convention, we simply write this as

$$\boldsymbol{x} = x_i\boldsymbol{e}_i.$$

Most vector, matrix and tensor expressions that occur in practice can be written very succinctly using this notation:

- Dot products: $\boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i$
- Cross products: $(\boldsymbol{u} \times \boldsymbol{v})_i = \epsilon_{ijk} u_j v_k$ (see below)
- Matrix multiplication: $(\boldsymbol{A} \cdot \boldsymbol{v})_i = A_{ij} v_j$
- Trace of a matrix: $\mathrm{tr}(\boldsymbol{A}) = A_{ii}$
- Tensor contraction: $\Delta = 2\mu\boldsymbol{e} : \boldsymbol{e} = 2\mu e_{ij} e_{ij}$
- Divergence: $\boldsymbol{\nabla} \cdot \boldsymbol{u} = \dfrac{\partial u_i}{\partial x_i}$
- Laplacian: $\boldsymbol{\nabla}^2 f = \dfrac{\partial^2 f}{\partial x_i^2}$ (but beware—the vector Laplacian $\boldsymbol{\nabla}^2\boldsymbol{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A})$ is given by $\dfrac{\partial^2 A_j}{\partial x_i^2}$ only when we work in Cartesian coordinates)

An index that appears exactly twice in a term is implicitly summed over; such an index is called a *dummy index.* The letter used for a dummy index is not important. An index that appears only once is called a *free index.*

No index may appear three times or more in an expression. For example, the expression $u_i v_i w_i$ is illegal in summation convention. If you do need the expression $\sum_{i=1}^{3} u_i v_i w_i$, write out the summation sign. Likewise, you can write 'no sum' after an expression such as $u_i v_i$ to indicate that i should be understood as a free index. These situations are fairly rare.

There are two special symbols in summation convention. The *Kronecker delta* δ_{ij} is defined by

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \tag{1}$$

The identity matrix is therefore $(\boldsymbol{I})_{ij} = \delta_{ij}$. The *Levi-Civita symbol* ϵ_{ijk} is defined by

$$\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = +1 \tag{2}$$

$$\epsilon_{132} = \epsilon_{213} = \epsilon_{321} = -1 \tag{3}$$

$$\epsilon_{ijk} = 0 \text{ otherwise.} \tag{4}$$

That is, ϵ_{ijk} is given by the parity of the permutation $(1, 2, 3) \to (i, j, k)$.

It can be shown that

$$\epsilon_{ijk} \epsilon ilm = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}. \tag{5}$$

The proof is long and tedious, but simply involves writing out all the terms and collecting them together carefully.

4 Exercises

Show that the above shorthands do give the expressions that they claim to.

What is the norm-squared of a vector, $|\boldsymbol{u}|^2$, in index notation?

What is the curl of a vector field, $\boldsymbol{\nabla} \times \boldsymbol{F}$, in index notation?

Show that the divergence theorem can be written as

$$\iint_{\partial V} F_j n_j \, \mathrm{d}S = \iiint_V \frac{\partial F_j}{\partial x_j} \, \mathrm{d}V.$$

How can Stokes' theorem be written?

Use the identity (5) to show that

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}.$$

Show that the advective derivative can be written as

$$(\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla} \left(\frac{1}{2} \boldsymbol{u}^2 \right) - \boldsymbol{u} \times (\boldsymbol{\nabla} \times \boldsymbol{u}).$$