Catch-up workshop for Part III General Relativity
Special relativity

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I. INTRODUCTION

This sheet contains a few exercises to refresh your memory on special relativity. This is the only major pre-requisite for the GR course as stated in the 'Guide to Courses'. I have tried to include exercises for everyone with different levels of difficulty. In general, the exercises should get progressively harder. Do not worry if you do not make it through the whole sheet.

While I have tried to make these sheets self-contained, please ask immediately if anything is unclear during the workshop.

Should you want to do some more reading, I strongly recommend David Tong's lecture notes on special relativity: `http://www.damtp.cam.ac.uk/user/tong/relativity/seven.pdf`. As a matter of fact this exercise sheet is strongly based on his notes.

Some of the questions are inspired by questions on the examples sheet of the Cambridge courses on Dynamics & Relativity and Electrodynamics. I have made a note if a question was stolen in full from one of the sheets. Should you feel the need to do more questions, you can find these sheets online at: `http://www.damtp.cam.ac.uk/user/examples/`.

On this sheet, Greek indices will range from 0 to $D-1$. Latin indices from the middle of the alphabet starting with i will run from 1 to $D-1$. We use the metric convention $-++++...$. Naturally, we use $c=1$.

II. SHORTER EXERCISES

1. Proper time τ is defined by $-d\tau^2 = -dt^2 + d\vec{x}^{\,2}$. Show from this definition that $\frac{dt}{d\tau} = \gamma$, where $\gamma = 1/\sqrt{1-v^2}$.
2. Show that the four velocity of a massive particle, $U^\mu \equiv \frac{dx^\mu}{d\tau}$, is $\gamma(1, v^i)$.
3. Show that $U \cdot U = -1$ in all frames.
4. The 4-momentum is defined as $P^\mu = mU^\mu$, where m is the particle's rest mass. By expanding γ for small v show that P^0 gives the classical formula for the kinetic energy and Einstein's famous formula $E_{\text{rest}} = m$.
5. Use the invariance of the inner product $P \cdot P$ to show that $E^2 = |\vec{p}|^2 + m^2$, where $E = P^0$ and $p^i = m\gamma v^i$, holds in all reference frames.
6. The 4-acceleration is $A^\mu = \frac{dU^\mu}{d\tau}$. Show that $U \cdot A = 0$.

III. LONGER EXCERCISES

7. Alice is on a head-on collision course with Bob. Their velocities are u and v respectively. By considering the invariant $U \cdot V$ derive the law of addition of velocities, i.e. find an expression for the velocity of Bob relative to Alice.

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8. A Lorentz boost along the x-direction is given by:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} . \tag{1}$$

Define the rapidity ϕ through $\cosh\phi = \gamma$. Find an expression for $\Lambda[\phi]$ and show that it satisfies $\Lambda^{\mathrm{T}}\eta\Lambda = \eta$. Explain how this fits together with $U \cdot V$ being Lorentz-invariant for two 4-vectors U and V. Show that rapidities add, i.e. $\Lambda[\phi_1]\Lambda[\phi_2] = \Lambda[\phi_1 + \phi_2]$.

9. To find how electric and magnetic fields transform under Lorentz transformations one has to consider the transformation of the electromagnetic tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} , \tag{2}$$

which transforms as $F'^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}F^{\rho\sigma}$.

For constant electric and magnetic fields, E and B, show that if $E \cdot B = 0$ and $E^2 B^2 \neq 0$ then there exist frames of reference where either E or B are zero, but not both. (It suffices to take just E_y and B_z non zero and consider Lorentz transformations along the x-direction (given in equation (1)) with speed $v < 1$.)
[Electrodynamics, sheet 1, Q5]