Catch-up workshop for Part III General Relativity
Index gymnastics

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I. INTRODUCTION

At several points in the GR course you will have to do a lot of index manipulations. While the basics of this will be introduced briefly in the lectures, it is strongly recommended that you make sure you are comfortable with the index gymnastics on this sheet before the lectures start.

On this sheet, Greek indices will range from 0 to $D-1$ and Latin indices from the middle of the alphabet starting with i will run from 1 to $D-1$.

As is customary, we denote symmetrization and antisymmetrization of a tensor $T_{\alpha\beta\ldots\gamma}$ by $T_{(\alpha\beta\ldots\gamma)}$ and $T_{[\alpha\beta\ldots\gamma]}$ respectively. Furthermore, we will denote partial derivatives of $T_{\alpha\beta\ldots\gamma}$ by $\partial_\alpha T_{\beta\gamma\ldots\delta} = T_{\beta\gamma\ldots\delta,\alpha}$.

I have tried to include exercises for everyone with different levels of difficulty. In general, the exercises should get progressively harder as the sheet goes on. Do not worry if you do not make it through the whole sheet.

While I have tried to make these sheets self-contained, please ask immediately if anything is unclear during the workshop.

II. SHORTER EXERCISES

1. Let $g^{\mu\nu}$ denote the inverse of the metric $g_{\mu\nu}$, i.e. the tensor for which $g^{\mu\nu}g_{\nu\sigma} = \delta^\mu_\sigma$. Evaluate $g^{\mu\nu}g_{\nu\mu}$.
2. Evaluate $g^{i\mu}g_{\mu i}$.
3. Prove $\partial_\mu g^{\nu\rho} = -g^{\nu\kappa}g^{\rho\lambda}\partial_\mu g_{\lambda\kappa}$.
4. Evaluate $T^{(\alpha\beta)}\omega_{[\alpha\beta]}$.
5. Evaluate $\omega_{[\alpha\beta]}U^\alpha U^\beta$. Hence, show that $\omega_{\alpha\beta}U^\alpha U^\beta = \omega_{(\alpha\beta)}U^\alpha U^\beta$.
6. The trace of a tensor $T_{\alpha\beta}$ is defined as $g^{\alpha\beta}T_{\alpha\beta}$. The trace-free part is $T^{\mathrm{TF}}_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{D}g_{\alpha\beta}T$, where T is the trace of $T_{\alpha\beta}$. Verify that this is indeed trace-free.

III. LONGER EXERCISES

7. Show that Einstein's equations,

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{1}$$

where R is the trace of $R_{\mu\nu}$, can be written in the form

$$R_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{2}{D-2}g_{\mu\nu}\left(\Lambda - 4\pi T\right), \tag{2}$$

where T is the trace of $T_{\mu\nu}$.

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8. Given $F_{\alpha\beta} = F_{[\alpha\beta]}$, show that $F_{[\alpha\beta,\gamma]} = 0$ holds if and only if $F_{\alpha\beta,\gamma} + F_{\beta\gamma,\alpha} + F_{\gamma\alpha,\beta} = 0$. Show that in special relativity (i.e. with $g_{\mu\nu} = \eta_{\mu\nu}$) the above condition together with $\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu$ implies that the electromagnetic stress-energy tensor

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu}{}_{\rho}F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right) \tag{3}$$

satisfies $T^{\mu\nu}{}_{,\mu} = -F^{\nu}{}_{\mu}J^{\mu}$.

9. Consider the action:

$$S = \int F^{\mu\nu}F_{\mu\nu}\, d^4x. \tag{4}$$

Assuming $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, show that requiring the variation δS to vanish for infinitesimal variations δA_μ which vanish at infinity leads to $\partial_\mu F^{\mu\nu} = 0$.

IV. LONGER EXERCISES WITH SOME GR NOTATION

In this section we are going to start introducing some GR notation. If you haven't seen this notation before you can just treat it as a definition for now. The meaning of everything will become clearer in the lectures.

Firstly, we introduce the *Christoffel symbols*:

$$\Gamma^{\mu}_{\nu\rho} = \tfrac{1}{2}g^{\mu\lambda}(g_{\lambda\nu,\rho} + g_{\lambda\rho,\nu} - g_{\nu\rho,\lambda}), \tag{5}$$

and define the so-called *covariant derivative* as

$$\nabla_\alpha \phi = \partial_\alpha \phi, \tag{6}$$

$$\nabla_\alpha V^\beta = \partial_\alpha V^\beta + \Gamma^{\beta}_{\alpha\gamma}V^\gamma, \tag{7}$$

$$\nabla_\alpha T_{\beta\gamma} = \partial_\alpha T_{\beta\gamma} - \Gamma^{\delta}_{\alpha\beta}T_{\gamma\delta} - \Gamma^{\delta}_{\alpha\gamma}T_{\beta\delta}. \tag{8}$$

Furthermore, we require that the covariant derivative satisfies the Leibniz rule, e.g. $\nabla_\alpha(U^\beta V_\gamma) = (\nabla_\alpha U^\beta)V_\gamma + U^\beta(\nabla_\alpha V_\gamma)$.

10. Given $ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -(1-r^{-1})dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$, find $g_{\mu\nu}$.
11. Show $\nabla_\alpha g_{\beta\gamma} = 0$.
12. Show that $\nabla_\alpha U_\beta = \partial_\alpha U_\beta - \Gamma^{\gamma}_{\alpha\beta}U_\gamma$. (Hint: Consider ∇_α acting on the scalar $V^\beta U_\beta$.)
13. Given $U^\alpha U_\alpha = -1$ show $U^\alpha \nabla_\beta U_\alpha = 0$.
14. Show that exercise 8 can be done in general relativity if $\eta_{\mu\nu}$ is replaced by $g_{\mu\nu}$ and all partial derivative are replaced by covariant derivatives.
15. Show that $\nabla^\nu \nabla_\nu t = -g^{\lambda\nu}\Gamma^{t}_{\lambda\nu}$.