Catch-up workshop for Part III General Relativity
Euler Lagrange equations

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(Dated: October 9, 2015)

I. INTRODUCTION

The GR course assumes that you are familiar with the Euler-Lagrange equations. There is not a lot you need to know about them and this sheet will either teach you everything you need to know or refresh your memory on it. The exercises will have a GR focus. In particular, we will calculate some particle paths for important spacetimes.

Some of the exercises on this sheet assume that you are familiar with the material on the index gymnastics sheet so I recommend looking at it to make sure you are comfortable with everything. I have tried to make all questions as self-contained if possible. However, please ask immediately if anything is unclear during the workshop. I have tried to include exercises for everyone with different levels of difficulty. In general, the exercises should get progressively harder.

On this sheet, Greek indices will range from 0 to $D-1$. Latin indices from the middle of the alphabet starting with i will run from 1 to $D-1$. We use metric convention $-+++ +...$. Naturally, we use $c = G = 1$.

II. THE EULER LAGRANGE EQUATIONS

Most of modern physics can be formulated in terms of a principle of stationary action. This means that the behaviour of a physical system can be derived from extremizing an action of the form:

$$S[x^i(t)] = \int_{t_0}^{t_1} L(x^i, \dot{x}^i, t)\, dt. \tag{1}$$

In this equation the integrand L is called the *Lagrangian*. One can show (exercise 5) that S is extremized exactly when then *Euler-Lagrange equation*,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}\ , \tag{2}$$

is satisfied. Given the action of a system one can use the Euler-Lagrange equation to derive the equations of motion.

III. EXERCISES WITHOUT GR

1. Given the Lagrangian $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}x^2$. Use the Euler-Lagrange equations to find the equations of motion. What physical system could this describe?
2. One of the most crucial and most interesting facts to remember is that a symmetry of the Lagrangian implies a conserved quantity. Show that if $\frac{\partial L}{\partial x^i} = 0$ then the system has the conserved quantity

$$\frac{\partial L}{\partial \dot{x}^i}. \tag{3}$$

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Show further that if $\frac{\partial L}{\partial t} = 0$ then the quantity

$$\frac{\partial L}{\partial \dot{x}^i}\dot{x}^i - L \tag{4}$$

is conserved.

3. Calculate the conserved quantity (4) for the Lagrangian in question 1. What is the physical interpretation of this conserved quantity?

4. Given the Lagrangian $L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + M/r$ derive the equations of motion and find and interpret the conserved quantities. What physical system could this describe?

5. Consider variations of the path $x(t) = x_0(t) + \delta x(t)$ in (1) such that the endpoints remain fixed, i.e. $\delta x^i(t_0) = \delta x^i(t_1) = 0$. By expanding L to linear order and integrating by parts show that linear variations of the action vanish if and only if the Euler-Lagrange equation is satisfied.

IV. EXERCISES WITH GR

In general relativity particle paths can be obtained from the Lagrangian

$$L = -g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}, \tag{5}$$

where τ denotes proper time defined by $-d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$. You can either just accept this Lagrangian for now or read chapter 3.3 of Harvey Reall's excellent GR notes. It is important to note that in the above Lagrangian, time is now included in the spacetime coordinates x^μ and the role of t in (2) is played by proper time τ. The Euler Lagrange equations are therefore

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu}\ , \tag{6}$$

where $\dot{}$ now denots derivative with respect to τ.

6. Use the results of question 2 to show that in General Relativity the Lagrangian (5) is conserved along particle paths. By comparing to the definition of proper time argue that for massive particles $L = 1$.

7. Consider flat space with the Minkowski metric. Construct the Lagrangian (5), find the conserved quantities and, hence, show that free particles move with constant 3-velocity dx^i/dt.

8. Consider the metric $ds^2 = -r^2dt^2 + dr^2/r^2$. Show that $E \equiv r^2\dot{t}$ is conserved and that

$$\dot{r}^2 = E^2 - Lr^2. \tag{7}$$

Finding an equation like (7) is usually the fastest way of analysing the motion of particles in a spacetime. For example, show that massive particles ($L = 1$) cannot travel to arbitrarily large values of r.

9. A Schwarzschild black hole of mass M has the metric

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{8}$$

where $F(r) = 1 - 2M/r$. Find the equation of motion for θ and show that a particle with initial position and velocity in the $\theta = \pi/2$ plane will remain in said plane. Use this to argue that for a particle that is initially at rest it is sufficient to consider the Lagrangian

$$L = F(r)\dot{t}^2 - F(r)^{-1}\dot{r}^2 - r^2\dot{\phi}^2. \tag{9}$$

Show that the quantities $E \equiv F(r)\dot{t}$, $H \equiv r^2\dot{\phi}$ and L itself are conserved. By using these conserved quantities in (9) show that

$$\tfrac{1}{2}\dot{r}^2 + V(r) = \tfrac{1}{2}E^2, \tag{10}$$

where the *effective potential* $V(r)$ is to be determined. Show that circular orbits correspond to stationary points of $V(r)$ and that they are only possible for $|H| > 2\sqrt{3}M$. Show further that stable circular orbits must have $r \geq 6M$. The case $r = 6M$ is called ISCO (innermost stable circular orbit).