Fractal Geometry and Additive Combinatorics(L16)

Non-Examinable (Graduate Level)

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Fractal Geometry, roughly speaking, studies sets with complicated but interesting infinitesimal structures. In this course, we will mainly focus on fractals with self-similar structures. We shall see that many GMT(geometric measure theory) properties of such fractals boil down to subtle additive combinatorial and Diophantine properties of numbers. In the other direction, there are several number theoretic problems that can be represented as certain geometric properties of fractals. This allows us to study those number theoretic/additive combinatorial problems by understanding the geometry of fractal sets.

The course is divided into three parts. For the first part (around 3-4 lectures), I plan to cover some basics in fractal geometry including various notions of dimensions, self-similar sets, and basic geometric properties of fractal sets, for example, projections and slicings. For the second part (4-6 lectures), I will introduce several challenging open problems which are of combinatorial nature, and some basic techniques to approach them. For the third part (all the rest), I will explain a result of Bourgain on sum-product estimates of fractal sets. If time permits, we will continue going along Bourgain's route and obtain a recent result of Hochman on the Hausdorff dimension of self-similar sets.

Part of the following topics will be covered (depends on the schedule):

- Part 1 (≤ 1 lecture per topic)

1 Self-similar sets, Hausdorff dimension, box counting dimensions

2 Marstrand's projection theorem and related results

3 Slicing properties of fractals

- Part 2 (≤ 2 lectures per topic)

1 Kakeya problem, Falconer's distance problem

2 Furstenberg's slicing problem

3 Bernoulli convolutions

- Part 3

1 Bourgain's sum-product theorem for fractal sets.

2 (If we have enough time left) Hochman's inverse entropy theorem and its application to the study of self-similar sets

Pre-requisites

Basic mathematical analysis, basic geometric measure theory, basic ergodic theory. Actually, I will recall most of the definitions.

Literature

Lecture notes, once finished, will be avaiable on the lecture's webpage. The following two books cover most of the basic theory in part 1 as well as some materials in part 2.

1. K. Falconer *Fractal Geometry: Mathematical Foundations and Applications.* 2nd edition. John Wiley & Sons, 2003.

2. P. Mattila *Fourier Analysis and Hausdorff Dimension*, CUP, 2015.

For part 3, we will focus on the following two articles.

1. J. Bourgain *The discretized sum-product and projection theorems* Journal d'Analyse Mathématique (112), 193-236 (2010).

2. M. Hochman *On self-similar sets with overlaps and inverse theorems for entropy*, Annals of Mathematics (180), issue 2, 773-822, (2014).