Multiplicative functions (M16)

Non-Examinable (Graduate Level)

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The classical proof of the prime number theorem, using Cauchy's residue theorem from complex analysis, is a beautiful piece of mathematics. It is also a little troubling. How can we really claim to understand the properties of the integers under multiplication if we have to resort to such witchcraft as the residue theorem?

In recent years there has been a movement, spearheaded by Andrew Granville and Kannan Soundararajan but building on earlier work of many people, to consider an alternative approach to the subject, one which avoids the use of the residue theorem. This has involved turning the focus away from the primes themselves and focussing instead on multiplicative functions. The programme has had numerous successes, not just in reinterpreting pre-existing theorems but in proving extremely surprising new results on multiplicative functions themselves.

In this course we will try to cover the following topics from the modern theory of multiplicative functions (in greater or lesser detail, depending on time constraints):

- pretentious multiplicative functions and Halász's theorem;
- Granville–Soundararajan's improvement on the Pólya–Vinogradov inequality;
- the Matomäki–Radziwiłł theorem;
- Tao's proof of the logarithmically-averaged Chowla conjecture.

Pre-requisites

I will assume familiarity with basic undergraduate real and complex analysis, including a little harmonic analysis (essentially just the Fourier inversion formula). It is *not* a pre-requisite to have previously attended a first course in analytic number theory, but it will certainly be helpful to have done so, not least for putting the results of this course into their full context.

Literature

1. Granville, A. and Soundararajan, K., *Large character sums: pretentious characters and the Pólya–Vinogradov theorem.* Journal of the American Mathematical Society, **20**(2), 357-384 (2007).
2. Granville, A., Harper, A. and Soundararajan, K., *A more intuitive proof of a sharp version of Halász's theorem.* Proceedings of the American Mathematical Society, 146(10), pp.4099-4104 (2018).
3. Matomäki, K. and Radziwiłł, M., *Multiplicative functions in short intervals.* Annals of Mathematics 1015-1056 (2016).
4. Soundararajan, K., *The Liouville function in short intervals [after Matomäki and Radziwiłł].* Séminaire Bourbaki (2016), p.68ème.
5. Tao, T., *The logarithmically averaged Chowla and Elliott conjectures for two-point correlations.* Forum of Mathematics, Pi (Vol. 4). Cambridge University Press 2016.
6. Walker, A. *Lecture notes*, to appear, 2020