Diophantine Analysis (L24)

Non-Examinable (Graduate Level)

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This course will discuss two classical methods in Diophantine Analysis. The first one can be traced back to Thue, who proved the following result in Diophantine Approximation. Let α be an algebraic number of degree d. Then for any

$$\kappa > \frac{d}{2} + 1,$$

there is a constant c such that

$$\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^\kappa}$$

for all $r, s \in \mathbf{Z}$. This result is a significant improvement of Liouville's bound, in which the exponent is $\kappa = d$. Thue used his result to bound the number of solutions of certain Diophantine equations. The method has been subsequently improved by Siegel, Dyson and finally Roth, who achieved the optimal exponent $\kappa = 2 + \varepsilon$. Schmidt generalized Roth's result to the setting of a system of inequalities for linear forms, which is known as the Subspace Theorem.

The second method that will be discussed in the course goes back to the work of Baker. Gelfond and Schneider independently proved that α^β is transcendental whenever $\alpha \neq \{0, 1\}$ and β is an irrational algebraic number, which was Hilbert's seventh problem. This can be reformulated as

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0$$

for any non-zero algebraic numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$, provided $\log \alpha_1$ and $\log \alpha_2$ are linearly independent over the rationals. Baker generalized this result to linear forms in arbitrarily many logarithms, and, moreover, he gave lower bounds for the absolute value of such a form. These estimates have been revisited and improved on by many authors.

Both methods have been utilized by many authors for a wide range of applications in number theory and beyond. In the course, we will discuss Roth's theorem and some estimates for linear forms in logarithms, and we will sample from their applications.

Pre-requisites

Some knowledge of calculus, complex analysis, linear algebra and Galois theory will be assumed.

Literature

The course will not follow any particular source, but there are many excellent books that discuss some of the course material including the following.

1. A. Baker, *Transcendental number theory.* Cambridge University Press, Cambridge, 1990.

2. E. Bombieri and W. Gubler, *Heights in Diophantine geometry.* Cambridge University Press, Cambridge, 2006.

3. Y. Bugeaud, *Linear forms in logarithms and applications.* European Mathematical Society (EMS), Zürich, 2018.

4. J. W. S. Cassels, *An introduction to Diophantine approximation.* Hafner Publishing Co., New York, 1972.

5. D. Masser, *Auxiliary polynomials in number theory.* Cambridge University Press, Cambridge, 2016.