Analytic Number Theory (L16, Reading course)

Thomas Bloom

Analytic number theory studies the properties of integers using techniques from analysis, both real and complex. This course studies the classical techniques of analytic number theory, focusing especially on the classical theory of the Riemann zeta function. A particular highlight is a proof of the Prime Number Theorem, which gives an asymptotic formula for the number of prime numbers, and which has an intimate relationship with the distribution of zeros of the zeta function. This course develops an understanding of the distribution of these zeros in some depth, including a discussion of the consequences of the infamous Riemann Hypothesis.

Topics will include:

- An introduction to Dirichlet series and the Riemann zeta function, including the functional equation and analytic continuation;
- Perron's formula and the proof of the Prime Number Theorem;
- Quantitative zero-free regions for the zeta function;
- An explicit formula for the prime counting function in terms of zeros of the zeta function;
- The distribution of the zeros of the zeta function in the critical strip, in particular the asymptotic formulas for the numbers of zeros up to a given height.
- The Riemann Hypothesis and its consequences, both on the properties of the zeta function and the error term in the prime number theorem.

Pre-requisites

A good understanding of undergraduate complex analysis will be essential. The graduate Part III course 'Multiplicative Functions' in Michaelmas Term offers an alternative, more modern, perspective on analytic number theory, and would complement this reading course well, but is not necessary.

Literature

1. T. F. Bloom, *Analytic Number Theory*, online lecture notes available at

   http://thomasbloom.org/teaching/ANT2020.pdf

   [Only Chapters 1, 2, 3, and 4 are covered in this course.]
2. H. M. Edwards, *Riemann's Zeta Function*, Dover Edition, 1974. [Only Chapters 1, 2, 3, 4, and 5 are covered in this course.]
3. S. J. Patterson, *An introduction to the theory of the Riemann Zeta-Function*, Cambridge University Press, 1988.
4. H. L. Montgomery and R. C. Vaughan, *Multiplicative Number Theory I. Classical Theory* Cambridge University Press, 2007. [Only Chapters 1, 2, 6, 10, 12, 13, and 14 are covered in this course.]
5. E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Second edition, Oxford University Press, 1986. [Only Chapters 1, 2, 3, 5, 9, 14 are covered in this course.]

Additional support

I will hold office hours three times, and a final time in Easter term. Examples sheets and complete solutions from the 2019/20 Analytic Number Theory course are available from `www.thomasbloom.org/teaching/ant2020.html`