Category Theory (M24)

Prof. P.T. Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]

Locally small categories. The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]

Limits. Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

Filtered colimits. Finitary functors, finitely-presentable objects. Applications to universal algebra.

Regular categories. Embedding theorems. Categories of relations, introduction to allegories.

Abelian categories. Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.

Monoidal categories. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

Fibrations. Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Pre-requisites

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. S. Mac Lane *Categories for the Working Mathematician.* Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.

2. S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.

3. T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.

4. E. Riehl *Category Theory in Context.* Dover Publications 2016. A new account of the subject by someone who first encountered it as a Part III student a dozen years ago.

5. F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.