The minimal surface equation and related topics (12E)

Non-Examinable (Part III Level)

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The minimal surface equation (MSE) is a quasi-linear elliptic partial differential equation satisfied locally by n-dimensional surfaces that minimize area locally in an $(n+1)$-dimensional space. More explicitly, if $\Omega \subset \mathbf{R}^n$ is a domain and $u : \Omega \to \mathbf{R}$ is a function of class C^2 having the property that for every ball $B = B_\rho(y) \subset\subset \Omega$,

$$\text{Area}(\text{graph } u|_B) = \inf_{v \in C^2(\overline{B}),\ v|_{\partial B} = u|_{\partial B}} \text{Area}(\text{graph}\, v),$$

then u satisfies the MSE: $\sum_{j=1}^n D_j \left(\frac{D_j u}{\sqrt{1+|Du|^2}} \right) = 0$ in Ω. The converse of this statement is also true. In the course of its long, rich history, the mathematical study of this equation has lead to the discovery of an array of powerful general techniques in linear and quasilinear elliptic PDEs, as well as to surprising global theorems about the MSE whose proofs are a beautiful blend of analytic and geometric ideas. The crown jewel of such theorems is the Bernstein theorem: *The only solutions to the MSE on all of* $\mathbf{R}^n$ *are affine functions whenever* $n \leq 7$*, but for* $n \geq 8$ *there are non-affine solutions!* This short lecture course will discuss the circle of ideas surrounding this equation, focusing more on the analysis/PDE side (than on the geometry side).

Pre-requisites

A PDE background such as that provided by the Part III course Analysis of PDE's. Little familiarity with geometry of submanifolds (of Euclidean space) will be helpful.

Literature

1. David Gilbarg & Neil Trudinger, *Elliptic Partial Differential Equations of Second Order.*

2. Leon Simon. *Lectures on Geometric Measure Theory.*

3. Leon Simon. *Schauder estimates by scaling.* Calc. Var. & PDE. **5**, 391–407 (1997).

4. Paul Minter's lecture notes from last year's course 'Elliptic PDEs,' available here:
   `https://minterscompactness.files.wordpress.com/2019/05/epde_notes.pdf`