An Introduction to Non Linear Analysis (M24)

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This class in an introduction to the basic analytic tools needed for the mathematical study of nonlinear problems arising from mathematical physics (non linear wave equations, fluid mechanics). A particular emphasis will be made on the classical description of fundamental non linear waves discovered in the 19th century: solitons or solitary waves. The exact role of these bubbles of energy in many systems is still mysterious and the subject of an intense research activity. The last twenty years have seen spectacular progress in the understanding of the stability properties of these objects, and their role in the description of *singularity formation mechanisms*.

We will review the main classical methods at hand for the mathematical description of these objects, and build bridges with very recent developments on singularity formation problems both for nonlinear waves and fluid mechanics problems. Lecture notes will be available online.

Syllabus

1. Basic tool box of modern analysis

- L^p spaces, Hölder, complex interpolation.
- Convolution, Young and Hardy-Littlewood-Sobolev inequality.
- Hilbert spaces: weak convergence and compact operators.
- Sobolev spaces $H^s(R^d)$: definition, Hilbertian structure.
- The Sobolev embedding and compactness.

2. A canonical model: the Non Linear Schrödinger equation

- The linear Schrödinger semi group in R^d.
- Dispersion and Strichartz estimates.
- Virial identity, scattering and blow up.
- Open problems and recent progress.

3. Solitons: an introduction to variational methods

- Phase portrait in dimension 1.
- Variational approach: existence of a ground state.
- Introduction to spectral theory: the case of the harmonic oscillator.
- Bifurcation and the Lyapounouv Schmidt theorem.
- The stability problem: the concentration compactness lemma.
- Solitons in fluid mechanics: the example of vortices.

4. Singularity formation in non linear evolution equations

- The minimal mass blow up solution for (NLS).
- Merle's classification theorem (1992).
- Self similar blow up: recent progress and perspectives.
- The blow up problem in fluid mechanics: recent progress and perspectives.

Pre-requisites

Basic notions of functional analysis (Hilbert and Banach spaces). Basis notion of distributions theory ($\mathcal{S}(R^d)$ and $\mathcal{S}'(R^d)$). Basic notion of continuous Fourier transform (Plancherel).

Literature

1. T. Cazenave : *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, **10**, NYU, CIMS, AMS 2003.
2. P. Raphaël: Concentration compacité à la Kenig-Merle, Séminaire Bourbaki, Exp. No. 1046, *Astérisque*, **352** (2013).
3. T. Tao: Nonlinear dispersive equations. Local and global analysis. CBMS Regional Conference Series in Mathematics, **106**, American Mathematical Society, Providence, RI, 2006.

Additional support

Examples sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.