Profinite Groups and Group Cohomology (L24)

Gareth Wilkes

Profinite groups are groups which are the limit of a family of finite groups in a certain sense, and arise naturally in algebraic geometry, geometric group theory and Galois theory. As such they link together several regions of mathematics, being at the same time infinite groups and complete metric spaces, and exhibit properties of the theory of both finite and infinite groups. This course will introduce profinite groups and their basic properties. We will go on to explore their relations with classical groups and certain algorithmic questions: for instance, the question of deciding whether two given infinite groups are isomorphic or not.

We will also introduce the immensely powerful and versatile theory of group cohomology. As well as exhibiting further links between group theory and topology, cohomology provides computable invariants which bring out the finer structure of a group. We introduce this theory both for classical groups and profinite groups, and discuss the similarities and differences of the two theories.

Course synopsis:

- Inverse and direct limits of groups. Definitions and basic properties of profinite groups. Gaschutz's Lemma.
- Residual finiteness properties of groups. Profinite completions. Constructing finite quotients of free groups. Distinguishing discrete groups through their finite quotients ('profinite rigidity' questions).
- Pro-p groups and the p-adic integers. Frattini subgroups. Hensel's lemma. Serre's theorem on uniqueness of topology.
- Elementary group cohomology: definitions for both discrete groups and profinite groups. Comparison of theory for discrete groups vs pro-p groups. Extensions and semi-direct products. Groups of cohomological dimension one.

Pre-requisites

Part IB Groups, Rings and Modules and Part IB Metric and Topological Spaces are essential. Concepts from Part II Algebraic Topology will be referenced, but detailed knowledge of proofs is not necessary.

Literature

Notes and example sheets will be made available on the lecturer's webpage https://www.dpmms.cam.ac.uk/~grw46/partiiiprofinite.html.

1. K. Brown, *Cohomology of Groups*. Springer 1982.
2. L. Ribes and P. Zalesskii, *Profinite Groups*. Springer 2000.
3. J-P. Serre, *Galois Cohomology*. Springer 2013.
4. J.S. Wilson, *Profinite Groups*. Clarendon Press 1998.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.