Finite dimensional Lie and associative algebras (M24)

Christopher Brookes

The main aim of this course is study Lie algebras that are finite dimensional as complex vector spaces. Lie algebras (and Lie groups) appear in many branches of mathematics and mathematical physics, the Lie algebra arising as the tangent space to the identity element in the associated Lie group. They consist of infinitesimal symmetries, a linearised approximation to the groups. One reason for their importance is that the finite dimensional complex representations of the simple Lie algebras correspond exactly to those of the groups. So instead of needing to study the topology and geometry of the simple Lie groups, or the algebraic geometry of the simple algebraic groups, we can concentrate on the Lie algebras using methods from linear algebra and still completely describe these representations.

The classification of the finite dimensional simple complex Lie algebras boils down to a study of root systems labelled by Dynkin diagrams. These root systems are fundamental combinatorial structures consisting of a collection of data accompanied by an associated Weyl group of symmetries.

Root systems and Dynkin diagrams also appear in classification problems when studying the representation theory of associative algebras. For example they arise in Gabriel's classification of the quivers (directed graphs, with multiple edges and loops allowed), and their associated path algebras, with only finitely many indecomposable complex representations. Thus root systems form the central theme of the course.

The Lie algebra material is entirely standard and can be found in many places. Serre's little book provides an excellent summary but with quite a bit of detail suppressed. I shall briefly introduce the definitions and basic structure theory. The core topics are:

Cartan subalgebras, root systems, Weyl groups, Dynkin diagrams, the finite simple Lie algebras. Classification of finite dimensional representations, Verma modules, Weyl character formula.

Similarly I shall run through the basic structure of finite dimensional associative algebras, before concentrating on the representation theory of quivers. A good source that covers all of this associative material is the book by Pierce. Core topics are:

Jacobson radical, Artin-Wedderburn theorem (classification of finite dimensional semisimple associative algebras, not necessarily complex). Quivers, path algebras and their representations. Gabriel's theorem.

Pre-requisites

Linear algebra. A first course on rings and modules (e.g. IB groups, rings and modules). Experience of some representation theory (e.g. the II course or equivalent) will be very useful but not essential.

Literature

1. K. Erdmann and M.J. Wildon, *Ingtroduction to Lie algebras.* Springer, 2006.

2. J.E. Humphries, *Introduction to Lie algebras and representation theory.* Springer, 1972.

3. N. Jacobson, *Lie algebras.* Dover, 1979.

4. R.S. Pierce, *Associative algebras.* Springer, 1982.

5. J.-P. Serre, *Complex semisimple Lie algebras* Springer, 1987.

Additional support

There will be four examples sheets, with associated examples classes, and a revision class in the Easter Term.