Faculty of Mathematics
Part III Essays: 2024-25

Titles 1 – 59

Department of Pure Mathematics
& Mathematical Statistics

Titles 60 – 110

Department of Applied Mathematics
& Theoretical Physics

Titles 111 – 138

Additional Essays

Introductory Notes

Overview

As explained in the Part III Handbook, students are expected to submit an essay (written during the year) by a deadline early in the Easter Term for 3 units of examination credit (equivalent to the credit available from a 3-hour written paper for a 24-lecture course). This Essay Booklet contains details of the approved essay titles, together with general guidelines and instructions for writing an essay. A timetable of relevant events and deadlines is included on page (v) of this document.

The essay is a key component of Part III. Experience has shown that the great majority of students find that working on their essay is an enjoyable change from learning from lectures and is valuable training for research as well as a range of other careers.

Essay Marks and Examination Credit

Each essay is awarded a numerical mark out of a maximum of 100 and a 'quality mark' (just as with written papers). Further details can be found in Appendix IV of the Part III Handbook and descriptors for the broad grade ranges of quality marks for essays are also reproduced below, in Appendix 2. The Faculty Board does not necessarily expect the mark distribution for essays to be the same as that for written examinations. Indeed, in recent years for many students their essay mark has been amongst their higher marks across all examination papers, both because of the typical amount of effort devoted to the essay and the different skill set tested (compared to a time-limited written examination). The Faculty Board wishes that hard work and talent thus exhibited should be properly rewarded.

Candidates should note that, since there is a maximum of 16 units of credit that can be obtained from examination papers for lecture courses, it is essential to submit an essay in order to achieve a total amount of examination credit in the range 17-19 units. See *Section 9 – Examinations and Assessment* in the Part III Handbook for further details.

Essay Titles

The titles of essays in this booklet have been approved by the Part III Examiners.[1] Additional titles may be approved by the Part III Examiners and will be added to this booklet not later than 1 March. Essay titles **cannot** be approved informally: the only allowed essay titles are those which appear in the final version of this document, available on the Faculty website.[2]

Requesting an Additional Title

If you wish to write an essay on a topic not covered in this booklet you should approach your Part III Subject Advisor/Departmental Contact or another member of the academic staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board (email: undergrad-office@maths.cam.ac.uk) **not later than 1 February** requesting that an essay on that topic be approved.[3] The new essay title will require the approval of the Part III Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. If you request an essay title you are under

[1] The titles are also published in the University's journal of record, the Cambridge University Reporter.
[2] All additional titles will also be published in the Cambridge University Reporter.
[3] See Regulation 17 of the Regulations for the Mathematical Tripos.

no obligation to write the corresponding essay. Once announced, an essay title is open to any student, subject to the guidance below.

Interaction with the Essay Setter

Normally candidates may consult the setter up to three times before the essay is submitted. This includes the first meeting described below, which may take the form of a group meeting at which the setter describes the essay topic and answers general questions. There is a range of practices across the Faculty for the other two meetings depending on the nature of the essay and whether, say, there is a need for further references and/or advice about technical questions. The setter may comment on an outline of the essay (for example in the second meeting), and may offer general feedback (for example, on mathematical style in general terms, or on whether clearer references to other sources are required) on a draft of the essay in the final meeting. The setter is not allowed to give students an expected grade for their essay.

Essay Title Choice

In order to ensure that individual setters are not oversubscribed and **all** students receive adequate guidance, students will be asked to nominate three preferred essay titles **by noon on Friday, 29 November 2024** via a dedicated Part III Essay Moodle. You must nominate three titles by at least two distinct essay setters and can expect to be notified of your allocated title during the following week. The allocation process will aim to assign first-preference titles to students wherever possible, subject to capacity of the essay setter. Where the number of students who name a given setter's title as their first preference exceeds the declared capacity of the setter, the selection of students up to the declared capacity will be made at random.

Before deciding on your preferred titles, you are strongly advised to meet the setters in person. For this purpose, a first meeting for each title will be arranged by the essay setter. These meetings will normally take place during the fifth and seventh week of lectures,[4] and will be advertised via the Part III Essay Moodle. Some essay setters may ask you to contact them to arrange a meeting. It is the responsibility of each student to ensure that they are aware of relevant meeting dates.

Changing Title after the Title-Choice Deadline

It is hoped that the vast majority of students will be allocated their preferred title, and that, with rare exceptions, students remain satisfied with their assigned title as they begin to work on the essay in earnest. However, some students may discover that their assigned title is not suitable for them after all. A subsequent change of title is therefore possible, subject to capacity of the relevant essay setter.

Students wishing to change title following the initial allocation must observe the following steps:

1) Consult the Part III Essay Moodle to identify essay setters with spare capacity.
2) Meet with the setters of any titles that appeal to establish suitability of the title.
3) Email partiii-essays@maths.cam.ac.uk with your current title, your desired title, with copy to **both** the setters of the current and the desired title. You should use the subject line "Part III Essay Title Change".

[4] Essay setters are discouraged from arranging essay meetings during the sixth week of lectures to avoid clashes with the Part III Progress Interviews.

Title change requests will be processed and approved on a first-come-first-served basis, using the time stamp of your email, normally within two working days.[5] Title change requests not respecting the above format will be returned.

It is not advisable to request a change of title if you have already invested significant time and effort in your essay. Title changes are possible until the end of the seventh week of lectures of the Lent term. Change requests received after this date will be noted, but the new essay setter will not be expected to provide the normal amount of guidance.

Choosing an Essay Title after the Title-Choice Deadline

Students who did not nominate three preferred titles but wish to choose a title after the deadline for essay-title choices on 29 November must observe the following steps:

1) Consult the Part III Essay Moodle to identify essay setters with spare capacity.
2) Meet with the setters of any titles that appeal to establish suitability of the title.
3) Email partiii-essays@maths.cam.ac.uk with your desired title, with copy to the setter of the desired title. You should use the subject line “Part III Essay Late Title Choice”.

Title allocation requests will be processed and approved on a first-come-first-served basis, using the time stamp of your email, normally within two working days.[5] Title allocation requests not respecting the above format will be returned.

Late title choices are possible until the end of the seventh week of lectures of the Lent term. Allocation requests received after this date will be noted, but the essay setter will not be expected to provide the normal amount of guidance.

Dropping an Essay Title after the Title-Choice Deadline

In order for all students to have fair access to one of their preferred titles, it is essential that you let us know if you are no longer interested in pursuing an allocated essay by emailing partiii-essays@maths.cam.ac.uk with your name and allocated title. You may request a new title at a later stage - see Choosing a Title after the Title-Choice Deadline above.

Content of the Essay and Originality

The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but it is not unusual for candidates to find novel approaches. All sources and references used should be carefully listed in a bibliography. Candidates are reminded that mathematical content is more important than style.

Presentation of the Essay

There is no prescribed length for the essay in the University Ordinances, but the Faculty Board Advice to the Part III Examiners suggests that 5,000-8,000 words is a normal length, and exceptionally long essays (i.e. more than twice this maximum) are strongly discouraged (see Appendix 2 in this booklet). In order to provide greater clarity, the Faculty Board now requires that all candidates use a standard LaTeX template, which will be provided. The length of an essay should normally then be between 20 and 30 pages and should only exceptionally be more than 35 pages. Any exceptions to this standard guidance will be noted

[5] The administrative offices of the University will close for a set period over the Christmas vacation. The closure dates will be clearly advertised nearer the time and you should not expect a response during this period.

below, alongside the description of the essay concerned. If you are in any doubt as to the length of your essay, please consult either the essay setter or your Part III Subject Advisor/Departmental Contact.

Academic Misconduct and Plagiarism

Before starting your essay, you **must** read both

- the University's statement on the Definition of Academic Misconduct
- the Faculty Guidelines on Plagiarism and Academic Misconduct, which are reproduced in Appendix 1 of this document.

The University takes a very serious view of academic misconduct in University examinations. The powers of the University Disciplinary Panels extend to the amendment of academic results or the temporary or permanent removal of academic awards, and the temporary or permanent exclusion from membership of the University. Fortunately, incidents of this kind are very rare.

Signed Declaration

The essay submission process includes signing the following declaration. It is important that you read and understand this before starting your essay.

> I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the *Definition of Academic Misconduct* and the *Faculty Guidelines on Plagiarism and Academic Misconduct* and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

If you are in any doubt as to whether you will be able to sign the above declaration, you should consult the member of staff who set the essay in the first instance. If the setter is unsure about your situation, you should consult the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) as soon as possible.

Viva Voce Examination

The Part III Examiners have power, at their discretion, to examine a candidate *viva voce* (i.e. to give an oral examination) on the subject of their essay, although this procedure is not often used.

Time Management

It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.

Confirming your Essay Submission

At the beginning of the Easter Term, you must state which written papers you have chosen, and confirm which essay you will be submitting for examination. At that point, you will be sent the appropriate *Paper Choice Form* to complete, to confirm both your paper choices and your essay title. Your Director of Studies must counter-sign this form, and you should then send it to the Chair of Part III Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 8 May 2025**. **This deadline must be strictly adhered to.**

Submitting your Essay

You should submit your essay via the Part III Essay Moodle **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 8 May 2025**. Alongside your essay you will need to submit a completed and signed Essay Submission Form as found on page (vi) of this document. Your Director of Studies must counter-sign this form, and you should then send it to the Chair of Part III Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 8 May 2025**. **This deadline (or any extension authorized by the University) will be strictly adhered to – see below.**

- The title page of your essay should bear **only** the essay title. Please **do not** include your name or any other personal details on the title page or anywhere else on your essay.
- Essays will need to be submitted in pdf format using the LaTeX template (no hard copies will be accepted). Students who believe they will be unable to use the LaTeX template should contact the UGO as soon as possible.

More detailed submission instructions will be provided closer to the submission date.

Extension of Submission Deadline

The essay submission deadline can be extended **only** if authorization is obtained by following the appropriate procedures set out by the University. According to a policy approved by the General Board's Education Committee for managing extensions for dissertations and coursework, candidates are now permitted to self-certify for a short extension of the essay submission deadline, up to seven days, for any reason (medical or non-medical). Candidates must request such an extension via the Faculty's Part III Essay Deadline Extension Request Form prior to the submission date, and must include evidence that they have informed their College Tutor of their request. While there is no requirement for any other supporting evidence to be provided, candidates are reminded that it is unlikely to be in their interest to spend excessive time on their essay at the expense of revision for their written papers. Candidates considering requesting an extension are strongly encouraged to discuss the matter with their College Director of Studies well in advance of the deadline. Applications for extensions beyond seven days must be made to the University's Examination Access and Mitigation Committee by the College on the student's behalf.

Consequences of Missing the Essay Submission Deadline

The submission deadline, including any authorised extension, will be strictly adhered to; candidates who miss the deadline are liable to receive no examination credit for their essay. Any candidate who thinks they may be at risk of missing the deadline, due to e.g. illness or other grave cause, should contact their College Tutor and Director of Studies as soon as possible.

Assessment of the Essay

The essay is marked by the setter, who is appointed as an Assessor for Part III, in accordance with the Essay Descriptors included as Appendix 2 in this booklet. Each essay mark is checked by a "standard checker" and may subsequently be moderated by the Part III Examiners for consistency within and across subject areas. Essay marks are expected to be released alongside the marks on written papers by the end of **Wednesday, 25 June 2025**.

Return of Essays

It is not possible to return essays to candidates. You are therefore advised to retain your own copy when submitting your essay.

Further Guidance

Advice on writing an essay is provided in two Wednesday afternoon talks listed below. Slides from these talks will subsequently be made available on the Part III Academic Support Moodle.

Feedback

If you have suggestions as to how these notes might be improved, please write to the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk).

Timetable of Relevant Events and Deadlines

Wednesday 6 November 4:15pm	Talk: *Planning your essay: reading, understanding, structuring*
Friday 29 November noon	Deadline for essay title choices
Saturday 1 February	Deadline for Candidates to request additional essays
Wednesday 12 February 4:15pm	Talk: *Writing your essay: from outline to final product*
Thursday 8 May noon	Deadline for Candidates to return form stating choice of papers and title of essay
Thursday 8 May noon	Deadline for Candidates to submit essays
Thursday 5 June	Part III Examinations expected to begin

Mathematical Tripos, Part III 2025
Essay Submission Form

To the Chair of Examiners for Part III of the Mathematical Tripos

I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the Definition of Academic Misconduct and the Faculty Guidelines on Plagiarism and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed (Candidate): .... Date: ....

Title of Essay: ………

………

Essay Number: ....

Candidate Name: ……………………………………………………………………………. College: ……………………………………………

Signed (Director of Studies): ……………………………………………………………. Date: …………………………………………….

Assessor Comments

The Assessor may provide comments on your essay which are intended to support your academic development. Any comments received by the Undergraduate Office will be sent to you by email as soon as possible following the publication of results. Please note that it is not mandatory for the Assessor to offer such comments, and that, where provided, comments do not represent a formal justification of the final mark.

If you would like to receive your comments, please provide a personal email address below (i.e. an email address other than CRSid@cam.ac.uk). Comments will not be sent to CRSid email addresses.

Email Address: ……….

Appendix 1: Faculty of Mathematics: Guidelines on Plagiarism and Academic Misconduct

For the latest version of these guidelines please see:

https://www.maths.cam.ac.uk/internal/faculty/plagiarism

University Resources

The University publishes information on Plagiarism and Academic Misconduct, including

- the University's definition of academic misconduct
- information for students, covering
    - Why does academic integrity matter?
    - Students' responsibilities
    - Collusion (including proofreading)
    - Artificial Intelligence
- information about Referencing and Study skills
- information about further online Resources and local sources of support
- Plagiarism FAQs.

There are references to the University's definition:

- in the Part IB and Part II Computational Project Manuals
- in the Part III Essay Booklet (linked from the Part III Essays page)
- in the Computational Biology Handbook (linked from the Computational Biology Course page).

Please read the University's definition of academic misconduct carefully.

The University has outlined Rules of Behaviour for both current and former registered students (Statutes and Ordinances 2022, Chapter II, Section 18; p.195). All registered students and formerly registered students are responsible for following the Rules of Behaviour.

Not knowing or forgetting about the rules or their consequences is not a justification for not following them.

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However, neither the University Statement nor the Faculty Guidelines supersede the University's Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

The scope of academic misconduct

Academic misconduct may be due to

- **plagiarism**
  this refers to using another person's language and/or ideas as if they are your own

- **collusion**
  this refers to collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed
- **contract cheating**
  this refers to contracting a third party to provide work, which is then used or submitted as part of a formal assessment as though it is the student's own work
- **use of artificial intelligence**
  this refers to the use of ChatGPT and other tools that can generate essay-style text, computer code, presentations, outlines and other content.

What is plagiarism?

Plagiarism can be defined as **the unacknowledged use of the work of others as if this were your own original work**. In the context of any University examination, this amounts to **passing off the work of others as your own to gain unfair advantage**.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.

Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to Turnitin, for example).

Where plagiarism or another form of academic misconduct is suspected, the Examiners of the relevant part of the Tripos may, at their discretion, examine a candidate *viva voce*.

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.) are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a 'ghost writing service'). Furthermore, you should not deliberately reproduce someone else's work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition, you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essay, project or coursework will be based on what you have read and learned from other sources, and it is important that in your essay, project or other coursework that you show exactly where, and how, your work is indebted to these other sources. The golden rule is that **the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else**.

A good guideline for avoiding plagiarism is not to repeat or reproduce other people's words, diagrams or computer programs. If you need to describe other people's ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay, project or other coursework, it is not sufficient merely to repeat or paraphrase someone else's view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

Quoting

A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:

- short quotations should be in inverted commas, and a reference given to the source
- longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

Paraphrasing

Paraphrasing means putting someone else's work into your own words. Paraphrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. "Smith (2001) goes on to argue that ..." or "Smith (2001) provides further proof that ..."). As with quotation, the full details of the source should be given in the bibliography or reference list.

General indebtedness

When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

Use of web sources

You should use web sources as if you were using a book or journal article. The above rules for quoting (including 'cutting and pasting'), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

Collaboration

Unless it is expressly allowed, collaboration is collusion and is considered academic misconduct. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

Use of artificial intelligence

Content produced by AI platforms, such as ChatGPT, is not original work and will be considered a form of academic misconduct to be dealt with under the University's disciplinary procedures. Several methods for detecting AI-generated text are already available and may be employed by the Examiners.

In addition to issues of academic integrity, students should be aware of several issues that have been reported:

- possible inaccuracy of the content generated, ranging from not being up-to-date to being entirely fictitious
- possibility of bias introduced and prejudicial views being perpetuated, based on existing online content
- ethical concerns around the gathering and use of user data, due to questionable consent and privacy practices of the platforms in question.

Appendix 2: Part III Essays and Grade Descriptors

The Assessor for each essay awards a numerical mark out of a maximum of 100 to each essay and in addition assigns a ‘quality mark’ (see Appendix IV of the Part III Handbook). The Faculty Board has specified that, just as with written papers, the minimum performance deserving of a distinction on a paper or an essay is associated with α-, while the minimum performance deserving of a pass is associated with β-.

The Faculty Board does not necessarily expect the mark distribution for essays to be the same as that for written examinations. Indeed, in recent years for many students the essay mark has been amongst their highest marks across all examination papers, both because of the typical amount of effort they have devoted to the essay and the different skill set being tested (compared to a time-limited written examination). The Faculty Board wishes that the hard work and talent thus exhibited should be properly rewarded.

There is no prescribed length for the essay in the University Ordinances and the Faculty Board recognises that the length of an essay is only a weak reflection of the quantity of work involved and bears no relation to the quality of the work done. However, it is anxious to prevent the essay absorbing too much of the candidate’s time. It is therefore perfectly content if a topic is set for which an excellent essay requires about 5000 words and would normally be unhappy if a topic were set for which an excellent essay required more than about 8000 words.

In order to provide greater clarity, the Faculty Board now requires that all candidates use a standard LaTeX template and has agreed that the expected length of an essay should normally then be between 20 and 30 pages and should only exceptionally be more than 35 pages.

Where an Assessor feels that a relaxation of the upper limit of 35 pages is justified for a particular essay title, they should specify an appropriate page range and upper limit, to be communicated to candidates in the essay booklet, and provide the Examiners with a clear explanation of why this is necessary, to be considered when the essay title is approved. The justification should explain why the increased upper limit does not represent an unreasonable increase in the time and effort that a candidate is expected to devote to this particular essay topic compared to others. Valid reasons might include the requirement to include an unusual number of diagrams, or the necessity to include data or code with the core of the essay.

In light of remarks above, as well as the comments of both internal and external Examiners over the years, the Faculty Board considers the following descriptors of the broad grade ranges for an essay to be appropriate. The Board trusts that these guidelines prove useful in guiding the judgement of the inevitably large number of Assessors marking essays, and thereby strengthen the mechanisms by which all essays are assessed uniformly. They are intended to be neither prescriptive nor comprehensive, but rather general guidance consistent with long-standing practice within the Faculty.

An Essay of α-Grade Standard (α-, α, α+)

Typical characteristics expected of an essay of α-grade standard include:

- Demonstration of clear mastery of the underlying mathematical content of the essay.
- Demonstration of thorough understanding and cogent synthesis of advanced mathematical concepts.
- A well-structured and well-written essay of appropriate length (5000-8000 words) with
    - few grammatical or presentational issues

    - a clear introduction demonstrating an appreciation of the context of the central topic of the essay
    - a coherent presentation of that central topic
    - a final section which draws the essay to a clear and comprehensible end, summarising well the key points while suggesting possible future work.

An essay of α-grade standard would be consistent with the quality expected of an introductory chapter of a PhD thesis from a leading mathematics department. A more elegant presentation and synthesis than that presented in the underlying papers, perhaps in the form of a shorter or more efficient proof of some mathematical result would be one possible characteristic of an essay of α-grade standard. Furthermore, it would be expected that an essay containing publishable results would be of α+ standard, but, for the avoidance of doubt, publishable results are **not necessary** for an essay to be of α+ standard. A mark in the α+ range should be justified by an explicit additional statement from the Assessor highlighting precisely which aspects of the essay are of particularly distinguished quality.

An Essay of β-Grade Standard (β-, β, β+)

Essays of β-grade standard encompass a wide range, but all should demonstrate understanding and synthesis of mathematical concepts at the level expected for a pass mark in a Part III lecture course.

Typical characteristics expected of an essay of β+ standard include:

- Demonstration of good mastery of most of the underlying mathematical content of the essay.
- A largely well-structured essay of appropriate length (5000-8000 words) with
    - some minor, grammatical or presentational issues
    - an introduction demonstrating an appreciation of at least some context of the central topic of the essay
    - a reasonable presentation of that central topic
    - a final section which draws the entire essay to a comprehensible end, summarising the key points.

Such essays would not typically exhibit extensive reading beyond the suggested material in the essay description, or original content.

Typical positive characteristics of an essay of β- (pass) standard include:

- Demonstration of understanding of some of the underlying mathematical content of the essay
- An essay consistent with the quality expected of an upper-second-class final-year project from a leading mathematics department

while negative characteristics might include some non-trivial flaws in presentation, for example:

- An inappropriate length
- Repetition or lack of clarity
- Lack of a coherent structure
- The absence of either an introduction or conclusion.

For the avoidance of doubt, a key aspect of the essay is that the important mathematical content is presented clearly in (at least close to) the suggested length. An excessively long essay is likely to be of (at best) pass standard.

Contents

1. Group Schemes and Representations .... **Professor S. Martin**

First let's begin with a motivating quotation: 'The very idea of scheme is of infantile simplicity—so simple, so humble, that no one before me thought of stooping so low. So childish, in short, that for years, despite all the evidence, for many of my erudite colleagues, it was really "not serious"!'(attributed to A. Grothendieck, Récoltes et Semailles, [3]).

The point of the essay is to understand group schemes and their role in representation theory. This will combine group theory, (some) algebraic geometry and representation theory.

Topics you might think about include: basic theory of algebraic groups and group schemes; fundamental structure of commutative groups schemes; infinitesimal theory, Lie algebras and distribution algebras; root data, Weyl groups and affine Weyl groups; the structure of reductive algebraic groups; representation theory of reductive algebraic groups.

Summary results and many more references can be found in the lecture notes of Williamson [6].

Relevant Courses

Essential:

Part II Representation Theory (or an equivalent introductory course on character theory of finite and/or Lie groups).

Part III Lie algebras and their representations

Useful:

Some background in algebraic geometry will be very helpful, but not essential.

Any course on Lie groups is of some utility.

References

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2. J. C. Jantzen. Representations of algebraic groups, volume 107 of Mathematical Surveys and Mono- graphs. American Mathematical Society, Providence, RI, second edition, 2003.

3. C. McLarty. The rising sea: Grothendieck on simplicity and generality. In Episodes in the history of modern algebra (1800–1950), volume 32 of Hist. Math., pp 301–325, Amer. Math. Soc., Providence, RI, 2007.

4. T. A. Springer. Linear algebraic groups, volume 9 of Progress in Mathematics. Birkhauser, Boston, Mass., 1981.

5. W. C. Waterhouse. Introduction to affine group schemes, volume 66 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, 1979.

6. G. Williamson. Sydney lecture notes available at

   https://www.maths.usyd.edu.au/u/geordie/GroupSchemesAndRepresentations/

2. Modular Character Theory .... **Professor S. Martin**

Pick some finite group G, not a p-group and not an abelian group (and not an example considered in lectures). Now pick a prime p dividing the order of the group (more ambitiously, pick a group and do the following for every prime p dividing the order of the group). A list of tasks all beginning research students should be able to do in this area includes: compute its ordinary character table, find its Brauer characters, find the characters of the projective indecomposable representations; compute the decomposition numbers and also construct the Cartan matrix.

Some of the concepts listed above were only mentioned briefly (if at all) in lectures so about a third of the essay will need to sketch more advanced theory of p-blocks and Brauer's main theorems. References [2] and [7] are gentle introductions.

Two particularly important classes of examples to think about are the simple groups and 'nearly simple'groups like $\mathrm{GL}_n(\mathbb{F}_q)$ or the symmetric group S_n that have a large (simple) composition factor. I would expect you to go through the above procedures for two non-trivial examples drawn from these types of groups. Namely

(a) The first example will involve one or two groups like $\mathrm{SL}_2(3)$, $\mathrm{SL}_2(5)$, $\mathrm{SL}_2(7)$, $\mathrm{SL}_3(2)$, $\mathrm{SL}_3(3)$ and $\mathrm{GL}_2(3)$ or you can present a general case like $\mathrm{SL}_2(p)$, for which see [1]. For further insights and inspiration you can do no better than Green's monumental construction of the ordinary characters of $\mathrm{GL}_n(\mathbb{F}_q)$, see [4].

(b) The second example involves the symmetric and alternating groups, for example S_5, S_6 or A_6. For this, a brief summary of the relevant theory of Young diagrams will be needed from [5].

Relevant Courses

Essential:

Part II Representation Theory and the Part III course on modular representations

Useful:

Basic courses on algebraic number theory and Galois theory.

References

1. J.L. Alperin, Local Representation Theory, Cambridge. (1986).
2. D.J. Benson. Representations and cohomology, Volume 1 CUP (1991).
3. D. Craven. Representation theory of finite groups: a guidebook. Universitext, Springer (2019).
4. J.A. Green. The characters of the finite general linear groups, Transactions of the American Mathematical Society **80** (1955), 402–447.
5. G.D. James The representation theory of the symmetric groups, Springer LNM 682 (1978).
6. P. Schneider. Modular representation theory of finite groups. Springer (2013).
7. P.J. Webb. A course in finite group representation theory. CUP (2016).

3. D-Modules and Hodge Theory .... **Professor I. Grojnowski**

The aim of this essay is to understand the basics of the theory of D-modules and some concrete applications to the topology of algebraic varieties. An ambitious essay will understand how D-modules with extra structure describe how the topological structure of algebraic varieties vary in families.

Begin by learning the definitions and properties of holonomic D-modules on algebraic varieties—coisotropic support, Kashiwara's lemma, Bernstein's lemma on b-functions, ..., and the consequences (the formalism of the six operations). This has applications throughout mathematics and physics.

An ambitious essay could then study the mixed Hodge structure on D-modules, beginning by computing the Kashiwara-Malgrange filtration on vanishing cycles in some interesting cases. If you do this, you will start learning about Hodge theory, too.

D-modules are a basic part of the language of modern representation theory as well as algebraic geometry. The Part III course on geometric representation theory will describe some of the representation theoretic applications of D-modules, will not overlap with this essay, but may provide another good motivation for learning the material.

As always, if you are interested in the essay we should meet and discuss the exact details of what you will learn.

References

Many textbook expositions of D-modules now exist. The two most striking are by the originators of the subject—Kashiwara and Bernstein. Bernstein's are printed notes, available on the web somewhere. Kashiwara's is

Masaki Kashiwara, *D*-modules and Microlocal Calculus, American Math Society Translations of Mathematical Monographs, Vol 217, 2003

For background on Hodge theory there are Deligne's extraordinary papers:

P. Deligne, Theorie des Hodges II, III. Inst. Hautes Etudes Sci. Publ. Math. No. 40 (1971), 5–57; Inst. Hautes Etudes Sci. Publ. Math. No. 44 (1974), 5–77.

P. Deligne, Travaux de Griffiths, Seminar Bourbaki 376, Lecture Notes in Math 180, Springer Verlag 1970, 213–237

4. The Geometry of Bun_G and Verlinde's Formula .... **Professor I. Grojnowski**

This is an essay about the geometry and representation theory of the stack of G bundles on an algebraic curve, where G is a reductive group. When $G = GL_n$, this is the moduli of vector bundles on the curve.

This is one of the central objects in the geometric Langlands program; it is a geometric avatar of one of the central objects of number theory, and it is also fundamental for modern physics.

How you should study this depends on how much algebraic geometry you know.

If your only exposure to algebraic geometry is from the Part III course, this essay is a great excuse to learn about concrete and subtle examples of algebraic varieties, and some of the advanced technology (GIT, moduli, and stacks) that modern geometers use.

In that case, you should take Mukai's book as your guide; you want to understand chapters 10 and 11, and some of the references quoted there. (This probably means you should read the entire book. It is a lovely book!)

If you already know a lot of algebraic geometry, the topics that an ambitious essay would cover:

The Drinfeld-Simpson theorem. Define the stack Bun_G, and its coarse quotients. Uniformise Bun_G analytically. Define the Harder-Narismhan stratification.

Explicitly describe the stack of G-bundles on $\mathbb{P}^1$ and on an elliptic curve E, and using this describe the cohomology of line bundles on the stack of bundles over these curves.

Degenerate curves to nodal rational curves, and state and prove the Verlinde formula.

Compute $Pic(Bun_G)$, and Pic of the coarse moduli spaces.

In any case, if you are interested in the essay we can discuss the exact details of what you will learn.

References for Bun_G.

0. Mukai, S. An Introduction to Invariants and Moduli, Cambridge University Press, 2012.

1. Drinfeld, V. G., Simpson, C. B-structures on G-bundles and local triviality. Math. Res. Lett. 2, 823–829 (1995).

2. Heinloth, J. Uniformization of $\mathcal{G}$-bundles. arXiv:0711.4450

3. Heinloth, J. Hilbert-Mumford stability on algebraic stacks and applications to $\mathcal{G}$-bundles on curves. arXiv:1609.06058

4. Faltings, G. Vector Bundles on curves. Bonn lecture notes, 1995. (Available on the internet)

5. Zhu, X. An introduction to affine Grassmannians and the geometric Satake equivalence. arXiv:1603.05593

5. The Moduli Stack of Vector Bundles on an Elliptic Curve .... Dr P. F. Kennedy-Hunt

In this essay you will understand, in the simplest interesting example, the geometry of the (algebro–geometric) moduli space of vector bundles. Despite being studied for over 70 years, spaces of vector bundles on curves remains an exciting area of research. Completing this essay will help make sophisticated ideas of moduli spaces, stacks, and stability conditions both concrete and accessible.

The key to a good essay will be showing an understanding of the geometry of the space of vector bundles on an elliptic curve, and most proofs should be only sketched. Whilst the information you need is readily accessible, it is not recorded in one place, so you might produce a genuinely helpful exposition. I expect good essays to include the following:

1. A classification of all vector bundles on an elliptic curve. This was originally worked out by Atiyah [1], but Mukai's approach gives geometric insight [2].

2. An account of slope stability in this context, and explaining stability/Harder–Narasimhan properties of the vector bundles classified in (1.). See for example [3, Section 1.2].

3. For which vector bundles V_1, V_2 does there exist a vector bundle on $\mathbb{A}^1 \times E$ which away from 0 in $\mathbb{A}^1$ is pulled back from V_1, and the fibre over 0 in $\mathbb{A}^1$ is the vector bundle V_2. To do this you can prove and apply [4, Example 1.7].

You don't have to use the language of stacks, though I am willing to help you to do so.

There are many directions a keen student could then go in, including but not limited to:

1. What do vector bundles on a nodal cubic look like? How can this space be compactified? See [5].

2. Why is the moduli space of vector bundles smooth?

3. What do Bridgeland stability conditions on an elliptic curve look like? See [6], especially the appendix.

4. What can you say about geometry of vector bundles on the deRham space X^{dR} of X?

For alternative inspiration you might look up the BunG seminar [7].

Relevant Courses

Essential: Algebraic Geometry

Useful: Abelian Varieties, Differential Geometry

References

1. Atiyah, M. F. *Vector bundles over an elliptic curve.* Proceedings of the London Mathematical Society 3.1 (1957): 414-452.

2. Mukai, S. *Semi-homogeneous vector bundles on an Abelian variety.* J. Math. Kyoto Univ. 18 (1978), no. 2, 239–272, DOI 10.1215/kjm/1250522574.

3. Huybrechts, D., and Manfred L. *The geometry of moduli spaces of sheaves.* Cambridge University Press, 2010.

4. Halpern-Leistner, D. *Theta-stratifications, Theta-reductive stacks, and applications.* arXiv:1608.04797.

5. Nagaraj, D.S., Seshadri, C.S. *Degenerations of the moduli spaces of vector bundles on curves.* I. Proc. Indian Acad. Sci. (Math. Sci.) 107, 101–137 (1997). https://doi.org/10.1007/BF02837721

6. Bridgeland, T. *Stability conditions on triangulated categories.* Annals of Mathematics (2007): 317-345.

7. 2023 section of https://math.uchicago.edu/ bundles/

6. Elliptic Divisibility Sequences .... Professor H. Krieger

Let $E : y^2 = x^3 + ax + b$ be an elliptic curve over $\mathbb{Q}$ and $P \in E(\mathbb{Q})$ a rational point. The nth multiple of P takes the form

$$[n]P = \left(\frac{a_n}{d_n^2}, \frac{b_n}{d_n^3}\right)$$

in reduced form, and the integer sequence $\{d_1, d_2, d_3, \dots\}$ is known as an *elliptic divisibility sequence* (EDS). These sequences satisfy non-linear recursion relations, have some arithmetic similarities to well-known sequences such as the Fibonacci and the Mersenne. Any EDS conjecturally contains only finitely many primes, though that is far out of reach.

Using Siegel's theorem on integral points on curves, Silverman [4] proved a much weaker statement: that the elements of any EDS eventually contain primitive prime divisors; that is, prime divisors which do not appear as divisors of previous elements of the sequence. This is known as 'Zsigmondy finiteness' for the EDS thanks to a similar result of Zsigmondy on sequences of the form $a^n - b^n$. Poonen [2] showed that the Zsigmondy finiteness of any EDS has implications for Hilbert's 10th problem. The result also has an analogue for dynamically-generated sequences with exponential growth rates, due to Ingram and Silverman [1].

This essay will develop a bit of the theory of height functions and provide a clear exposition of the proof of Siegel's theorem from the weak Mordell-Weil theorem (a good source text is [3]), and the deduction of Silverman's result. The essay can then move in one of two directions. The first option is Poonen's theorem, including background of Hilbert's tenth and discussion of Poonen's use in [2] of the EDS Zsigmondy finiteness. The second option is Ingram-Silverman's dynamical generalization [1], including Silverman's dynamical analogue of Siegel's theorem [5], dynamical height functions, and the strategy of Ingram-Silverman. A student with further interest is welcome to engage with any of the substantial literature on elliptic divisibility sequences or dynamical Zsigmondy finiteness that they find appealing.

Relevant Courses

Essential: Elliptic Curves.

Useful: Local Fields or at least some basic algebraic number theory.

References

1. Ingram, P. and Silverman, J.H., 2009, March. Primitive divisors in arithmetic dynamics. In Mathematical Proceedings of the Cambridge Philosophical Society (Vol. 146, No. 2, pp. 289-302). Cambridge University Press.

2. Poonen, B., 2003. Hilbert's tenth problem and Mazur's conjecture for large subrings of $\mathbb{Q}$. Journal of the American Mathematical Society, 16(4), pp.981-990.

3. Serre, J.P., Brown, M. and Waldschmidt, M., 1989. Lectures on the Mordell-Weil theorem. F. Vieweg.

4. Silverman, J.H., 1988. Wieferich's criterion and the abc-conjecture. Journal of number theory, 30(2), pp.226-237.

5. Silverman, J.H., 1993. Integer points, Diophantine approximation, and iteration of rational maps.

6. Silverman, Joseph H. The arithmetic of elliptic curves. Vol. 106. New York: Springer, 2009.

7. Dual Complexes of Algebraic Varieties .... Professor D. Ranganathan

Let X be a smooth, but possibly non-proper, algebraic variety over the complex numbers. By theorems of Nagata and Hironaka, there is always a smooth and proper algebraic variety $\overline{X}$ that (i) contains X as a dense open, and (ii) such that $D := \overline{X} \setminus X$ is a union of smooth hypersurfaces meeting transversely. The combinatorics of the irreducible components of this D can be encoded in a simplicial complex $\Delta(D)$. While the precise simplicial complex depends on the choice of $\overline{X}$, remarkably, the homotopy type $\Delta(D)$ is an invariant of X itself. It is called the *dual complex*.

The rational cohomology of the dual complex encodes subtle information about the mixed Hodge theory on X and can be used to study the topology of X using combinatorial methods. For example, recent progress on the understanding of the cohomology of the moduli spaces of curves and abelian varieties have come via a careful study of their associated dual complexes.

This essay will explore the construction and topology of dual complexes. The essay should start by giving a proof of the theorem above – that the homotopy type of the dual complex is an invariant of the starting variety X. This proof relies on basic tools in birational geometry, including resolution of singularities and toroidal weak factorization for birational maps. These tools may be black boxed and treated via examples. The relationship with mixed Hodge theory should also be explained.

After this, the essay should choose a direction to explore in detail. Possibilities include the study dual complexes of hyperplane arrangements complements or Calabi–Yau pairs. A different option would be to show that every simplicial complex can appear as the dual complex of some compactification of some smooth variety, following Kollár.

A good essay should include concrete, small examples, showing the types of behaviour that can be seen in the topology of $\Delta(D)$ in low dimension.

Relevant Courses

Essential: Part III Algebraic Geometry, Part III Algebraic Topology.

References

1. Abramovich, Dan, Karu Kalle, Matsuki, Kenju, and Wlodarczyk, Jaroslaw. *Torification and factorization of birational maps.* Journal of the American Mathematical Society 15.3 (2002): 531-572.

2. Danilov, Vladimir I. *Polyhedra of schemes and algebraic varieties.* Mathematics of the USSR-Sbornik 26.1 (1975): 137.

3. Kollár, János. *Simple normal crossing varieties with prescribed dual complex.* Algebraic Geometry 1.1 (2014): 57-68.

4. Kollár, János, and Chenyang Xu. *The dual complex of Calabi–Yau pairs.* Inventiones mathematicae 205 (2016): 527-557.

5. Payne, Sam. *Boundary complexes and weight filtrations.* Michigan Mathematical Journal 62.2 (2013): 293-322.

8. Brill–Noether Theory .... Professor D. Ranganathan

Let C be a smooth projective algebraic curve of genus g. Brill–Noether theory studies what kind of maps can be constructed from C to complex projective spaces. Precisely, the basic question is: for which r and d does there exist a map

$$C \to \mathbb{P}^r$$

of degree d that does not factor through a hyperplane? The *Brill–Noether theorem*, proved by Griffiths–Harris, Kempf, and Kleiman–Laksov in the 1970s and 80s, asserts that if C is *general* then such a map exists if and only if the number

$$\rho(g, r, d) = g - (r + 1)(g - d + r)$$

is non-negative. For particular choices of C the statement is false, but only in one direction – there can be maps of unexpectedly low degree from special curves. A typical example is given by hyperelliptic curves – one shouldn't expect a curve of genus at least 3 to double cover $\mathbb{P}^1$, but there exist curves of every genus that do.

There are now many proofs of this theorem, using a range of techniques – degeneration methods, tropical geometry, vector bundles on K3 surfaces, and stability conditions on moduli of sheaves. There have also been several important recent generalisations, including ones that explore what can happen when C fails to be general in a controlled way.

This essay will be an exposition of aspects of Brill–Noether theory. The essay should start with some discussion of the variety parameterizing maps from a curve to projective space and use the geometry of the Jacboian to explain where the number $\rho(g, r, d)$ comes from. You can then focus on either one of the two aspects of the theorem – the existence of maps when $\rho(g, r, d)$ is non-negative, which can be derived using intersection theory, or the non-existence when $\rho(g, r, d)$ is negative, which is proved by degeneration techniques or K3 geometry.

A good essay should also include several examples, such as giving geometric constructions in low genus that realise the relevant maps, or using elementary tools such as Riemann–Hurwitz and Riemann–Roch to prove non-existence in specific examples. An ambitious essay can also examine some of the recent developments, including the Hurwitz–Brill–Noether theorem, maximal rank conjecture, or interpolation problem.

Relevant Courses

Essential: Part III Algebraic Geometry

Useful: Intersection Theory

References

1. Cools, Filip, Draisma, Jan, Payne, Sam, and Robeva, Elina. *A tropical proof of the Brill–Noether theorem.* Advances in Mathematics 230.2 (2012): 759-776.

2. Eisenbud, David, and Joe Harris. *A simpler proof of the Gieseker-Petri theorem on special divisors.* Inventiones mathematicae 74 (1983): 269-280.

3. Griffiths, Phillip, and Joseph Harris. *On the variety of special linear systems on a general algebraic curve.* (1980): 233-272.

4. Kleiman, Steven L., and Dan Laksov. *On the existence of special divisors.* American Journal of Mathematics 94.2 (1972): 431-436.

5. Lazarsfeld, Robert. *Brill-Noether-Petri without degenerations.* Journal of Differential Geometry 23.3 (1986): 299-307.

9. Montgomery's Conjecture and Kakeya Sets .... Dr D. Maldague

In Fourier restriction theory, we study exponential sums $f_S(x) = \sum_{n=1}^{N} b_n e^{ix\cdot\xi_n}$ with coefficients $b_n \in \mathbb{C}$ and restricted frequency set $\{\xi_n\} = S$. Properties of the frequency set S, like curvature or randomness, lead to strong L^p bounds for these exponential sums. The celebrated ℓ^2 decoupling theorem of Bourgain and Demeter [2] proves sharp L^p estimates for a variety of S, resolving longstanding problems like the main conjecture of Vinogradov's Mean Value Theorem and proving sharp Strichartz estimates for the Schrödinger equation on irrational tori. However, there are still many important L^p exponential sum conjectures for which decoupling techniques only provide limited information.

One important example of a Fourier restriction setup from number theory is Dirichlet polynomials $D(t) = \sum_{n=N}^{2N} b_n e^{it\log n}$. When $b_n \equiv 1$, $D(t)$ is a partial sum of the Riemann zeta function. Chapter 7 of Montgomery's notes [4] provides an excellent background on L^p estimates for $D(t)$. Montgomery's L^q conjecture (MC1) is Conjecture 1 on p. 129 and and the stronger Montgomery's Large Value conjecture (MC2) is Conjecture 2 on p. 142. An essay could provide an overview of Chapter 7 and report on the history of limited progress on MC1 and MC2. This includes Bourgain's counterexample [1] to the ℓ^2 Montgomery's conjectures as well as the very recent paper of Guth-Maynard [3] which improves MC2 bounds for $D(t)$ that held since the 1940s. To understand the difficulty of Montgomery's conjectures, it is fascinating to note that they relate to another famous conjecture from the area of fractal geometry called the Kakeya conjecture. The Kakeya conjecture asserts that a Borel subset of $\mathbb{R}^n$ which contains a unit line segment in every direction must have full Hausdorff dimension. This implication was shown by Bourgain in [1].

There are additional open-ended questions that are possible to explore. (1) If one assumes the Kakeya conjecture, is it possible to make progress on Montgomery's conjectures? (2) What are the challenges of using wave packets from Fourier restriction/decoupling in this setting? (3) What do we expect when the frequencies $\log n$ are replaced by a more general sequence a_n satisfying $a_{n+1} - a_n \sim N^{-1}$ and $(a_{n+2} - a_{n+1}) - (a_{n+1} - a_n) \sim N^{-2}$? (4) How do L^q or Large Value estimates for $D(t)$ give information about the distribution of primes (see §13 of [3])? Some of these questions are very hard, and it is expected to meet me to discuss how to approach them.

Relevant Courses

Essential: A solid undergraduate analysis course. Part III Fourier Restriction Theory and Applications.

Useful: Part III Analytic Number Theory.

References

1. J . Bourgain, *Remarks on Montgomery's conjectures on Dirichlet series*, Geometric Aspects of Functional Analysis (1989-1990), 1991, p. 153–165.

2. J. Bourgain and C. Demeter, *The proof of the l^2 decoupling conjecture*, Ann. of Math. (2) 182, 2015, p. 351–389.

3. L. Guth and J. Maynard, *New large value estimates for Dirichlet polynomials*, available at arXiv:2405.20552, 2024.

4. H. Montgomery, *Ten Lectures On The Interface Between Analytic Number Theory And Harmonic Analysis*, No. 84. American Mathematical Soc., 1994.

10. General Relativity as a Dynamical PDE .... Dr. R. Teixeira da Costa

Einstein's equations in General Relativity can be seen a second order PDE whose solution is a Lorentzian manifold. This PDE has no concrete character until one breaks the diffeomorphism invariance of the equations. In doing so with a clever choice of (wave) gauge, Choquet-Bruhat [3] showed that Einstein's equations have a wave-like character and are locally well-posed as dynamical equations. This classical result has been revisited and improved [2,4]. As in the case of wave equations, there is a natural notion of maximal development predicted by the initial data (i.e. globally hyperbolic) [1,6]. See also the book [5].

The goal of this essay is to discuss the classical proof of existence, uniqueness and continuous dependence on initial data of the maximal globally hyperbolic development of initial data for the Einstein equations.

Relevant Courses

Essential: Part III Analysis of Partial Differential Equations ; *either* Part III Differential Geometry *or* Part III General Relativity

References

1. Choquet-Bruhat, Y., & Geroch, R. (1969). Global aspects of the Cauchy problem in General Relativity. Communications in Mathematical Physics, 14(4), 329–335. https://doi.org/10.1007/BF01645389

2. Fischer, A. E., & Marsden, J. E. (1972). The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system, I. Communications in Mathematical Physics, 28(1), 1–38. https://doi.org/10.1007/BF02099369

3. Fourès-Bruhat, Y. (1952). Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. Acta Mathematica, 88, 141–225. https://doi.org/10.1007/BF02392131

4. Hughes, T. J. R., Kato, T., & Marsden, J. E. (1977). Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. Archive for Rational Mechanics and Analysis, 63(3), 273–294. https://doi.org/10.1007/BF00251584

5. Ringström, H. (2009). The Cauchy Problem in General Relativity. European Mathematical Society Publishing House. https://doi.org/10.4171/053

6. Sbierski, J. (2016). On the Existence of a Maximal Cauchy Development for the Einstein Equations: a Dezornification. Annales Henri Poincaré, 17(2), 301–329. https://doi.org/10.1007/s00023-015-0401-5

11. The (Riemannian) Penrose Inequality ....

Dr. R. Teixeira da Costa

The Penrose inequality is a conjecture [6] which asserts that, in an asymptotically flat $(1+3)$-dimensional Lorentzian manifold satisfying the Einstein vacuum equations, a black hole region whose horizon has area A contributes $\sqrt{A/(16\pi)}$ (in appropriate units) to the total mass. Since every such spacetime arises from the evolution of initial data for the Einstein equation, an asymptotically Euclidean $3-\textit{dimensional}$ Riemannian manifold satisfying some constraint equations, one can formulate an analogous conjecture for initial data sets in General Relativity.

The Riemannian version of the Penrose inequality was first shown in [4,5] using geometric flows. Recently, a more elementary proof was given in [1], based on the analysis of a suitable boundary value elliptic problem. These techniques are quite powerful, see e.g. [2,3]. The goal of this essay is to present some of the key ideas of the proof in a self-contained manner.

Relevant Courses

Essential: Part III Analysis of PDE, Part III Differential Geometry

Useful: Part III Elliptic PDE

References

1. Agostiniani, V., Mantegazza, C., Mazzieri, L., & Oronzio, F. (2022). Riemannian Penrose inequality via Nonlinear Potential Theory. 1–27. http://arxiv.org/abs/2205.11642

2. Agostiniani, V., & Mazzieri, L. (2017). On the Geometry of the Level Sets of Bounded Static Potentials. Communications in Mathematical Physics, 355(1), 261–301. https://doi.org/10.1007/s00220-017-2922-x

3. Agostiniani, V., Mazzieri, L., & Oronzio, F. (2021). A green's function proof of the positive mass theorem. 2, 1–22.https://arxiv.org/pdf/2108.08402

4. Bray, H. L. (2001). Proof of the Riemannian Penrose inequality using the positive mass theorem, Journal of Differential Geometry 59, 177–267.

5. Huisken, G., & Ilmanen T. (2001). The inverse mean curvature flow and the Riemannian Penrose inequality, Journal of Differential Geometry 59, 353–437.

6. Penrose, R. (1973). Naked singularities. Annals of the New York Academy of Sciences, 224(1), 125–134. https://doi.org/10.1111/j.1749-6632.1973.tb41447.x

12. Ginzburg-Landau Vortices ....

Professor C. Warnick

Type II semiconductors can occur in a phase, the mixed state, in which a superconducting bulk is threaded by small tubes of normally conducting material, called vortex filaments. This can be modelled mathematically using Ginzburg-Landau theory. In particular [1, 2], the dynamics of the vortices is determined by the behaviour of solutions to a non-linear reaction–diffusion equation in a limit where the energy of the field concentrates on a point (in 2d) or line (in 3d). In this limit the vortices behave as interacting point particles. The purpose of the essay is to study the mathematical literature dealing with this singular PDE limit [3, 4, 5].

Relevant Courses

Essential: Analysis of Partial Differential Equations

References

1. E Weinan, "Dynamics of vortices in Ginzburg-Landau theories with applications to superconductivity", *Phys. D* **77**, (1994) 383-404
2. L. Peres, J. Rubinstein, "Vortex dynamics in U(1) Ginzburg-Landau models", *Phys. D* **64**, (1993) 299-309,
3. Hua Lin, F. "Some dynamical properties of Ginzburg-Landau vortices". *Comm. Pure Appl. Math.*, **49** (1996): 323-359.
4. Jerrard, R., Soner, H. "Dynamics of Ginzburg-Landau Vortices", *Arch Rational Mech Anal* **142** (1998), 99-125
5. Lin, F.H., "Vortex dynamics for the nonlinear wave equation". *Comm. Pure Appl. Math.*, **52** (1999) 737-761.

13. Concentration Inequalities for Lipschitz Maps on Infinite Graphs ...... Dr A. Zsák

There has been considerable interest in the problem whether every metric space of bounded geometry coarsely embeds into a uniformly convex Banach space due to the work of Kasparov and Yu that established a connection between such embeddings and the Novikov conjecture.

Brown and Guentner proved that a metric space with bounded geometry coarsely embeds into a reflexive Banach space. However, it was unknown whether there was *any* metric space which did not coarsely embed into a reflexive space. The analogous problem for uniform embeddings was also a well known open problem. In [1] Kalton proved two landmark results. He showed that every stable metric space embeds coarsely and uniformly into a reflexive space. He then showed that the Banach space c_0 does not embed coarsely or uniformly into any reflexive space. Kalton proved the latter result by introducing a coarse and uniform invariant, called property Q, in the form of a concentration inequality for Lipschitz maps on the class of interlacing graphs. Shortly thereafter, a similar concentration inquality for Lipschitz maps on Hamming graphs was used by Kalton and Randrianarivony in their study in [2] of the coarse geometry of $\ell_p \oplus \ell_q$.

These concentration inequalities for Lipschitz maps represented a genuine breakthrough and led to the solution of a number of longstanding open problems. One notable example is the result of Baudier, Lancien and Schlumprecht in [3] showing that ℓ_2 does not embed coarsely into every infinite-dimensional space. In another remarkable paper [4], Baudier, Lancien, Motakis and Schlumprecht used the concentration inequality of [3] to give the first example of a coarsely rigid *unrestricted* class of infinite-dimensional Banach spaces.

A successful essay should present Kalton's result in [1] that every stable metric space coarsely and uniformly embeds into a reflexive space, develop Kalton's concentration inequality for Lipschitz maps on interlacing graphs together with applications demonstrating how Kalton's property Q provides an obstruction to the coarse embeddability of a Banach space X or the uniform embeddability of its closed unit ball B_X into a reflexive or uniformly convex space. The essay should then develop the concentration inequalities in [2], [3] and [4] with applications that should include proofs of the uniqueness of uniform structure on $\ell_p \oplus \ell_q$ for $1 < p < 2 < q < \infty$ [2], the coarse non-embeddability of ℓ_2 into Tsirelson's original space [3] and the construction of the new coarsely rigid class in [4].

Relevant Courses

Essential: Functional Analysis

References

1. Kalton, N. J., *Coarse and uniform embeddings into reflexive spaces*, Q. J. Math. **58** (2007), no. 3, 393–414
2. Nigel J. Kalton and N. Lovasoa Randrianarivony, *The coarse Lipschitz geometry of* $l_p \oplus l_q$, Math. Ann. **341** (2008), no. 1, 223–237
3. F. Baudier, G. Lancien, and Th. Schlumprecht, *The coarse geometry of Tsirelson's space and applications*, J. Amer. Math. Soc. **31** (2018), no. 3, 699–717
4. F. Baudier, G. Lancien, P. Motakis, and Th. Schlumprecht, *A new coarsely rigid class of Banach spaces*, J. Inst. Math. Jussieu **20** (2021), no. 5, 1729–1747

14. A Proof of the Kahn-Kalai Conjecture .... Professor W. T. Gowers

A family $\mathcal{A}$ of subsets of $\{1, 2, \ldots, n\}$ is called *increasing* if whenever $A \in \mathcal{A}$ and $A \subset B$ we have also that $B \in \mathcal{A}$. The *threshold* for an increasing family is the unique p such that if the elements of a set A are chosen independently and randomly with probability p, then the probability that A belongs to $\mathcal{A}$ is $1/2$. (It is easy to see that this exists except in trivial cases that $\mathcal{A}$ is empty or equal to the power set of $\{1, 2, \ldots, n\}$.)

Thresholds are of central importance in probabilistic combinatorics and statistical physics, but can be difficult to estimate. The Kahn-Kalai conjecture is a bold conjecture (sufficiently bold that Kahn and Kalai expected it to be false) that asserts that a different quantity that is much simpler to estimate, known as the expectation threshold, differs from the actual threshold by at most a logarithmic factor. A remarkable and surprisingly short proof of this was discovered by Jinyoung Park and Huy Tuan Pham, which was shortened yet further by Phuc Tran and Van Vu. The result will be the centerpiece of the essay.

The shortness of the proof means that on its own it is not sufficient for an essay. There are several possibilities for additional material, such as earlier results that motivated the conjecture, a "fractional version" of the result proved in 2019, or results about thresholds for particular natural monotone families (an example of such a family is the set of all graphs on a given set of vertices that contain a Hamilton cycle). Candidates doing this essay will discuss the possibilities with me and I will help them make a suitable selection.

Relevant Courses

There are no essential Part III courses, but Combinatorics is likely to contain tangentially related material, as is Entropy Methods in Combinatorics.

References

1. Keith Frankston, Jeff Kahn, Bhargav Narayanan and Jinyoung Park, *Thresholds versus fractional expectation-thresholds*, Ann. of Math. (2) **194** (2021), pp. 475-495, or https://arxiv.org/abs/1910.13433

2. Jeff Kahn and Gil Kalai, *Thresholds and expectation thresholds*, Combin. Probab. Comput. **16** (2007), pp. 495-502, or https://arxiv.org/abs/math/0603218.

3. Jinyoung Park and Huy Tuan Pham, *A proof of the Kahn-Kalai conjecture*, https://arxiv.org/abs/2203.17207

4. Phuc Tran and Van Vu, *A short proof of the Kahn-Kalai conjecture*, https://arxiv.org/abs/2303.02144

15. Poset Ramsey Numbers ....

Dr M.-R. Ivan

Background. Poset saturation has been introduced in 2017 by Ferarra, Kay, Krammer, Martin, Reiniger, Smith and Sullivan [1], and interest in it has grown considerably ever since. The definitions are extremely similar to those used in the area of graph saturation, which was introduced in 1964 by Erdős, Haijnal and Moon. Given a finite poset $\mathcal{P}$, we say that a family $\mathcal{F}$ of subsets of $[n]$ is $\mathcal{P}$-saturated if it does not contain an *induced copy* of $\mathcal{P}$, but adding any other set to $\mathcal{F}$ creates such a copy. The smallest size such an $\mathcal{F}$ can have is called the *saturation number* of $\mathcal{P}$, denoted by $sat^*(n, \mathcal{P})$.

Determining even the correct rate of growth for even simple posets, such as the diamond (4 point poset with one minimal and one maximal element, and 2 incomparable other elements), has proven to be very challenging, showing that whilst the definitions are very similar, saturation for graphs is intrinsically different from poset saturation.

As a consequence, a natural question is what about Ramsey-type questions for posets? For fixed posets $\mathcal{P}_1$ and $\mathcal{P}_2$ the *poset Ramsey number* of $\mathcal{P}_1$ versus $\mathcal{P}_2$, denoted by $\mathcal{R}(\mathcal{P}_1, \mathcal{P}_2)$, is the minimum n such that every 2-colouring of the power set of $[n]$ contains either a blue copy of $\mathcal{P}_1$, or a red copy of $\mathcal{P}_2$.

For the diagonal setting $\mathcal{P}_1 = \mathcal{P}_2 = Q_n$, it is known that $2n \leq \mathcal{R}(Q_n, Q_n) \leq n^2 + 2n$ [2]. These bounds were gradually improved to $2n+1 \leq \mathcal{R}(Q_n, Q_n) \leq n^2 - n + 2$, see chronologically Cox and Stolee [3], Lu and Thompson [4], and Bohman and Peng [5]. Whilst determining the exact asymptotic behaviour is likely quite challenging, both the lower and the upper bound have potential for significant improvements, which is a potential direction enthusiastic students can explore, although it is certainly not expected.

In the off-diagonal setting $\mathcal{P}_1 = Q_m$, $\mathcal{P}_2 = Q_n$, it is known that $\mathcal{R}(Q_1, Q_n) = n + 1$ and $\mathcal{R}(Q_2, Q_n) = n + \Theta\big(\frac{n}{\log(n)}\big)$, see [6] and [7]. For $m \geq 3$ only rough estimates are known [4]. However it is believed that, for every fixed m, $\mathcal{R}(Q_m, Q_n) = n + o(n)$, which is one of the main conjectures in this area. Note that every poset is contained in some Q_m, so this conjecture is equivalent to $\mathcal{R}(\mathcal{P}, Q_n) = n + o(n)$ for every fixed poset $\mathcal{P}$.

Essay. The aim of the essay is to first explore in detail the proofs of the main results in both the diagonal and the off-diagonal case, highlighting the core ideas behind each result. After that, one should choose either the diagonal or the off-diagonal case to focus on, or in exceptional circumstances (which should be agreed on with the assessor), another related open problem. At this stage, the essay is expected to delve deep into the already existing proofs and methods. On top of the mathematical clarity and completeness of the essay, it is also encouraged that the results are accompanied by some mathematical discussions such as providing some intuition as to why the methods stop working at a certain point (if applicable), or making a parallel to the classical Ramsey numbers. An essay of excellent standards is not expected to contain any original research, but students are very welcome to explore that possibility if they wish to (e.g. improving on one of the existing bounds, even by an additive constant).

Relevant Courses

Essential: Numbers and Sets (IA).

Useful: Graph Theory (II), Logic and Set Theory (II), Ramsey Theory (III).

References

1. M. Ferrara, B. Kay, L. Kramer, R. R. Martin, B. Reiniger, H. C. Smith, and E. Sullivan. The saturation number of induced subposets of the Boolean lattice. Discrete Mathematics, 340(10):2479–2487, 2017.

2. M. Axenovich and S. Walzer. Boolean lattices: Ramsey properties and embeddings. Order, 34(2):287–298, 2017.

3. C. Cox and D. Stolee. Ramsey numbers for partially-ordered sets. Order, 35(3):557–579, 2018.

4. L. Lu and C. Thompson. Poset Ramsey numbers for Boolean lattices. Order, 39(2):171–185, 2022.

5. T. Bohman and F. Peng. A construction for Boolean cube Ramsey numbers. Order, 2022.

6. D. Grósz, A. Methuku, and C. Tompkins. Ramsey numbers of Boolean lattices. Bulletin of the London Mathematical Society, 2023.

7. M. Axenovich and C. Winter. Poset Ramsey numbers: large Boolean lattice versus a fixed poset. Combinatorics, Probability and Computing, pages 1–16, 2023.

16. Hamiltonian Cycles and Spheres in Hypergraphs .... Professor I. Leader

The notion of a Hamilton cycle (a cycle through all the vertices) in a graph also makes sense for a hypergraph. There are various versions. One is that we may list the vertices cyclically in such a way that every interval of length k (where the hypergraph consists of k-sets) belongs to the hypergraph. Is there an analogue of the well-known Dirac theorem for graphs, which states that if a graph has minimum degree at least $n/2$ then it has a Hamilton cycle?

The aim of the essay is to focus on some classical results on this question, and also on some recent work on 'Hamilton spheres'.

Relevant Courses

Useful: Combinatorics

References

1. An approximate Dirac-type theorem for k-uniform hypergraphs, V.Rodl, A.Rucinski and E.Szemeredi, Combinatorica volume 28 (2008), 229-260.

2. Spanning surfaces in 3-graphs, A.Georgakopoulos, J.Haslegrave, R.Montgomery and B.Narayanan, available at arXiv:1808.06864

17. Notions of Rank in Additive Combinatorics .... Professor J. Wolf

This essay explores various notions of tensor rank that have been exploited in additive combinatorics over the last twenty years. One of these is *slice rank*, which goes back to work of Schmidt in the 1980s and was rediscovered by Tao in the course of refining the polynomial method of Croot-Lev-Pach and Ellenberg-Gijswijt; a generalisation known as *partition rank* was subsequently introduced by Naslund; a third notion is *analytic rank*, due to Gowers and Wolf. In the case of degree-2 tensors, they all reduce to the usual rank of a matrix.

It is not difficult to see that the analytic rank of a tensor is always bounded above by its partition rank. A long-standing conjecture asserts that the converse also holds, up to a constant. The reading list below includes some of the recent work towards this.

An essay on this topic should (a) motivate and clearly set out the main notions of rank, (b) give an overview of the literature to date on the relationships between them, (c) cover at least one argument towards the aforementioned conjecture for general-degree tensors in depth, and (d) discuss an application.

Relevant Courses

Essential: Introduction to Additive Combinatorics (Michaelmas)

Potentially useful: Algebraic Geometry (Michaelmas)

References

1. Cohen A. and Moshkovitz, G. *Partition and Analytic Rank are Equivalent over Large Fields*, Duke Math. J. 172, 2433-2470 (2023)

2. Gowers, W.T. and Wolf, J. *Linear forms and higher-degree uniformity for functions on $\mathbb{F}_p^n$.* Geom. Funct. Anal. 21 (1), 36-69 (2011)

3. Janzer, O. *Polynomial bound for the partition rank vs the analytic rank of tensors.* Discrete Analysis 2020:7, 18pp (2020)

4. Lampert, A. *Slice rank and analytic rank for trilinear forms.* arXiv:2404.19704 (2024)

5. Lovett, S. *The analytic rank of tensors and its applications.* Discrete Analysis 2019:7, 10pp (2019)

6. Milićević, L. *Polynomial bound for partition rank in terms of analytic rank.* Geom. Funct. Anal. 29 (5), 1503–1530 (2019)

7. Moshkovitz, G. and Zhu, D. G. *Quasi-linear relation between partition and analytic rank.* arXiv:2211.05780 (2022)

8. Naslund, E. *The partition rank of a tensor and k-right corners in $\mathbb{F}_q^n$.* J. Combin. Theory Ser. A 174, 105-190 (2020)

18. Isoperimetric Functions in Group Theory ....
Dr M. Arenas

In geometry, an isoperimetric function is a way of measuring, for each loop in a space, the 'optimal area' of discs filling-in that loop. In group theory, isoperimetric functions are related to the complexity of the word problem – the question of deciding whether, given a generating set S for a group G, a word on S represents the identity element in G.

In this essay, the student will define isoperimetric functions and Dehn functions (optimal isoperimetric functions) for finitely presented groups, study their possible growth types and how these are related to the geometry of the group, and introduce the isoperimetric spectrum, which is the set **IP** of growth types of Dehn functions of finitely presented groups. There is some flexibility on what the students might cover, but possible results include characterising groups with linear Dehn function, proving that certain important classes of groups have quadratic Dehn function, or showing that the isoperimetric spectrum is a dense subset of $\{1\} \cup [2, \infty)$.

Relevant Courses

Essential: IA Groups, II Algebraic Topology

Useful: IB Geometry, IB Groups, Rings and Modules, III Geometric Group Theory

References

1. Brady N Riley T Short H. The Geometry of the Word Problem for Finitely Generated Groups. Basel: Birkhauser; 2007.

2. M. R. Bridson, The geometry of the word problem, Invitations to geometry and topology, Oxf. Grad. Texts Math., vol. 7, Oxford University Press, Oxford, 2002, pp. 29—91. Available at https://people.maths.ox.ac.uk/bridson/papers/bfs/bfs.pdf

3. Section 4 of Bridson, M. R. (2006, August). Non-positive curvature and complexity for finitely presented groups. In International Congress of Mathematicians (Vol. 2, pp. 961-987). European Math. Soc.

4. Gersten, S. M. (1993). Isoperimetric and isodiametric functions of finite presentations. Geometric group theory, 1, 79-96.

19. Aspects of Artin Groups and their Classifying Spaces ....
Dr M. Arenas

Artin groups are infinite groups that are easy to define combinatorially. They generalise free groups, free abelian groups, and braid groups, are closely related to Coxeter groups, and have relevance in many aspects of geometric group theory and symplectic and algebraic geometry. Despite their seemingly simple descriptions, Artin groups remain in general poorly understood, and fundamental questions, like the question of whether all Artin groups are torsion-free and admit compact classifying spaces (the "$K(\pi, 1)$ conjecture"), or whether they have solvable word problem, are yet to be fully resolved.

In this essay, the student will define Artin groups and survey their connection to Coxeter groups and hyperplane complements, describe their associated Deligne and Salvetti complexes, and sketch, via examples, Ven der Lek's proof that ever Artin group arises as the fundamental group of a certain quotient of a hyperplane complement by the action of its associated Coxeter group. Further topics could include a discussion of Artin monoids, an account of the solution to the word problem for extra-large type Artin groups due to Appel and Schupp, an account of the solution of the Tits' conjecture due to Crisp and Paris, or an account of the proof of the $K(\pi, 1)$ conjecture for FC-type Artin groups, due to Charney and Davis.

Relevant Courses

Essential: IA Groups, II Algebraic Topology

Useful: IB Geometry, IB Groups, Rings and Modules, III Geometric Group Theory, III Coxeter Groups

References

1. K. I. Appel and P. E. Schupp, Artin groups and infinite Coxeter groups, In¨ent. Math. 72 Ž . 1983 , 201]220.

2. Charney, R. (2016). Problems related to Artin groups. American Institute of Mathematics.

3. R. Charney and M. Davis, The $K(\pi, 1)$-problem for hyperplane complements associated to infinite reflection groups, J. Amer. Math. Soc. 8 (1995), 597–627.

4. R. Charney and M. Davis, Finite $K(\pi, 1)$'s for Artin groups, In: F. Quinn (ed.), Prospects in Topology, Ann. of Math. Stud. 138, Princeton Univ. Press, Princeton, 1995, pp. 110–124.

5. Crisp, J., Paris, L. The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group. Invent. math. 145, 19–36 (2001). https://doi.org/10.1007/s002220100138

6. H. van der Lek, The Homotopy Type of Complex Hyperplane Complements, Ph.D. Thesis, University of Nijmegen (1983)

20. SYZ Mirror Symmetry .... Professor A. M. Keating

Mirror symmetry is a deep geometric phenomenon with roots in string theory. Loosely speaking, it consists of a range of dualities between pairs of manifolds, say X and $X^\vee$, equipped with different additional structures. The spaces X and $X^\vee$ are often objects of symplectic or algebraic geometry; mirror symmetry has grown into one of the guiding paradigms in those fields. Striking early evidence for mirror symmetry was a landmark paper by string theorists Candelas, de la Ossa, Green and Parkes, for the case where X is the quintic threefold in complex projective four-space. This gave a prediction for certain invariants of X (called counts of holomorphic curves) by calculating some very different invariants for $X^\vee$ (namely period integrals).

One formulation of mirror symmetry, originally due to Strominger-Yau-Zaslow (SYZ), is a duality between singular torus fibrations on X and $X^\vee$. In the case where X is equipped with a symplectic structure (i.e. a nowhere degenerate closed 2-form), the torus fibres are usually taken to be Lagrangian (i.e. half dimensional, such that the symplectic form vanishes). The goal of the essay is to explore SYZ mirror symmetry from this view-point.

The essay should start with an introduction to Lagrangian torus fibrations, including an account of the Arnol'd–Liouville theorem and Duistermaat's result on global action-angle coordiates [1, 2, 6]. It should proceed with a discussion of the 2-dimensional setting (i.e. when smooth fibres are 2-tori); in particular, when the base of the fibration is a topological disc, and there are only isolated, nodal singular fibres, it should explain why X and $X^\vee$ are diffeomorphic [1, 5, 6]. Finally, there should be a substantial discussion of some 3-dimensional cases; this could include the two dual local models for Y-shaped singular loci [4, 3], and / or Gross' constructions of topological torus fibrations on X and its dual when X is the quintic threefold [5].

Relevant Courses

Essential: Part III Algebraic Topology, Part III Differential Geometry

Useful: Part III Algebraic Geometry, Part III Symplectic Topology

References

1. J. J. Duistermaat, On global action-angle coordinates, *Comm. Pure Appl. Math. 33 (1980), no.6, 687–706*
2. J. D. Evans, Lectures on Lagrangian torus fibrations, *London Math. Soc. Stud. Texts, 105*
3. J. D. Evans and M. Mauri, Constructing local models for Lagrangian torus fibrations, *Ann. H. Lebesgue 4 (2021), 537–570*
4. M. Gross, Special Lagrangian fibrations. I. Topology, *Integrable systems and algebraic geometry (Kobe / Kyoto, 1997), 156–193.*
5. M. Gross, Topological mirror symmetry, *Invent. Math. 144 (2001), no.1, 75–137*
6. M. Symington, Four dimensions from two in symplectic topology, *Proc. Sympos. Pure Math., 71 (2003)*

21. The Hodge Decomposition Theorem ....

Dr A. G. Kovalev

The concept of Laplace operator $\Delta = -(\partial/\partial x_1)^2 - \ldots - (\partial/\partial x_n)^2$ for functions on the Euclidean space $\mathbf{R}^n$ can be extended to oriented Riemannian manifolds. The construction uses a certain Hodge star 'duality' operator, and the resulting Laplace–Beltrami operator (or Hodge Laplacian) is well-defined on differential forms. The celebrated Hodge decomposition theorem implies a natural isomorphism between the kernel of this Laplacian (i.e. the space of harmonic differential forms) and the de Rham cohomology, for a compact oriented manifold without boundary [1]. This theory admits nice extensions to compact manifolds with boundary [2] and to non-compact

Riemannian manifolds with tubular ends [3]. The essay could explore some of the latter results. Interested candidates are welcome to contact me (`A.G.Kovalev@dpmms`) and discuss further; section 3.5 of `http://www.dpmms.cam.ac.uk/~agk22/riem1.pdf` could be a good preliminary reading.

Relevant Courses

Essential: Differential Geometry

Useful: Algebraic Topology

References

1. F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer 1983. Chapter 6.

2. S. Cappell, D. DeTurck, H. Gluck, E. Y. Miller, *Cohomology of harmonic forms on Riemannian manifolds with boundary*, arxiv:math.DG/0508372 or Forum Math. **18** (2006), 923–931.

3. R.B. Lockhart, *Fredholm, Hodge and Liouville theorems on noncompact manifolds*, Trans. Amer. Math. Soc. **301** (1987), 1–35.

22. Yau's Solution of the Calabi Conjecture .... Dr A. G. Kovalev

The subject area of this essay is compact Kähler manifolds. Very informally, a Kähler manifold is a complex manifold admitting a metric and a symplectic form, both nicely compatible with the complex structure. The Ricci curvature of a Kähler manifold may be equivalently expressed as a differential form which is necessarily closed. Furthermore, the cohomology class defined by this form depends only on the complex manifold, but not on the choice of a Kähler metric. The Calabi conjecture determines which differential forms on a compact complex manifold can be realized by Ricci forms of some Kähler metric. Substantial progress on the conjecture was made by Aubin and it was eventually proved by Yau. This result gives, among other things, a powerful way to find many examples of Ricci-flat manifolds. The essay could discuss aspects of the proof and possibly consider some applications and examples. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` for further details.

Relevant Courses

Essential: Differential Geometry, Complex Manifolds

Useful: Algebraic Topology, Elliptic Partial Differential Equations

References

1. D. Joyce, *Riemannian holonomy groups and calibrated geometry*, OUP 2007. Chapters 6 and 7.

2. S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.*, **31** (1978), 339–411.

3. a good text on Kähler complex manifolds, e.g. D. Huybrechts, *Complex geometry. An introduction.* Springer 2005.

23. Manifolds with Non-Negative Scalar Curvature .... Dr P. Minter

A classical question in Riemannian geometry asks how the geometry of a closed Riemannian manifold (M^n, g) influences its topology and vice versa. A well-known instance of such a result is the Gauss–Bonnet theorem stating that, when $n = 2$,

$$\int_M K \, \mathrm{d}A = 2\pi\chi(M),$$

where K is the Gauss curvature of (M, g) and $\chi(M)$ is the Euler characteristic of M. In particular, a closed 2-manifold admitting a metric of positive Gauss curvature is diffeomorphic to a sphere. Over the years the interaction between geometry and topology in higher dimensional manifolds has been intensely studied under various curvature assumptions of varying strengths. One of the weakest invariants one can consider is the scalar curvature of (M, g), namely the sum of all possible sectional curvatures at a given point of M. In [1] it was shown that there is no topological obstruction to a compact manifold M^n, $n \geq 3$, carrying a metric with *negative* scalar curvature, even if just on a small set. A natural question is then: *is there any topological obstruction for M to admit a metric with non-negative scalar curvature?*

Over the years numerous obstructions have been found using two seemingly unrelated methods: the index theory of Dirac operators, and variational methods utilising geometric measure theory. The aim of this essay is to investigate results proven using the latter method in the influential work by Schoen and Yau [2, 3], including the resolution of the *Geroch Conjecture* when $n \leq 7$: *the n-torus does not admit a metric with positive scalar curvature.* The essay should provide a detailed proof of the main results in [3, Section 1]. A good essay should also overview [1, Theorem 1.1] on the existence of metrics with negative scalar curvature. An ambitious essay can also pursue other developments, including a contrast with the methods surrounding the Dirac operator [4], or more recent developments utilising Gromov's μ-bubbles [5].

Relevant Courses

Essential: Differential Geometry

Useful: Some background in functional analysis, such as Analysis of Functions (Part II)

References

1. J. Kazdan & F. Warner, *Scalar curvature and conformal deformation of Riemannian structure*, Journal of Differential Geometry, 1975.

2. R. Schoen & S.T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Annals of Mathematics, 1979.

3. R. Schoen & S.T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Mathematica, 1979.

4. M. Gromov & H. Lawson, *Spin and scalar curvature in the presence of a fundamental group. I*, Annals of Mathematics, 1980.

5. O. Chodosh & C. Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, Annals of Mathematics, 2024.

24. Geometric Applications of Stable Currents .... Dr P. Minter

Techniques from the calculus of variations are one of the key tools used to probe Riemannian manifolds. One first sees this with the notion of a *geodesic*, namely critical points of the length functional. Using geodesics one can prove many interesting results, such as:

(i) *Bonnet–Myers' theorem*: if (M^n, g) is a complete, connected Riemannian manifold with Ricci curvature bounded below by $\frac{n-1}{r^2}$ for some $r > 0$, then the diameter of M is $\leq \pi r$.

(ii) *Synge's theorem*: if (M^{2n}, g) is a closed, orientable, even-dimensional Riemannian manifold with positive sectional curvature, then M is simply connected.

The basic idea behind these proofs is to capitalise on the curvature terms appearing within the second variation of the length functional to reach the desired conclusion. Geodesics are, however, insufficient for many problems, in particular those pertaining to higher dimensional homology and homotopy classes. For these, one can attempt to use higher dimensional analogues of geodesics, known as *minimal submanifolds.* The issue is that, unlike in the case of curves, elements of higher homology classes cannot always be represented by *smooth* minimal submanifolds, and one needs to work within a larger class of 'generalised' or 'weak' submanifolds, known as *currents.* This need to work with 'weak' objects is analogous to the need to work with Sobolev spaces in PDE theory.

One can then attempt to exploit the second variation for area for these minimal currents in the same way as for geodesics, however things are significantly more complicated. In certain instances, this approach has yielded interesting geometric consequences. Work by Lawson and Simons [2] showed that manifolds immersed within spheres with certain constraints on the extrinsic geometry must be homotopy spheres. They also provide a variational characterisation for C^1 submanifolds of $\mathbb{CP}^n$ to be complex submanifolds.

The essay should begin with the proof of Synge's theorem (following e.g. [1]). Then, the essay should establish the main theorems in [2], providing an exposition of the relevant background on currents (see also [3]). A good essay should also provide a discussion on the inability to represent homology classes by smooth submanifolds (let alone minimal submanifolds) and thus the need for singular representatives. An ambitious essay can also investigate further the various existence results from geometric measure theory, such as the Federer–Fleming compactness theory [4].

Relevant Courses

Essential: Differential Geometry, Measure Theory

Useful: Some knowledge of Analysis of PDEs, in particular the concept of a *weak solution*, would be useful.

References

1. P. Peterson, *Riemannian Geometry*, Graduate Texts in Mathematics, Springer

2. H. Lawson & J. Simons, *On stable currents and their application to global problems in real and complex geometry*, Annals of Mathematics, 1973.

3. L. Simon, *Lectures on Geometric Measure Theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University.

4. H. Federer & W. Fleming, *Normal and integral currents*, Annals of Mathematics, 1960.

25. Regularity of Solutions to the One-Phase Free Boundary Problem ..... Dr P. Minter

Free boundary problems are a class of variational problems used to model heat flow and fluid interfaces. The simplest example, known as the *one-phase free boundary problem* or *Alt–Caffarelli problem*, focuses on the simplest possible case where the solutions $u : B_1(0) \to [0, \infty)$ are critical points of the energy

$$E(u) := \int_{B_1(0)} |Du(x)|^2 + \mathbf{1}_{\{u>0\}}(x)\,\mathrm{d}x,$$

where $B_1(0) \subset \mathbb{R}^n$ is the unit ball centred at the origin and $\mathbf{1}_A$ denotes the indicator function of the set A. One can derive the Euler–Lagrange equation for critical points to see that they obey $\Delta u = 0$ on the set $\{u > 0\}$, and $|Du| = 1$ on $B_1(0) \cap \partial\{u > 0\}$. This boundary condition is on a set which depends on the solution u, giving rise to the name "free boundary problem".

Existence of minimisers of the Alt–Caffarelli problem is possible through standard methods in PDE theory. The question left concerns the regularity of these minimisers at the free boundary $\partial\{u > 0\}$ (the regularity in the interior $\{u > 0\}$ is immediate from the regularity of harmonic functions). To date, it is known that the set of possible singular points has codimension *at least* 5, and in general *at most* 7 in the free boundary. It is also known to be an integer, which leaves the possible codimension lying in $\{5, 6, 7\}$; determining its exact value remains a very interesting open question. The method to establish these bounds is based on classifying model solutions, which turn out to be those which are homogeneous of degree one.

The aim of this essay is to investigate the problem of the optimal dimension of the singular set in the Alt–Caffarelli problem. The initial reference for this is the original work of Alt and Caffarelli [1], which established existence of minimisers as well as regularity in the two-dimensional setting. The work of Caffarelli, Jerison, and Kenig [2] then established that the singular set has codimension at least 4, which was then followed by the work of Jerison and Savin [3] to get this to codimension at least 5. Work of De Silva and Jerison [4] then provided an example to illustrate that the best one can do was codimension 7. A good essay should provide an overview of the existence theory and a lower bound of 5 on the codimension of the singular set. An ambitious essay can also examine the upper bound of 7 on the codimension as well as the differences between minimisers and critical points.

Relevant Courses

Essential: Analysis of PDEs, Elliptic PDEs

Useful: Some knowledge of Differential Geometry would be useful (e.g. mean curvature).

References

1. H. Alt & L. Caffarelli, *Existence and regularity for a minimum problem with free boundary*, 1981.

2. L. Caffarelli, D. Jersion, & C. Kenig, *Global energy minimizers for free boundary problems and full regularity in three dimensions*, Contemporary Mathematics, 2004.

3. D. Jerison & O. Savin, *Some remarks on stability of cones for the one-phase free boundary problem*, Geometric and Functional Analysis, 2015.

4. D. De Silva & D. Jerison, *A singular energy minimizing free boundary*, Journal für die reine und angewandte Mathematik, 2009.

26. The Local Properties of Diffusion on Manifolds .... Dr R. Gross

The heat kernel is the fundamental solution to the heat equation

$$\frac{\partial}{\partial t} f + \Delta f = 0.$$

Among other things, it can be utilized to solve boundary-value problems. In $\mathbb{R}^d$, it takes the form

$$p^{\mathbb{R}^d}(t, x, y) = \frac{1}{(4\pi t)^{d/2}} e^{-\|x-y\|^2/4t}.$$

Incidentally (or not), this is also the probability density of finding Brownian motion at position y at time t, given that it started at position x at time 0.

The heat equation can also be defined on a general Riemannian manifold M, using the Laplace-Beltrami operator; the corresponding heat kernel $p^M(t, x, y)$ then describes the transition density of Brownian motion on the manifold. In this case, the heat kernel rarely has an explicit closed form. However, since manifolds locally look like Euclidean space $\mathbb{R}^d$, and since Brownian motion does not move very far away in a small time t, we can expect that for small times, $p^M(t, x, y) \approx p^{\mathbb{R}^d}(t, x, y)$. One example which makes this intuition precise is the Minakshisundaram–Pleijel expansion:

$$p^M(t, x, x) \sim \frac{c_0(x)}{t^{d/2}} + \frac{c_1(x)}{t^{d/2-1}} + \dots$$

as $t \to 0$. The constants $c_i(x)$ tell us something about the local geometry and curvature of the manifold M; integrating over all $x \in M$ then gives a relation between some global constants of the manifold to the heat kernel trace $\int_M p^M(t, x, x)\, dx$.

The goal of this essay is to understand the small-time asymptotics of the heat kernel for compact manifolds, and their relation to the local and global geometry of the manifold. It is possible to explore the relation to diffusion processes and other operators.

Relevant Courses

Essential: Differential Geometry

Useful: Advanced Probability

References

1. M. R. Levitin, D. Mangoubi and I. Polterovich, *Topics in spectral geometry*, Graduate Studies in Mathematics, 237, Amer. Math. Soc., Providence, RI, [2023] ©2023; MR4655924

2. H. P. McKean Jr. and I. M. Singer, Curvature and the eigenvalues of the Laplacian, J. Differential Geometry **1** (1967), no. 1, 43–69; MR0217739

3. S. A. Molchanov, Diffusion processes and Riemannian geometry, Uspehi Mat. Nauk **30** (1975), no. 1(181), 3–59; MR0413289

27. The Madsen–Weiss Theorem .... Professor O. Randal-Williams

This essay concerns moduli spaces of oriented surfaces of some fixed genus. A good model for this is the space of all subsurfaces of $\mathbb{R}^\infty$ having that genus, where the topology is such that the surfaces can move around in the way your intuition says they should. The Madsen–Weiss theorem, proved in 2002 [4], concerns the topology of this space as the genus of the surfaces is allowed to tend to infinity. More precisely, it identifies the limiting (co)homology of this sequence of spaces.

This aim of this essay is to study the proof of this theorem, following [2]. It should begin with some background in the differential topology of spaces of smooth functions, embeddings, and diffeomorphisms. It should then describe spaces of non-compact manifolds, cobordism categories, and their relation, and identify the "space of long manifolds" with the infinite loop space of a Thom spectrum. Finally, specialising to the case of surfaces, it should explain parameterised surgery, the statement and application of the group-completion theorem [5], and the calculation of the rational cohomology of the associated infinite loop space. There are several sources of notes, such as [1, 3], from which to obtain different perspectives.

Relevant Courses

Essential: Part III Algebraic Topology

References

1. S. Galatius, *Lectures on the Madsen-Weiss Theorem.* In Moduli Spaces of Riemann Surfaces, eds. B. Farb, R. Hain, E. Looijenga. IAS/Park City Mathematics Series 20 (2013). `http://web.math.ku.dk/~wahl/Galatius.ParkCity.pdf`

2. S. Galatius, O. Randal-Williams, *Monoids of moduli spaces of manifolds.* Geometry & Topology 14 (2010) 1243–1302.

3. A. Hatcher, *An exposition of the Madsen–Weiss theorem.* `https://pi.math.cornell.edu/~hatcher/Papers/MW.pdf`

4. I. Madsen, M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture.* Ann. of Math. (2) 165 (2007), 843–941.

5. D. McDuff, G. Segal, *Homology Fibrations and the "Group-Completion" Theorem.* Invent. math. 31 (3) (1976) 279–284.

28. Stable Homotopy Groups of Spheres .... Professor O. Randal-Williams

One of the basic and motivating problems in algebraic topology is to understand the homotopy groups of spheres $\pi_{n+k}(S^k)$, meaning the continuous maps $S^{n+k} \to S^k$ up to deformation. The problem can be simplified by taking a limit as $k \to \infty$ to obtain a sequence π_n^{s} of abelian groups. The first few of these are

$$\mathbb{Z} \quad \mathbb{Z}/2 \quad \mathbb{Z}/2 \quad \mathbb{Z}/24 \quad 0 \quad 0 \quad \mathbb{Z}/2 \quad \mathbb{Z}/240 \quad \mathbb{Z}/2 \oplus \mathbb{Z}/2 \quad \mathbb{Z}/2 \oplus \mathbb{Z}/4 \quad \mathbb{Z}/6 \quad \mathbb{Z}/504 \quad \cdots$$

and no particular pattern is visible.

This essay will be an introduction to the main tool for analysing these groups, the *Adams spectral sequence*. This applies one prime at a time, and at the prime 2 has the form

$$\mathrm{Ext}_A^s(\mathbb{F}_2, \mathbb{F}_2)_t \Longrightarrow {}_{(2)}\pi_{t-s}^{\mathrm{s}}.$$

Your essay should begin by describing the Steenrod algebra and its properties, a brief introduction to spectra, and then constructing the Adams spectral sequence. It should also discuss the multiplicative structure on this spectral sequence. References include parts of [1, 2, 3].

There are two aspects to using this tool. The first is computing the left-hand side, which is the *cohomology of the Steenrod algebra* and is algorithmic, but is not easy. You should explain how to compute it for $t - s \leq 5$ by hand, and may then use various pieces of software [4, 5] for computing more of it.

The second is the behaviour of the spectral sequence, as the left-hand side is only an "upper bound" to the answer. You will see that for $t-s \leq 13$ the behaviour can be formally determined, letting you compute ${}_{(2)}\pi_n^{\mathrm{s}}$ for $n \leq 13$, but at $t - s = 14$ ambiguities begin to arise. Your essay should discuss in depth methods to resolve such ambiguities: this will be a significant proportion of the essay. There are many ways to do so, which you should discuss in detail with me.

Relevant Courses

Essential: Part III Algebraic Topology

Useful: Part III Simplicial Homotopy Theory

References

1. D. Ravenel *Complex cobordism and stable homotopy groups of spheres.* Pure Appl. Math., 121 Academic Press, Inc., 1986. xx+413 pp.

2. J. Rognes, `https://www.mn.uio.no/math/personer/vit/rognes/papers/notes.050612.pdf`

3. J. McCleary, *A user's guide to spectral sequences.* Second edition, Cambridge Stud. Adv. Math., 58 CUP, 2001. xvi+561 pp.

4. R. Bruner's Ext program. `http://www.rrb.wayne.edu/papers/index.html`

5. `https://github.com/SpectralSequences/sseq`

29. Barcodes .... Professor I. Smith

Singular homology arises from a chain complex. The Morse homology of a manifold with a Morse function is naturally a *filtered* chain complex; such filtered complexes also arise extensively in symplectic geometry, where the classical action functional plays the role of the Morse function. A 'barcode' (or persistence module) is a combinatorial way of encoding such a filtered chain complex. The filtration gives rise to additional invariants ('spectral invariants', 'boundary depth'), which go beyond homology, and which are often of dynamical importance. The essay will summarise the basic theory of barcodes and then explore a symplectic application, e.g. to numbers of periodic points of Hamiltonian diffeomorphisms, or to applications to Hamiltonian homeomorphisms using the 'continuity' of barcodes.

Relevant Courses

Essential: Algebraic Topology.

Useful: Differential Geometry. Symplectic Topology (non-examinable).

References

1. *Framed Morse complex and its invariants.* S. Barranikov, Adv. Soviet. Math. 21 [available online]
2. *Persistent homology and Floer-Novikov theory*, M. Usher and J. Zhang. arXiv:1502.07928.
3. *Topological persistence in geometry and analysis.* L. Polterovich et al, AMS Univ. Lecture Series, 2020.
4. *Autonomous Hamiltonian flows, Hofer's geometry and persistence modules.* L. Polterovich, E. Shelukhin. arXiv:1412.8277.
5. *Barcodes and area-preserving homeomorphisms.* F. Le Roux, S. Seyfaddini and C. Viterbo. arXiv:1810.03139.
6. *On the Hofer-Zehnder conjecture.* E. Shelukhin, arXiv:1905.04769.

30. String Topology .... Professor I. Smith

If M is a smooth manifold, the homology of the free loop space $H_*(\mathcal{L}X)$ carries a rich algebraic structure, including a product and a Lie bracket; this package goes broadly by the name of 'string topology'. Much but not all of the structure is preserved by homotopy equivalences, even though the definitions are simplified by appealing to the smooth structure on the manifold. This essay will discuss some of this structure, including computations in some basic examples (spheres, oriented surfaces and the 'Goldman bracket'). It will then explore connections to at least one other area: symplectic topology, where the string topology package can be used to constrain Lagrangian embeddings and can also be generalised to the symplectic cohomology of any Weinstein manifold (e.g. any smooth complex affine variety); or to Hochschild homology, where there are purely algebraic analogues of the string product and coproduct.

Relevant Courses

Essential: Algebraic Topology

Useful: Symplectic Topology

References

1. *String topology.* M. Chas, D. Sullivan. arXiv:9911159.
2. *String topology in three flavours.* F. Naef, M. Rivera, N. Wahl. arXiv:2203.02429.
3. *An elementary proof of the string topology structure of compact oriented surfaces.* A. Kupers, arXiv:1110.1158.
4. *A biased view of symplectic cohomology.* P. Seidel, arXiv:0704.2055.
5. *Fukaya's work on Lagrangian embeddings.* J. Latschev, arXiv:1409.6474.

31. Graph Complexes and Moduli Spaces of Tropical Curves .... Dr J. Steinebrunner

Kontsevich's graph complex GC is a chain complex over $\mathbb{Q}$ roughly given by

$$\cdots \longrightarrow \mathbb{Q}\left\langle \begin{matrix}\text{connected graphs}\\ \text{with } n+1 \text{ edges}\end{matrix} \right\rangle \longrightarrow \mathbb{Q}\left\langle \begin{matrix}\text{connected graphs}\\ \text{with } n \text{ edges}\end{matrix} \right\rangle \longrightarrow \mathbb{Q}\left\langle \begin{matrix}\text{connected graphs}\\ \text{with } n-1 \text{ edges}\end{matrix} \right\rangle \longrightarrow \cdots$$

where the differential sends a graph G to the alternating sum of all graphs G/e that can be obtained from G by collapsing a non-loop edge $e \subset G$. More care with signs and graph symmetries is needed to make this precise, but the definition remains elementary. Nevertheless, little is known about the homology of GC.

The goal of this essay would be twofold:
Firstly, to establish an isomorphism (following [1, sections 2-4])

$$H_*(\mathrm{GC}) \cong \bigoplus_{g\geq 2} H_*(\Delta_g; \mathbb{Q})$$

between the homology of GC and the homology of the moduli spaces of tropical curves Δ_g, which appear in the study of families of complex curves in algebraic geometry, but can also be defined topologically. The above isomorphism thus describes a geometric object in combinatorial terms, making it an effective tool for computations.

Secondly, to establish an additional differential on GC and to deduce from this the existence of a spectral sequence, starting from $H_*(\mathrm{GC})$ and converging to almost 0 (following [2]). This will involve learning what a spectral sequence is (an indispensable tool in algebraic topology) and what it means for it to converge. From this spectral sequence one can deduce that there must be higher degree classes in the homology of GC (and thus of Δ_g) to cancel certain easy to construct "loop classes" for $g = 1$. It would also be interesting to understand explicitly how the "wheel classes" interact with the differential [2, appendix B].

Relevant Courses

Essential: Part III Algebraic Topology

References

1. *Tropical curves, graph complexes, and top weight cohomology of $\mathcal{M}_g$*, Melody Chan, Søren Galatius, and Sam Payne, J. Amer. Math. Soc. 34 (2021), 565-594.

2. *Differentials on graph complexes*, Anton Khoroshkin, Thomas Willwacher, and Marko Živković, Adv. Math. 307 (2017), 1184-1214.

32. Rigidity Theorems for Hyperbolic Groups .... Professor H. Wilton

A finitely generated group is called *word-hyperbolic* if triangles in its Cayley graph are uniformly thin. This condition defines a vast class of groups, first introduced by Gromov [5], and enables the geometric techniques developed by Thurston when studying hyperbolic 3-manifolds to be applied in a much wider setting.

The idea of the essay is to explore rigidity theorems for hyperbolic groups. A typical result is Paulin's theorem, which says that the outer automorphism group of a torsion-free hyperbolic group Γ is infinite if and only if Γ splits as an amalgamated free product or HNN extension over a cyclic subgroup [3, 6].

A successful essay, after describing some of the basic theory of hyperbolic groups [4, Chapters III.H and III.Γ], will explain the basic strategy used to prove theorems like Paulin's theorem (sometimes called the *Bestvina–Paulin method*): if rigidity fails (in this case, if the outer automorphism group is infinite), then one can apply a limiting argument to extract an action on an $\mathbb{R}$-tree [1]; one then applies Rips' classification of actions on $\mathbb{R}$-trees [2] to deduce a contradiction. More advanced essays will go into other results in the same vein, such as Rips–Sela's proof that rigid hyperbolic groups are co-Hopfian [7].

Relevant Courses

Essential: Part III Geometric Group Theory

References

1. Mladen Bestvina. $\mathbb{R}$-trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002.
`http://www.math.utah.edu/~bestvina/eprints/handbook.ps`

2. Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. *Inventiones Mathematicae*, 121(2):287—321, 1995.

3. M. R. Bridson and G. A. Swarup. On Hausdorff–Gromov convergence and a theorem of Paulin. *Enseign. Math. (2)*, 40(3-4):267–289, 1994.

4. Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin, 1999.

5. M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.

6. Frédéric Paulin. Outer automorphisms of hyperbolic groups and small actions on **R**-trees. In *Arboreal group theory (Berkeley, CA, 1988)*, volume 19 of *Math. Sci. Res. Inst. Publ.*, pages 331–343. Springer, New York, 1991.

7. E. Rips and Z. Sela. Structure and rigidity in hyperbolic groups. I. *Geometric and Functional Analysis*, 4(3):337–371, 1994.

33. Locally Presentable and Accessible Categories .... Professor P. T. Johnstone

Locally presentable categories were introduced by Gabriel and Ulmer [1,2], and were an early attempt to capture the essential categorical structure of the category of models of a theory. The fact that they succeeded in doing just this, for a particular (very natural) class of 'essentially algebraic' theories, was proved by M. Coste [3]. More recently, attention has focused on the much larger class of accessible categories [4,5], which are categories of models of theories in a much broader sense; locally presentable categories are precisely those accessible categories which are complete as categories. An essay on this topic could either take as its goal the main theorem characterizing accessible categories as categories of models, or it could survey the way in which particular properties of the axiomatization of a theory are reflected in properties of its category of models. (Some examples of the latter may be found in [6].)

Relevant Courses

Essential: Category Theory

References

1. F. Ulmer, Locally α-presentable and locally α-generated categories, in *Reports of the Midwest Category Seminar V*, Lecture Notes in Math. vol. 195 (Springer–Verlag, 1971), 230–247. (This is a summary in English of the main results of [2].)

2. P. Gabriel and F. Ulmer, *Lokal präsentierbare Kategorien*, Lecture Notes in Math. vol. 221 (Springer–Verlag, 1971).

3. M. Coste, Localisation, spectra and sheaf representation, in *Applications of Sheaves*, Lecture Notes in Math. vol. 753 (Springer–Verlag, 1979), 212–238.

4. M. Makkai and R. Paré, *Accessible Categories: the Foundations of Categorical Model Theory*, Contemporary Math. vol. 104 (Amer. Math. Soc., 1989).

5. J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, L.M.S. Lecture Notes Series no. 189 (C.U.P., 1994).

6. P.T. Johnstone, *Sketches of an Elephant: a Topos Theory Compendium*, chapters D1–2, Oxford Logic Guides 44 (O.U.P., 2002), 861–889.

34. Large Cardinals .... Professor B. Löwe

A cardinal κ is called *regular* if every unbounded subset of κ has cardinality κ; successor cardinals, i.e., cardinals of the form $\aleph_{\alpha+1}$, are always regular; the usual limit cardinals, e.g., $\aleph_\omega$, $\aleph_{\omega+\omega}$, or $\aleph_{\omega_1}$, are not. Thus, the following is a natural question:

"Are there any uncountable regular limit cardinals?".

If they exist, they must be very large. It turns out that this question is intricately connected with the incompleteness phenomenon in set theory: if there is an uncountable regular limit cardinal, then there is a model of ZFC; therefore, ZFC is consistent, and hence (by Gödel's Second Incompleteness Theorem) ZFC cannot prove the existence of these cardinals (unless, of course, it is inconsistent).

Regular limit cardinals (a.k.a. *weakly inaccessible cardinals*) are the smallest examples of set-theoretic notions called *large cardinals*: cardinals with properties that imply that they must be very big and whose existence cannot be proved in ZFC.

The theory of these objects was the subject of a Part III course in the academical years 2021/22, 2022/23 and 2023/24 and the lecture material is available online [4].

The goal of this essay title is to pick one particular large cardinal notion (or a cluster of closely related large cardinal notions) and give an overview of what is known about it. Examples would be notions of indescribability, partition cardinals (weakly compact cardinals, Ramsey cardinals, Jónsson cardinals, Rowbottom cardinals), notions of compactness for infinitary languages (weakly and strongly compact cardinals), supercompact cardinals, huge cardinals, cardinals close to the Kunen inconsistency (I3, I2, and I1), and others, working mainly with the textbooks [1,2,3] and the papers referenced there.

Relevant Courses

Essential: Part II *Logic & Set Theory* (or equivalent), Part III *Forcing & the Continuum Hypothesis.*

Useful: Part III *Logic & Computability*, Part III *Model Theory.*

References

1. F. R. Drake. *Set theory—an introduction to large cardinals.* Studies in Logic and the Foundations of Mathematics, 76. North-Holland Publishing Co., Amsterdam, 1974.

2. T. Jech. *Set theory. The third millennium edition, revised and expanded.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

3. A. Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994.

4. B. Löwe. Large Cardinals, Part III course in Lent term 2023, University of Cambridge, website. https://www.math.uni-hamburg.de/home/loewe/Lent2023

35. Applications of Forcing .... Professor B. Löwe

The technique of forcing is the most versatile model construction technique in set theory. It was invented in 1962 by Paul Cohen to prove the independence of Cantor's *Continuum Hypothesis* from ZFC and was very quickly adopted as a method to prove a large range of independence results. Many open questions in set theory were shown to be independent from ZFC within a decade of Cohen's invention.

In the Part III lecture course *Forcing & the Continuum Hypothesis*, we develop the basic theory of forcing and prove Cohen's result. This essay title aims to look at the large number of independence results that were proved in the decade after Cohen's result.

The following are examples for a focus of the essay. The student will pick one of these and give a detailed account of the results and the underlying mathematics.

(i) The consistency of any violation of GCH consistent with monotonicity and Kőnig's Lemma: Easton's theorem [1].
(ii) The characterisation of regularity properties in the form of generics: Solovay-style characterisations and Judah-Shelah-style characterisations [2,6].
(iii) Solovay's construction of a model of ZF in which all sets are Lebesgue measurable [6].
(iv) Suslin's Hypothesis SH: Solovay-Tennenbaum theorem [7].
(v) Consistency of Martin's Axiom MA (plus $\neg\mathsf{CH}$).
(vi) Consistency of Borel's Conjecture [5].

Relevant Courses

Essential: Part II *Logic & Set Theory* (or equivalent), Part III *Forcing & the Continuum Hypothesis.*

Useful: Part III *Model Theory & Non-Classical Logic*, Part III *Model Theory.*

References

1. W. B. Easton. Powers of regular cardinals. Ann. Math. Log. 1 (1970), 139–178.

2. J. Ihoda & S. Shelah. Δ^1_2 sets of reals. Ann. Pure Appl. Log. 42 (1989), 207–223.

3. T. Jech. *Set Theory.* The Third Millenium Edition, Revised and Expanded. Springer-Verlag 2003 [Springer Monographs in Mathematics].

4. A. Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994.

5. R. Laver. On the consistency of Borel's conjecture. Acta Math. 137 (1976), 151–169.

6. R. M. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. Ann. Math. 92 (1970), 1–56.

7. R. M. Solovay & S. Tennenbaum. Iterated Cohen extensions and Souslin's problem. Ann. Math. 94 (1971), 201–245.

36. Algebraic Theories and Distributive Laws ....
Dr J. Siqueira

An *algebraic theory* is classically understood as a finitary first-order equational theory, that is, its only axioms are equalities between terms. Naturally, there are multiple languages and axiomatisations one can propose that yield the exact same models: they are to be thought as *presentations* of the same fundamental concept. Category theory brings a few ways to think of theories in a syntax-invariant manner; chiefly among them are the notions of *Lawvere theory* [1] and of *monad*. This essay examines such notions and categorical relationships between theories called *distributive laws*.

In its most classical incarnation, a distributive law of monads allows one to combine two algebraic theories into a new one: the usual distributive law of multiplication over addition combines the theory of monoids with the theory of abelian groups to form the theory of rings, for instance. More generally, a theorem by Jon Beck [2] establishes that distributive laws between monads T_1 and T_2, liftings of T_1 into T_2-algebras, and multiplications one can put onto $T_2 \circ T_1$ to make it a monad with the natural choice of unit are all mutually equivalent.

There are a few directions to choose from in this topic. The core essay will explain how the classical and categorical takes on algebraic theories relate, establish the relationship between Lawvere theories and monads [3], and prove Beck's Theorem on distributive laws. To take it further, one can then delve into higher categorical aspects of the theory and prove Street's Theorem that distributive laws correspond to monads in the 2-category of monads [4], explain the relationship with factorisation systems [5, 6], or address generalisations of distributive laws [7, 8, 9].

Relevant Courses

Essential: III Category Theory

Useful: II Logic & Set Theory (or equivalent), III Model Theory, III Logic and Computability

References

1. F. W. Lawvere. Functorial Semantics of Algebraic Theories and Some Algebraic Problems in the context of Functorial Semantics of Algebraic Theories, Reprints in Theory and Applications of Categories No. 5 (2004), pp. 1-121 (originally published as: Ph.D. thesis, Columbia University, 1963 and in Reports of the Midwest Category Seminar II, 1968, 41-61).

2. J. Beck. Distributive laws, Reprints in Theory and Applications of Categories, No. 18, 2008, pp. 1-304 (originally published in: Seminar on Triples and Categorical Homology Theory Lecture Notes in Mathematics, Volume 80, Springer-Verlag, 1969).

3. F. E. J. Linton. Some Aspects of Equational Categories. In: S. Eilenberg, D. K Harrison, S. MacLane, H. Röhrl (eds), Proceedings of the Conference on Categorical Algebra (1966). Springer, Berlin, Heidelberg. https://doi.org/10.1007/978-3-642-99902-4_3

4. R. Street. The formal theory of monads, Journal of Pure and Applied Algebra, Volume 2, Issue 2, (1972), pp. 149-168, ISSN 0022-4049, https://doi.org/10.1016/0022-4049(72)90019-9

5. R. Rosebrugh, R. J. Wood, Distributive laws and factorization, Journal of Pure and Applied Algebra 175 (2002), pp. 327-353. https://doi.org/10.1016/S0022-4049(02)00140-8

6. G. Böhm, Factorization Systems Induced by Weak Distributive Laws, Applied Categorical Structures, Volume 20 (2012), pp. 275–302. https://doi.org/10.1007/s10485-010-9243-y

7. E. Cheng, M. Hyland, J. Power, Pseudo-distributive Laws, Electronic Notes in Theoretical Computer Science, Volume 83 (2003), pp. 227-245, ISSN 1571-0661 https://doi.org/10.1016/S1571-0661(03)50012-3.

8. P. F. Faul, G. Manuell, J. Siqueira, 2-dimensional bifunctor theorems and distributive laws, Theory and Applications of Categories, Vol. 37, No. 34 (2021), pp. 1149–1175.

9. G. Lobbia, Distributive Laws for Relative Monads, Applied Categorical Structures, Volume 31, No. 19 (2023). https://doi.org/10.1007/s10485-023-09716-1

10. E. Cheng, Distributive laws for Lawvere theories, Compositionality, Volume 2 Issue 1 (2020), ISSN 2631-4444. https://doi.org/10.32408/compositionality-2-1

37. Model-Theoretic Games .... Dr J. Siqueira

A well-known result of Cantor that is repeatedly given as a problem to students in logic in several forms is that there is only one countable dense linear order without endpoints (up to isomorphism); the proof revolves around a back-and-forth argument that is a very replicable and useful technique in Model Theory.

Ehrenfeucht provided a game semantics for this sort of reasoning [1]. An Ehrenfeucht-Fraïssé game is played in turns to decide whether two structures are elementary equivalent: one player, the *Spoiler*, attempts to show that they are different, whereas their opponent, the *Duplicator*, argues that they are the same. Ehrenfeucht's Theorem relates the game configuration upon a win of the Duplicator in m rounds to the equivalence of structures built during the game with respect to sentences of quantifier-rank $\leq m$.

Games such as the Ehrenfeucht-Fraïssé game and the pebbling game provide a convenient way to perform non-definability arguments for different fragments of logic, and are particularly useful in the realm of finite Model Theory, where most of the traditional model-theoretic tools are unavailable. The essay is expected to introduce the games, explain the claims above in more detail, and prove the essential results for a game of choice. There are a few other directions a student can take to go beyond the fundamentals: those categorically minded are invited to explore the game comonads of [2] and the notion of *arboreal category* [3] meant to axiomatically capture back-and-forth systems, but those who find joy in set theory may be more interested in discussing long games with ω_1 rounds following [4], or ground an essay on [5].

A primary source for this topic (and finite model theory more broadly) is [6].

Relevant Courses

Essential: III Model Theory, II Logic & Set Theory (or equivalent)

Useful: III Forcing and the Continuum Hypothesis, III Category Theory, III Logic and Computability

References

1. A. Ehrenfeucht. An application of games to the completeness problem for formalized theories. Fundamenta Mathematicae, 49:129–141, 1961.

2. S. Abramsky, N. Shah. Relating Structure and Power: Comonadic Semantics for Computational Resources. Journal of Logic and Computation, Volume 31, Issue 6, September 2021, pp. 1390-1428, https://doi.org/10.1093/logcom/exab048

3. S. Abramsky, L. Reggio. Arboreal Categories: An Axiomatic Theory of Resources. Logical Methods in Computer Science, Volume 19, Issue 3, 2023, pp. 14:1–14:36.

4. A. Mekler, S. Shelah and J. Väänänen. The Ehrenfeucht-Fraïssé-Game of Length ω_1. Transactions of the American Mathematical Society, Vol. 339, No. 2, October 1993, pp. 567-580.

5. T. Wang, J. Väänänen. An Ehrenfeucht-Fraïssé Game for $L_{\omega_1\omega}$. Mathematical Logic Quarterly, 59 (4-5):357-370, 2013.

6. H-D. Ebbinghaus, J. Flum. Finite model theory. Second Edition. Springer Monographs in Mathematics, Springer-Verlag Berlin Heidelberg, 2005 (originally published in the series: Perspectives in Mathematical Logic, 1995). https://doi.org/10.1007/3-540-28788-4

38. Two-Descent on the Jacobian of a Hyperelliptic Curve .... Professor T. A. Fisher

The Mordell-Weil theorem states that the group of K-rational points $A(K)$ on an abelian variety A, defined over a number field K, is a finitely generated abelian group. Following the proof (a "descent calculation") gives an upper bound for the rank of $A(K)$.

This essay should begin by describing the classical "number field" or "direct" method for 2-descent on an elliptic curve. References for this include [1] and [2]. The essay should then explain how the method generalises to 2-descent on the Jacobian of a hyperelliptic curve: see [3], [4], [5], [6]. If time and space permit, the representation of 2-Selmer group elements as explicit 2-coverings could also be discussed.

Relevant Courses

Essential: Elliptic Curves

Useful: Local Fields

References

1. A. Brumer and K. Kramer, The rank of elliptic curves, *Duke Math. J.* 44 (1977), no. 4, 715–743.

2. J.W.S. Cassels, *Lectures on elliptic curves*, LMS Student Texts 24, CUP, 1991.

3. J.W.S. Cassels and E.V. Flynn, *Prolegomena to a middlebrow arithmetic of curves of genus 2*, LMS Lecture Note Series 230, CUP, 1996.

4. E.V. Flynn, B. Poonen and E.F. Schaefer, Cycles of quadratic polynomials and rational points on a genus 2 curve, *Duke Math. J.* 90 (1997), no. 3, 435–463.

5. E.F. Schaefer, 2-descent on the Jacobians of hyperelliptic curves, *J. Number Theory* 51 (1995), no. 2, 219–232.

6. M. Stoll, Implementing 2-descent for Jacobians of hyperelliptic curves, *Acta Arith.* 98 (2001), no. 3, 245–277.

39. Isogeny Volcanoes .... Professor T. A. Fisher

The isomorphism classes of elliptic curves over a finite field, together with the isogenies of fixed prime degree between them, may be described by a graph, certain components of which have been called "volcanoes" (see [1],[4],[6]). Understanding these graphs helps improve some of the algorithms (of particular interest in cryptography) for computing with elliptic curves over finite fields. The essay should develop this theory (including background on complex multiplication: see [2],[5]) and give applications to problems such as: computing endomorphism rings, computing modular polynomials, computing Hilbert class polynomials, the explicit isogeny problem [3], and computing elliptic curves with a given number of rational points. Ideally the essay should be illustrated with your own numerical examples.

Relevant Courses

Essential: Elliptic Curves

Useful: Local Fields

References

1. R. Bröker, K. Lauter and A.V. Sutherland, Modular polynomials via isogeny volcanoes, *Math. Comp.* 81 (2012), no. 278, 1201–1231.

2. D.A. Cox, *Primes of the form $x^2 + ny^2$, Fermat, class field theory and complex multiplication*, John Wiley & Sons, New York, 1989.

3. L. De Feo, C. Hugounenq, J. Plût and E. Schost, Explicit isogenies in quadratic time in any characteristic, *LMS J. Comput. Math.* 19 (2016), suppl. A, 267–282.

4. M. Fouquet and F. Morain, Isogeny volcanoes and the SEA algorithm, *Algorithmic number theory* (Sydney, 2002), 276–291, Springer, Berlin, 2002.

5. J.H. Silverman, *Advanced topics in the arithmetic of elliptic curves*, Springer-Verlag, New York, 1994.

6. A.V. Sutherland, Isogeny volcanoes, *ANTS X, Proceedings of the Tenth Algorithmic Number Theory Symposium*, 507–530, Open Book Ser., 1, Math. Sci. Publ., 2013.

40. Counting Number Fields .... Dr J. Laga

Given a transitive permutation group $G \subset S_n$, Malle's conjecture proposes asymptotics for the number of degree-n number fields $K/\mathbb{Q}$ of bounded discriminant whose Galois closure has Galois group G. For example, it predicts that the total number of degree-n number fields with $|\Delta_K| < X$ is $c_n X + o(X)$ for some constant $c_n > 0$.

Then essay should start with a precise statement of Malle's conjecture and a survey of results in the literature, using [2,3,5] as a starting point. In particular, for every transitive subgroup G of S_n for $n \leq 5$, you should include the current best upper and lower bounds for counting G-extensions of $\mathbb{Q}$.

The second part of the essay should contain a detailed proof of one of the following two known cases of Malle's conjecture:

- The $D_4 \subset S_4$ case [4], using class field theory and analytic number theory.
- The S_3 case, due to Davenport–Heilbronn, using binary cubic forms and geometry-of-numbers techniques. You should follow the more streamlined proof given in [1].

Relevant Courses

Essential: Part II Number fields or equivalent.

Useful: Local fields, Analytic Number Theory

References

1. Bhargava, Manjul and Shankar, Arul and Tsimerman, Jacob, *On the Davenport-Heilbronn theorems and second order terms*, Invent. Math. 193 (2013), no. 2, 439–499.
2. Cohen, Henri and Diaz y Diaz, Francisco and Olivier, Michel, *A survey of discriminant counting*, Algorithmic number theory (Sydney, 2002), 80–94, Lecture Notes in Comput. Sci., 2369, Springer, Berlin, 2002.
3. Cohen, Henri and Diaz y Diaz, Francisco and Olivier, Michel, *Counting discriminants of number fields*, J. Théor. Nombres Bordeaux 18 (2006), no. 3, 573–593.
4. Cohen, Henri and Diaz y Diaz, Francisco and Olivier, Michel, *Enumerating quartic dihedral extensions of* $\mathbb{Q}$, Compositio Math. 133 (2002), no. 1, 65–93.
5. Wood Matchett, Melanie, *Asymptotics for number fields and class groups*, available at `https://people.math.harvard.edu/ mmwood/Publications/ArithStats.pdf`

41. Galois Cohomology ....

Professor J. A. Thorne

Group cohomology assigns to any group G and $\mathbb{Z}[G]$-module M a series of abelian groups $H^i(G, M)$. When $G = \mathrm{Gal}(L/K)$ is the Galois group of a field extension and M is a module of arithmetic interest (for example, $M = L^\times$), these groups have arithmetic meaning, and are commonly referred to as Galois cohomology groups. When K is a number field and G is the absolute Galois group of K, determination of the Galois cohomology of the units and of the idèle class group is essentially equivalent to class field theory.

The goal of this essay will be to give an exposition of the basics of group and Galois cohomology, and the main theorems describing the Galois cohomology of local and global fields, before giving some application. Possible applications include:

(a) The proof, after Golod and Shafarevich, of the existence of infinite class field towers;

(b) or the proof, after Tate, that the truth of the Birch–Swinnerton-Dyer for a given abelian variety is invariant under isogeny.

Relevant Courses

Essential: Local Fields

Useful: Elliptic Curves

References

1. Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, *Cohomology of number fields.* Grundlehren der Mathematischen Wissenschaften, 323. Springer-Verlag, Berlin, 2008.
2. Emil Artin and John Tate, *Class field theory.* AMS Chelsea Publishing, Providence, RI, 2009.
3. J. S. Milne, *Arithmetic duality theorems.* Second edition. BookSurge, LLC, Charleston, SC, 2006.
4. John Tate, *On the conjectures of Birch and Swinnerton-Dyer and a geometric analog.* Séminaire Bourbaki, Vol. 9, Exp. No. 306, pp. 415–440, Soc. Math. France, Paris, 1995.
5. Peter Roquette, *On class field towers.* in *Algebraic Number Theory* (Proc. Instructional Conf., Brighton, 1965), pp. 162–203, Thompson, Washington, D.C., 1967.

42. Distribution of Conjugate Algebraic Integers ....

Professor P. P. Varjú

An algebraic integer is totally positive if all of its Galois conjugates are positive real numbers. Write λ_{SSS} for the smallest real number such that for all $\varepsilon > 0$, there are only finitely many totally positive algebraic integers α with

$$\mathrm{Tr}(\alpha) < (\lambda_{SSS} - \varepsilon) \deg(\alpha).$$

Here Tr stands for the field trace from $\mathbb{Q}(\alpha)$ to $\mathbb{Q}$, and SSS stands for Schur, Siegel and Smyth.

Schur proved that $e^{1/2} \leq \lambda_{SSS} \leq 2$ in 1918. The lower bound has been improved multiple times over the course of the past hundred years, but the upper bound stood as best known until the recent paper of Smith [3], where $\lambda_{SSS} < 1.89831$ is proved. The current records are $1.80203 < \lambda_{SSS} < 1.8216$. The upper bound is due to Orloski and Sardari [2], the lower bound is due to them and Smith.

The problem of estimating λ_{SSS} is closely related to the following one. Which probability measures on $\mathrm{R}_{\geq 0}$ may arise as the limiting distribution of uniform measures on the set of conjugates of totally positive algebraic integers? Indeed λ_{SSS} is the smallest value of $\int x d\mu$ with μ running through the possible limiting measures.

Smith [3] gave a description of the limiting measures by showing that all measures that satisfy certain constraints introduced earlier by Smyth may arise as a limiting distribution. Orloski and Sardari [1,2] generalized and refined Smith's result in different ways.

The essay will give an exposition of some of the results in [1,2,3].

Relevant Courses

None, but basic knowledge of Galois Theory, Complex Analysis and Measure Theory is desirable.

References

1. B. J. Orloski, N. T. Sardari, Limiting distributions of conjugate algebraic integers, arXiv:2302.02872

2. B. J. Orloski, N. T. Sardari, A quantitative converse of Fekete's theorem, arXiv:2304.10021

3. A. Smith, Algebraic integers with conjugates in a prescribed distribution, Ann. of Math. (2)200(2024), no.1, 71–122.

43. Schubert Varieties and Frobenius Splitting .... Dr R. Zhou

The geometry of flag varieties plays an important role in the study of representations of a reductive group. A basic example is given by the projective line $\mathbb{P}^1$, which is just the flag variety for the group $SL_2(\mathbb{C})$. This is equipped with an action of $SL_2(\mathbb{C})$, and the induced action on the global sections of the line bundle $\mathcal{O}(n)$ then gives rise to representations of this group.

In the 80's, Mehta and Ramanathan introduced a powerful tool to study geometric and cohomological properties of flag varieties and their Schubert subvarieties; this is the notion of a Frobenius split variety. The definition of such a variety, which only makes sense in characteristic p, is surprisingly simple, yet can be used to show the variety has many remarkable properties, eg. all the higher cohomology groups of ample line bundles are zero.

The goal of this essay is to prove that Schubert varieties associated to a reductive group in characteristic p are Frobenius split, and to derive some geometric consequences of this result. For example, a nice application would be to give a proof of the normality and Cohen–Macaulayness of Schubert varieties; results of this kind have many modern applications such as in the study of Shimura varieties. The essay could begin with a brief recall of reductive groups and the geometry of the associated flag variety; there are lots of nice examples you could play around with here. One can then move on to discuss Frobenius split varieties in a general setting, before moving on to the proof of the main result for Schubert varieties and its consequences.

Relevant Courses

Useful: Algebraic Geometry, Lie Algebras and their Representations.

References

1. *Linear Algebraic Groups,* T. Springer, Birkhäuser, Progress in Mathematics 9, (2008).

2. *Frobenius splitting and cohomology vanishing for Schubert varieties*, V. B. Mehta and A. Ramanathan, Annals of Mathematics (1985), pp. 27-40.

3. *Schubert varieties are arithmetically Cohen-Macaulay*, A. Ramanathan, Inventiones Math. (1985), pp. 283-294.

4. *Normality of Schubert Varieties,* V. B. Mehta and V. Srinivas, Amer. J. Math. (1987), pp. 987-989.

5. *Frobenius Splitting Methods in Geometry and Representation Theory*, M. Brion and S. Kumar, Birkhäuser, Progress in Mathematics 231 (2004).

44. Singular Moduli ....

Dr R. Zhou

Singular moduli are the values taken by the modular j-invariant at imaginary quadratic numbers in the upper half plane, and are intimately related to the class field theory of imaginary quadratic fields via the theory of complex multiplication (CM). In their paper *On singular moduli,* Gross and Zagier discovered a remarkable factorization property for the norm of the difference of two singular moduli, and gave two proofs of this result, an algebraic proof and an anlytic proof. The computations contained in this paper directly motivated their later breakthrough work relating heights of Heegner points to derivatives of L-series.

The aim of the essay is to give an exposition of the proof of this factorization property. The original paper of Gross–Zagier is probably still the best source to use, but will require you to learn some background on modular and elliptic curves, including reductions of elliptic curves and some CM theory. The essay should begin with some of this background, including a proof of the algebraicity of the j-invariant evaluated at imaginary quadratic numbers. One can then move on to the main part of the essay, where you should cover the proof. You could focus more on the algebraic proof or the analytic proof, or both, depending on your taste.

Relevant Courses

Useful: Local Fields, Elliptic Curves.

References

1. *On singular moduli*, B. Gross and D. Zagier, J. Reine Angew. Math. (1984).

2. *Heegner points and derivatives of L-series*, B. Gross and D. Zagier, Inventiones (1986).

3. *Advanced Topics in the Arithmetic of Elliptic Curves,* J. Silverman, Graduate Texts in Mathematics, 151.

45. Entropy and Information in Ergodic Theory ....

Professor I. Kontoyiannis

Roughly and somewhat incorrectly speaking, for the purposes of this essay an ergodic system is the most general form of a stochastic process that satisfies the strong law of large numbers. Ergodic theory has its origins in the study of abstract dynamical systems. It is mostly analytical in flavor and it also shares many tools with probability theory and information theory. The *entropy* has played a central role in the development of ergodic theory, and it has provided deep and strong connections with probability and information theory.

This essay will explore and describe the foundations of the ergodic theory of discrete sample paths as outlined in the first four sections of [1], and the connections with entropy and information theory in the rest of Chapter 1 and in Chapter 2 of [1]. The material is sections I.9 and I.10 is interesting but not essential. More ambitious essays can consider discussing material from Chapter 5 of [2], Section 14 of [3], the more recent work [4], or further papers from the recent literature after consulting with the essay setter.

Relevant Courses

Essential: Probability and Measure, Linear Analysis

Useful: Analysis of Functions, Coding and Cryptography

References

1. P.C. Shields, *The ergodic theory of discrete sample paths.* Vol. 13. American Mathematical Soc., 1996.

2. D.J. Rudolph, *Fundamentals of measurable dynamics: Ergodic theory on Lebesgue spaces.* Oxford, UK: Clarendon Press, 1990.

3. P. Billingsley, *Ergodic theory and information.* John Wiley & Sons, New York, 1965.

4. E. Lindenstrauss and M. Tsukamoto. "From rate distortion theory to metric mean dimension: variational principle." *IEEE Transactions on Information Theory*, **64**, no. 5, pp. 3590-3609, 2018.

46. The Joint Calibration Problem .... Professor M. R. Tehranchi

Many models in quantitative finance take as inputs the observed price of some given asset and an unobserved (possibly multi-dimensional) parameter, and they return the prices of contingent claims written on the given asset. For example, the Black–Scholes model predicts that the price of a European call option is a certain function of the price of the underlying asset and its volatility parameter σ.

Given a model, a practitioner must calibrate it by finding the parameter such the model reproduces, at least approximately, the observed prices of liquid contingent claims on the underlying asset.

Two assets where there exist well-calibrated models are the S&P 500 index and the VIX volatility index. However, the VIX is itself a contingent claim on the S&P 500, so in principle a single model should be able to *jointly* calibrate options on both the S&P 500 and on the VIX simultaneously.

A successful essay will survey the literature on models that can be jointly calibrated to both S&P 500 and VIX options. There is scope to explore various topics, such as the technical mathematical challenges in the more practical aspects of efficiently dealing with high dimensionality in the numerical implementation of a forward variance model.

Relevant Courses

Essential: Stochastic Calculus & Applications to Finance

Useful: Advanced Probability

References

1. Julien Guyon. The joint S&P 500/VIX smile calibration puzzle solved. *Risk Magazine*: 2020

2. Jim Gatheral, Paul Jusselin, Mathieu Rosenbaum. The quadratic rough Heston model and the joint S&P 500/VIX smile calibration problem. arXiv:2001.01789

3. Alessandro Bondi, Sergio Pulido, Simone Scott. The rough Hawkes Heston stochastic volatility model. *Mathematical Finance*: 2024

4. Sigurd Rømer. Empirical analysis of rough and classical stochastic volatility models to the SPX and VIX markets. *Quantitative Finance* 22(10): 2022

47. Points of Infinite Multiplicity of Planar Brownian Motion .... Professor W. Werner

The goal of the essay is to study the proof of the following result (and of some extensions) due to Le Gall: Consider a planar Brownian motion $(B_t, t \in [0,1])$. *There almost surely exists a (random) uncountable set M of times in $[0,1]$ such that B is at the same (random) position at all the times in M.*

One of the key concepts behind the proof is that of intersection local times between Brownian paths that will have to be presented as well.

Relevant Courses

Essential: Advanced Probability, Recommended: Stochastic Calculus

Main reference

1. Jean-François Le Gall, Some properties of planar Brownian motion, in Ecole d'été de probabilités de Saint-Flour XX, 1990, L.N. in Math. 1527, Springer – also available on https://www.imo.universite-paris-saclay.fr/∼jean-francois.le-gall/LeGallStFlour.pdf

48. Super-Brownian Motion and its Genealogical Structure(s) .... Professor W. Werner

Super-Brownian motion is a measure-valued Markov process that can be loosely speaking described as the evolution of a cloud of infenitesimal partical that move independently using Brownian displacements and undergo a constant critical branching vs. killing mechanism. It can be shhown to be closely related to solutions of nonlinear PDEs such as $\Delta u = u^2$. The goal of this essay is to present its definition, its construction via decorated continuous trees (that would correspond to its genealogy) and to explore questions such as the conditional law of the genealogical structure given the measure-valued process (questions about "indistinguishability" of particles would pop up).

Relevant Courses

Advanced Probability [Essential], Stochastic Calculus [strongly recommended], Random structures in d-dimensional space [strongly recommended]

Main reference

Jean-François Le Gall, Spatial branching processes, random snakes and partial differential equations, Lectures in Mathematics ETH Zürich, Birkhäuser, 1999 – also available on

https://www.imo.universite-paris-saclay.fr/ jean-francois.le-gall/Book-Zurich.pdf

49. Random Planar Maps and the Stable Gasket .... Dr Y. Yuan

Planar maps are (roughly speaking) graphs drawn on a plane. Random planar maps are planar maps sampled according to a probability distribution. Large random planar maps are viewed as natural models for random surfaces. Therefore it is natural to ask whether one can obtain limiting objects by rescaling large random planar maps.

For a large class of random planar maps, it has been shown that they converge to a universal object called the Brownian map (see e.g. [4]). The role of the latter can be compared to Brownian motion as a universal scaling limit of random walks. On the other hand, there are also classes of random planar maps that have heavy-tailed face degrees and converge to a different class of universal limits. (This can be compared to stable processes.) These limiting objects retain some of the large faces of the planar map. It has been shown in [1] that the laws of such random planar maps, viewed as metric spaces and properly rescaled, are tight. In a current work (see [3]), they identify the scaling limit, called stable gasket.

The goal of this essay is to outline the steps that lead to proving the tightness result [1]. In particular, a bijection between certain planar maps and a type of labeled trees plays an essential role. The student is invited to explain particular ingredients of the proof, and put the problem into context.

Relevant Courses

Essential: Advanced Probability

References

1. Jean-François Le Gall and Grégory Miermont (2011). Scaling limits of random planar maps with large faces. *Ann. Probab.*. 39(1):1–69.

2. J. Bouttier, P. Di Francesco, and E. Guitter (2004). Planar maps as labeled mobiles. *Electron. J. Combin.*. 11(1):Research Paper 69, 27.

3. Nicolas Curien, Grégory Miermont, and Armand Riera. Slides available at `https://www.cirm-math.fr/RepOrga/2528/Slides/riera.pdf`.

4. Grégory Miermont. Aspects of random maps. Available at `http://perso.ens-lyon.fr/gregory.miermont/coursSaint-Flour.pdf`

50. The Chemical Distance in Critical 2D Percolation .... Dr Y. Yuan

Consider an infinite planar graph and delete each edge (or vertex) independently with probability p. At a certain critical value for p, the remaining graph forms interesting structures. Nowadays a lot is known about the shape of the structure, however the metric structure of the graph (also called chemical distance) remains poorly understood. For instance, it is expected that the shortest length of a path crossing a rectangle of Euclidean diameter R grows like $R^{\alpha+o(1)}$ for some exponent α. However there is not even a prediction what the exact value of α might be. It has been shown in [1] that α (if it exists) has to be strictly larger than 1. More recently, it has been shown in [2, 3] that $\alpha < 4/3$.

The goal of this essay is to explain the background on this problem and some proofs of the aforementioned results.

Relevant Courses

Useful: Advanced Probability, Random Structures in Finite-dimensional Space

References

1. M. Aizenman and A. Burchard (1999). Hölder regularity and dimension bounds for random curves. *Duke Math. J.*, 99(3):419–453.

2. Michael Damron, Jack Hanson, and Philippe Sosoe (2017). On the chemical distance in critical percolation. *Electron. J. Probab.*, 22:Paper No. 75, 43.

3. Michael Damron, Jack Hanson, and Philippe Sosoe (2021). Strict inequality for the chemical distance exponent in two-dimensional critical percolation. *Comm. Pure Appl. Math.*, 74(4):679–743.

51. Convex Formulations of Finite-Width Neural Networks .... Dr S. A. Bacallado

Recent advancements in deep learning have spurred significant interest in understanding the surprising effectiveness of neural networks trained via gradient descent on non-convex loss functions. While traditional machine learning often relies on convex optimisation, the empirical success of neural networks appears to defy this paradigm. This essay delves into a burgeoning research area that aims to explain this phenomenon by exploring the intricate relationship between over-parameterised ReLU networks and convex optimisation.

The essay will focus on a recent line of work by Ergen, Pilanci, and others, which elegantly demonstrates that the global optima of over-parameterised ReLU networks can be characterised as solutions to high-dimensional convex optimisation problems. This framework, initially introduced by Ergen and Pilanci in [2] and further elaborated in [3], provides a powerful lens through which to analyse the training process of neural networks. More recent work extends the analysis to deeper networks [4] and analyses the optimality gap of randomised algorithms which approximate the original high-dimensional convex optimisation problems [5]. Finally, in [6], Pilanci employs the language of Clifford's Geometric Algebra to simplify convex reformulations of various regularised ReLU networks.

The essay should present a limited number of mathematical results from [2-6] and other relevant papers, introduce key proof ideas, and discuss the relevance of these results in the context of the broader literature, and in relation to convex formulations of neural networks in the infinite-width limit (see [1], for example). Numerical experiments characterising the approximate algorithms introduced by Pilanci et al. are encouraged.

Relevant Courses

Essential: Modern Statistical Methods.

References

1. Chizat, Lénaïc, and Francis Bach. "On the global convergence of gradient descent for over-parameterized models using optimal transport." Advances in neural information processing systems 31 (2018).

2. Pilanci, Mert, and Tolga Ergen. "Neural networks are convex regularizers: Exact polynomial-time convex optimization formulations for two-layer networks." International Conference on Machine Learning. PMLR, 2020.

3. Ergen, Tolga, and Mert Pilanci. "Convex geometry and duality of over-parameterized neural networks." Journal of machine learning research 22.212 (2021): 1-63.

4. Ergen, Tolga, and Mert Pilanci. "Global optimality beyond two layers: Training deep ReLU networks via convex programs." International Conference on Machine Learning. PMLR, 2021.

5. Kim, Sungyoon, and Mert Pilanci. "Convex Relaxations of ReLU Neural Networks Approximate Global Optima in Polynomial Time." arXiv preprint arXiv:2402.03625 (2024).

6. Pilanci, Mert. "From Complexity to Clarity: Analytical Expressions of Deep Neural Network Weights via Clifford's Geometric Algebra and Convexity." arXiv preprint arXiv:2309.16512 (2023).

52. Sampling, Diffusions, and Stochastic Localisation .... Dr S. A. Bacallado

Denoising diffusions are a successful technique for sampling from high-dimensional distributions. They operate by inverting a stochastic process that gradually adds noise to an input signal. This process can be used to sample from distributions that are either explicitly given, such as Bayesian posteriors, or learned from a collection of samples. Learning the target distribution from samples has been particularly successful in generative modelling, leading to breakthroughs in generators for complex data like images and protein structures.

Recent works have established rigorous guarantees for diffusion algorithms that generate samples from certain probability distributions. The work of El Alaoui, Montanari, and Sellke [1] presents a novel approach to sampling from the Sherrington–Kirkpatrick spin glass model, a challenging problem in statistical physics. Their work provides a practical algorithm for sampling from the Gibbs measure at high temperatures, establishing bounds on the Wasserstein distance between the sampled distribution and the true distribution. Their work is based on algorithmic versions of *stochastic localisation*, a powerful technique for proving mixing properties of Markov chains and functional inequalities in high dimensions. In [2], Montanari generalises this approach, providing several examples and useful insights. Similar techniques, combining stochastic localisation and approximate message passing, were used in [3] to obtain a practical algorithm for sampling from the posterior distribution in certain spiked models for sparse regression and low-rank matrix estimation.

Your essay should provide a brief introduction to diffusion models, synthesise the results of these papers, and highlight some key proofs. The essay should contextualise these methods in the larger literature on denoising diffusions. Numerical experiments are encouraged, but not necessary.

Relevant Courses

Helpful: Modern Statistical Methods, Advanced Probability, Mixing Times of Markov Chains.

References

1. El Alaoui, Ahmed, Andrea Montanari, and Mark Sellke. "Sampling from the Sherrington-Kirkpatrick Gibbs measure via algorithmic stochastic localization." 2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS). IEEE, 2022
2. Montanari, Andrea. "Sampling, diffusions, and stochastic localization." arXiv preprint arXiv:2305.10690 (2023).
3. Montanari, Andrea, and Yuchen Wu. "Posterior sampling from the spiked models via diffusion processes." arXiv preprint arXiv:2304.11449 (2023).

53. Statistical Inference in Interacting Particle Models .... Professor R. Nickl

Interacting particle models play a crucial role in statistical physics and kinetic theory. They can be described by a high-dimensional diffusion process coupled by an 'interaction potential'. The 'mean field limit' distribution of the interacting particle system is the solution of a non-linear PDE known as the McKean-Vlasov equation, see [2] for a classical reference.

A challenging problem in high-dimensional statistics is to identify the interaction potential based on discrete observations of the interaction system. Recent progress has been made in [1], where many further references can be found.

The purpose of this essay is to review the basic probabilistic model as well as the different approaches to conducting statistical inference on the underlying infinite-dimensional parameter.

Relevant Courses

Essential: Principles of Statistics, Probability and Measure

Useful: Advanced Probability, Stochastic Calculus

References

1. R. Nickl, G. Pavliotis, K. Ray; Bayesian Nonparametric Inference in McKean-Vlasov models, https://arxiv.org/abs/2404.16742
2. A.S. Sznitman, Topics in propagation of chaos, Springer, 1991.

54. Assumption-Lean Inference in Semiparametric Statistics .... Mr E. H. Young and Prof R. J. Samworth

The theoretical justification for the performance of many classical procedures in statistics, e.g. maximum likelihood estimators, assumes a finite-dimensional parametric model. However, in modern applications such a restrictive class may be badly misspecified, resulting in poor performance in terms of estimation accuracy and confidence interval coverage, for instance. Such issues contribute in part to the 'reproducibility crisis' in the applied sciences. This observation has led to a move towards semiparametric and nonparametric statistics ([1], [2], [3], [4], [5]), where milder assumptions on the data generating mechanism are made for theoretical results, and where methods often outperform classical approaches on real world datasets ([3], [6], [7]).

Semiparametric models ([1], [2], [3]) involve infinite-dimensional parameters (e.g. functions satisfying minimal smoothness assumptions), possibly in addition to a finite-dimensional component. Surprisingly, in a number of settings, parametric rates of convergence ([8], [9], [10], [11]), or minimax rates for function estimation ([12]) are achievable, despite the extra flexibility that these models allow.

This essay could take a number of forms. One approach would be to explore an estimand of interest in a specific semiparametric model, which could be studied either via theoretical analyses (e.g. studying optimality) and/or practical considerations on real datasets. Such estimand(s) could take the form of a regression functional ([4], [10]) or a causal parameter/function ([8], [9]). Another direction could focus on retaining valid inference under the relaxation of a specific (semi)parametric model, e.g. incorporating uncertainty in the link function used in a generalised (partially) linear model.

Relevant Courses

Useful: Topics in Statistical Theory, Modern Statistical Methods, Causal Inference

References

1. van der Vaart, A. W. (1998) *Asymptotic Statistics.* Cambridge University Press.

2. Tsiatis, A. A. (2006) *Semiparametric Theory and Missing Data.* Springer Series in Statistics. Springer, New York.

3. van der Laan, M. J. and Rose, S. (2018) *Targeted Learning in Data Science Causal Inference for Complex Longitudinal Studies.* Springer Series in Statistics. Springer, New York.

4. Vansteelandt, S. and Dukes, O. (2022) Assumption-lean inference for generalised linear model parameters. *J. Roy. Statist. Soc., Ser. B*, **84**, 657–685.

5. Samworth, R. J. and Shah, R. D. (2024+) *Modern Statistical Methods and Theory.* Cambridge University Press, to appear.

6. Lundborg, A. R., Kim, I., Shah, R. D. and Samworth, R. J. (2024) The Projected Covariance Measure for assumption-lean variable significance testing. *arXiv preprint arXiv:2211.02039v4.*

7. Young, E. H. and Shah, R. D. (2024+) Sandwich boosting for accurate estimation in partially linear models for grouped data. *J. Roy. Statist. Soc., Ser. B*, to appear.

8. Kennedy, E. H. (2022) Semiparametric doubly robust targeted double machine learning: a review. *arXiv preprint arXiv:2203.06469.*

9. Chernozhukov, V., Chetverikov, D., Demirer, M., Duflo, E., Hansen, C., Newey, W. and Robins, J. (2018) Double/debiased machine learning for treatment and structural parameters. *Econom. J.*, **21**, C1–C68.

10. Young, E. H. and Shah, R. D. (2024) Robust Semiparametric Efficient Estimation. *Forthcoming.*

11. Robins, J. M., Li, L., Mukherjee, R., Tchetgen Tchetgen, E. J. and van der Vaart, A. W. (2017) Minimax estimation of a functional on a structured high dimensional model. *Ann. Statist.*, **45**, 1951–1987.

12. Kennedy, E. H., Balakrishnan, S., Robins, J. M. and Wasserman, L. (2024) Minimax rates for heterogeneous causal effect estimation. *Ann. Statist.*, **52**, 793–816.

55. Tensor Methods in Statistics ....

Dr K. Verchand and Professor R. J. Samworth

In many modern statistical applications, it is common to collect data in multi-way comparison tables (tensors), including brain imaging ([1]), hyperspectral imaging ([2]) and recommender system design ([3]). Tensors can be thought of as generalisations of matrices to multi-way arrays, but while some concepts carry over naturally, others may be more complicated (e.g. the notion of the rank of a tensor can be defined in more than one way).

Recent years have witnessed the development of a great deal of statistical methodology and theory for tensor data, including for problems such as tensor regression ([4]), tensor PCA ([5]) and tensor completion ([6]). The theoretical analysis of such methods has required the development of new tools, for instance in tensor perturbation theory ([7]), in a similar spirit to the Davis–Kahan theorem for matrix data ([8]).

Nevertheless, a great deal of work remains to be done, for instance in developing and analysing appropriate statistical models for tensor data that go beyond ‘signal plus homoscedastic noise’, addressing computational issues ([9]) and handling missing data. After surveying the state-of-the-art in tensor methods in Statistics, a candidate may wish to explore one of the directions mentioned above in their essay.

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods

References

1. Zhou, H., Li, L., and Zhu, H. (2013) Tensor regression with applications in neuroimaging data analysis. *J.Amer. Statist. Assoc.*, **108**, 540–552.

2. Li, N. and Li, B. (2010) Tensor completion for on-board compression of hyperspectral images. *Proc. IEEE Int. Conf. Image Proc.*, 517–520.

3. Bi, X., Qu, A. and Shen, X. (2018) Multilayer tensor factorization with applications to recommender systems. *Ann. Statist.*, **46**, 3308–3333.

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7. Auddy, A. and Yuan, M. (2023) Perturbation bounds for (nearly) orthogonally decomposable tensors. *Information and Inference*, to appear.

8. Yu, Y., Wang, T. and Samworth, R. J. (2015) A useful variant of the Davis–Kahan theorem for statisticians. *Biometrika*, **102**, 315–323.

9. Wein, A. S. (2023) Average-case complexity of tensor decomposition for low-degree polynomials. In *Proceedings of the 55th Annual ACM Symposium on Theory of Computing (STOC '23).*

56. Statistical Inference with Random Forests ....
Dr W. G. Underwood and Professor R. J. Samworth

Decision trees and random forests form a broad class of procedures for statistical estimation in classification and regression settings [1, 2]. Due their recent empirical successes, interpretability and computational tractability, they have become some of the most popular general-purpose machine learning tools [3, 4], with applications in healthcare, finance, online commerce, text analysis, and bioinformatics, to name just a few. However, the development of inferential methodology for constructing confidence sets and goodness-of-fit tests [5] is still under active development. Recently proposed techniques include sample-splitting approaches such as 'honest' trees [6, 7], bootstrapping procedures [8], U-statistic central limit theorems [9], and robust bias correction [10]; many of these analyses make substantial changes to the underlying partitioning algorithms in order to permit a theoretical analysis.

One approach to this essay might involve surveying existing methods for random forest inference, comparing their relative strengths and weaknesses both in theory and in empirical studies. An ambitious essay may address potential improvements to these methods, possibly by weakening the assumptions imposed, generalising to dependent data settings, considering causal inference models, or proposing techniques for mitigating the computational burden.

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods, Statistical Learning in Practice

References

1. Breiman, L., Friedman, J., Olshen, R. A. and Stone, C. J. (1984). *Classification and Regression Trees.* Chapman and Hall/CRC, New York.

2. Breiman, L. (2001). Random forests. *Machine learning*, **45**, 5–32.

3. Biau, G. and Scornet, E. (2016). A random forest guided tour. *Test*, **25**, 197–227.

4. Chi, C. M., Vossler, P., Fan, Y. and Lv, J. (2022). Asymptotic properties of high-dimensional random forests. *The Annals of Statistics*, **50**, 3415–3438.

5. Janková, J., Shah, R. D., Bühlmann, P. and Samworth, R. J. (2020). Goodness-of-fit testing in high dimensional generalized linear models. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, **82**, 773–795.

6. Athey, S. and Imbens, G. (2016). Recursive partitioning for heterogeneous causal effects. *Proceedings of the National Academy of Sciences*, **113**, 7353–7360.

7. Wager, S. and Athey, S. (2018). Estimation and inference of heterogeneous treatment effects using random forests. *Journal of the American Statistical Association*, **113**, 1228–1242.

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57. New Advances in Conformal Prediction .... Professor R. D. Shah

Black-box machine learning methods have been hugely successful in prediction problems across a range of application areas. However, as well providing accurate predictions, often it is also important to quantify uncertainty the in our predictions. To fix ideas, consider a setting where given i.i.d. response–predictor pairs $(Y_1, X_1), \dots, (Y_n, X_n)$, we would like to construct a prediction set $\hat{C}$ based on the data, such that for a new i.i.d. pair (Y_{n+1}, X_{n+1}), $\mathbb{P}(Y_{n+1} \in \hat{C}(X_{n+1})) \geq 1 - \alpha$, for a chosen level α. Conformal prediction is a framework that given any flexible regression method, can produce such a $\hat{C}$, and remarkably, the confidence guarantee is non-asymptotic and holds regardless of the true underlying distribution of the data.

While some of the original ideas behind conformal prediction were worked out in the late 1990s, it was only much more recently that attention was drawn to this in the statistical community [1] where it has since become a topic of intense research. One line of work has aimed to modify the general procedure to yield smaller prediction sets [2,3] or understand and enhance its guarantees in terms of conditional coverage [4]. Other developments have sought to extend the approach to handle more challenging settings involving distribution shift [5,6], that is where the 'test point' (Y_{n+1}, X_{n+1}) does not follow the same distribution as the rest of the data or non-exchangeable data [7]. Conformal prediction has also found a number of applications in statistical problems that are not directly to do with prediction, such as causal inference [8], missing data [9] and the detection of outliers [10], for example.

This essay could start by reviewing some of the basic principles of conformal prediction, and then take a number of directions. One option would be to focus on the basic problem of producing prediction sets and overview some of the advances relating to efficiency of confidence sets and/or conditional coverage. An alternative would be to look at one or more of the application areas, and review some of the recent developments. For each of these options, one could aim to understand differences between different versions of the methods (for instance using different so-called conformal scores) through numerical experiments. There may also be scope for suggesting minor modifications to the procedures, or combining existing methods in new ways, though this is certainly not required.

Relevant Courses

Essential: Modern Statistical Methods

Useful: Causal Inference

References

1. J. Lei, M. G'Sell, A. Rinaldo, R. J. Tibshirani, and L. Wasserman, "Distribution-free predictive inference for regression," Journal of the American Statistical Association, vol. 113, no. 523, pp. 1094– 1111, 2018.

2. Ruiting Liang and Wanrong Zhu and Rina Foygel Barber, "Conformal prediction after efficiency-oriented model selection," https://arxiv.org/abs/2408.07066

3. Ran Xie, Rina Foygel Barber, Emmanuel J. Candès, "Boosted Conformal Prediction Intervals," https://arxiv.org/pdf/2406.07449

4. Isaac Gibbs, John J. Cherian, Emmanuel J. Candès, "Conformal Prediction With Conditional Guarantees," https://arxiv.org/abs/2305.12616

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6. Aabesh Bhattacharyya, Rina Foygel Barber, "Group-Weighted Conformal Prediction ," https://arxiv.org/abs/2401.17452

7. R. F. Barber, E. J. Candès, A. Ramdas, and R. J. Tibshirani, "Conformal prediction beyond exchangeability," arXiv:2202.13415, 2022.

8. L. Lei and E. J. Candés, "Conformal inference of counterfactuals and individual treatment effects," arXiv:2006.06138, 2020.

9. Yonghoon Lee, Edgar Dobriban, Eric Tchetgen Tchetgen, "Simultaneous Conformal Prediction of Missing Outcomes with Propensity Score ε-Discretization," https://arxiv.org/abs/2403.04613

10. Stephen Bates, Emmanuel Candès, Lihua Lei, Yaniv Romano, Matteo Sesia, "Testing for Outliers with Conformal p-values," https://arxiv.org/abs/2104.08279

58. Graphs, Causality, Time .... Professor Q. Zhao

Graphs have become an indispensable tool to model causality and statistical data measured at different times. This essay is about how these three concepts—graphs, causality, and time—could be linked in a mathematically coherent way. It is recommended that the student chooses one of the following topics and study it in detail:

1. Causal interpretations of directed acyclic graphs ("graph $\times$ causality") [1,2].

2. Cyclic graphical models ("graph × time") [3,4,5].

3. Sequential decision problems ("causality × time") [6,7,8].

Thus, the essay can explore the intersection of two of these three concepts and develop a view of this deep yet practically important area of research.

Relevant Courses

Useful: Causal Inference. Statistics in Medical Practice. Advanced Probability. Modern Statistical Methods.

References

1. J Pearl (2009). *Causality: Models, Reasoning, and Inference.* 2nd ed. Cambridge University Press. https://doi.org/10.1017/CBO9780511803161.

2. T S Richardson and J M Robins (2013). *Single World Intervention Graphs (SWIGs): A Unification of the Counterfactual and Graphical Approaches to Causality.* Techical Report 128, University of Washington. https://csss.uw.edu/files/working-papers/2013/wp128.pdf

3. S L Lauritzen and T S Richardson (2002). Chain graph models and their causal interpretations. *Journal of the Royal Statistical Society (Series B, Statistical Methodology).* https://doi.org/10.1111/1467-9868.00340

4. V Didelez (2008). Graphical models for marked point processes based on local independence. *Journal of the Royal Statistical Society (Series B, Statistical Methodology).* https://doi.org/10.1111/j.1467-9868.2007.00634.x.

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6. M A Hernán and J M Robins (2020). *Causal Inference: What If.* Chapman & Hall/CRC. https://www.hsph.harvard.edu/miguel-hernan/causal-inference-book/. [Particularly Part III of the book.]

7. B Chakraborty and E E M Moodie (2013). *Statistical Methods for Dynamic Treatment Regimes: Reinforcement Learning, Causal Inference, and Personalized Medicine.* Springer. https://doi.org/10.1007/978-1-4614-7428-9.

8. A A Tsiatis, M Davidian, S T Holloway, and E B Laber (2020). *Dynamic Treatment Regimes: Statistical Methods for Precision Medicine.* Chapman and Hall/CRC. https://doi.org/10.1201/9780429192692.

59. Experimental Design with Network Structure .... Dr P. Zhao, Professor Q. Zhao

The classical approach to identifying and estimating causal effects relies on the Stable Unit Treatment Value Assumption (SUTVA), which posits that an individual's potential outcomes are unaffected by the treatment assignments of others [1]. However, SUTVA is often violated in real-world settings, where individuals are part of complex social networks that facilitate communication, influence, or the spread of diseases. This leads to network interference, where the outcomes of individuals become interdependent [2]. In such scenarios, experimental designs—potentially data-adaptive—can help collect more relevant data efficiently and better identify causal effects of interest [3,4].

The aim of this essay is to provide an overview of common experimental designs within the framework of network interference. The essay could outline the key features of these strategies, compare their strengths and weaknesses in various settings, and explore how experiments with interference can be analyzed [5,6]. Alternatively, it may focus on new theoretical and methodological contributions, such as addressing challenges when the underlying network that mediates interference is unknown or costly to estimate [7], or exploring selective randomization inference techniques [8].

Relevant Courses

Useful: Causal Inference

References

1. Imbens GW, Rubin DB. *Causal Inference for Statistics, Social, and Biomedical Sciences: An Introduction.* Cambridge University Press; 2015.

2. Hudgens, Michael G, and M. Elizabeth Halloran. 2008. Toward Causal Inference With Interference. *Journal of the American Statistical Association*, 103 (482): 832–42.

3. Ugander, Johan and Yin, Hao. Randomized graph cluster randomization, *Journal of Causal Inference*, vol. 11, no. 1, 2023, pp. 20220014.

4. Yuchen Hu, Shuangning Li, Stefan Wager, Average direct and indirect causal effects under interference, *Biometrika*, Volume 109, Issue 4, December 2022, Pages 1165–1172.

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60. Schrödinger Representations and the Canonical Commutation Relations

Professor D. M. A. Stuart

The Heisenberg commutation relations in their simplest form read

$$[Q, P] = i\,,$$

and only have solutions as operators on an infinite dimensional vector space. In particular there is the Schrödinger representation, in which the vector space is the Hilbert space $L^2(\mathbb{R}, dx)$ and the operators are the unbounded operators given by $Qf(x) = xf(x)$ and $Pf(x) = -if'(x)$ (on appropriate dense domains). Weyl gave an alternative framework for discussing the commutation relations in terms of bounded operators, by consideration of one parameter unitary groups of operators $U(s), V(t)$ on a Hilbert space which verify

$$U(s)V(t) = e^{ist}V(t)U(s)\,.$$

It was proved by von Neumann that any representation of the Weyl relations may be constituted from a family of Schrödinger representations (with the one parameter groups U, V being generated by self-adjoint operators P, Q solving the Heisenberg relation.) However this uniqueness theorem does not hold for systems with an infinite number of degrees of freedom, in particular for quantum fields.

An essay should provide a proof of the von Neumann theorem for a finite number of degrees of freedom, and discuss its failure in the case of an infinite number of degrees of freedom. The Fock-Cook solution in the latter case could then be discussed, together with its relation to infinite dimensional Schrödinger representation provided by Gaussian measures on infinite dimensional spaces.

Relevant Courses

Essential: Quantum Field Theory, Advanced Probability, Functional Analysis

References

1. *Representations of the commutation relations* by Gårding, L. and Wightman, A., Proc. Nat. Acad. Sci. U.S.A. 40 (1954), 622–626.
2. *Gaussian Hilbert Spaces* by S. Janson, CUP (1997)
3. *Functional Analysis* by M. Reed and B. Simon, Academic Press (1980)
4. *Tensor algebras over Hilbert spaces. I* by I. Segal. Trans AMS 81 (1956) pp 106-134.
5. *Distributions in Hilbert space and canonical systems of operators* by I. Segal. Trans AMS 88 (1958) pp 12-41.

61. Spectral Methods for Spectral Computations .... Dr M. Colbrook

Computing the spectra of operators acting on infinite-dimensional spaces is fundamental in the sciences and has a wide range of applications. However, the naive approach of truncating to a matrix and computing eigenvalues can fail spectacularly. Despite this, a growing number of methods now offer rigorous convergence guarantees. The goal of this essay is to connect these methods with state-of-the-art spectral techniques, with a particular focus on the Dedalus project: http://dedalus-project.org. Spectral methods are a class of techniques used in applied mathematics and numerical analysis to solve differential equations by expanding the solution in terms of a set of basis functions, often selected for their global properties. This essay will necessarily engage with aspects of both pure and applied mathematics.

The essay can be structured in the following stages:

- **Literature Review:** A thorough review of the papers [1,2,3], along with additional literature surrounding these areas, which will also be discussed and provided.
- **Algorithm Development:** The development of algorithms that utilize sparse spectral methods to compute spectra with error control. This will involve combining the techniques presented in the above papers. There are further algorithms in additional papers that could also be addressed.
- **Applications or Examples:** There are numerous applications of this project, including black holes, resonance theory, fluid dynamics, and quantum mechanics.

Relevant Courses

Essential: Linear Analysis Courses

Useful: Numerical Analysis Courses

References

1. Colbrook, Matthew J., Bogdan Roman, and Anders C. Hansen. "How to compute spectra with error control." Physical Review Letters 122.25 (2019): 250201.
2. Colbrook, Matthew J., and Anders C. Hansen. "The foundations of spectral computations via the solvability complexity index hierarchy." Journal of the European Mathematical Society 25.12 (2022): 4639-4718.
3. Olver, Sheehan, and Alex Townsend. "A fast and well-conditioned spectral method." SIAM Review 55.3 (2013): 462-489.

62. Energy-Preserving Neural Networks for Dynamical Systems .... Dr D. Murari, Professor C.-B. Schönlieb

Dynamical systems in the form of differential equations have long been the cornerstone in describing physical and scientific phenomena. The modelling and simulation of such systems has been a long-standing scientific challenge. In recent years, we have seen an upsurge in deep learning (DL) techniques for such tasks [1-3]. DL-aided simulations are beneficial over classical numerical methods when dealing with differential equations that are, e.g., expensive to evaluate or with dynamics only partially known. For any numerical solver, however, care must be taken that the numerical simulation respects key properties (structures) of the dynamical systems, such as energy preservation for the simulation of a free rigid-body, as otherwise this could lead to results that are not meaningful, e.g., because they are non-physical or qualitatively inaccurate. Thus, when using DL techniques to solve or discover a differential equation, designing and deploying conservative strategies for the simulations to get valuable results is fundamental.

Despite this need, there is no overall strategy for how to systematically build networks that preserve an energy function exactly, neither a comprehensive analysis of the importance of doing so when approximating the solutions of differential equations. There are some recent works [4,5] that explore neural networks that preserve linear and quadratic invariants, designing them as numerical approximations for parametric differential equations with a linear or quadratic conserved energy. However, there is no general approach as well as no theoretical analysis of even these existing works.

In this essay, we would like to explore the topic of energy-preservation and deep learning. Starting with a general review on solving and learning dynamical systems with DL techniques, the essay should then discuss existing work on the preservation of linear and quadratic energies and provide a perspective onto their theoretical analysis and their potential extension to other energy classes. For those who are computationally inclined, the addition of numerical experiments to demonstrate the discussed approaches would be advantageous.

Relevant Courses

Essential: Numerical Analysis, Dynamical Systems

Useful: Functional Analysis, Partial Differential Equations

References

1. Raissi, M., Perdikaris, P., & Karniadakis, G. E. (2019). Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. Journal of Computational physics, 378, 686-707.

2. Champion, K., Lusch, B., Kutz, J. N., & Brunton, S. L. (2019). Data-driven discovery of coordinates and governing equations. Proceedings of the National Academy of Sciences, 116(45), 22445-22451.

3. Li, Z., Kovachki, N., Azizzadenesheli, K., Liu, B., Bhattacharya, K., Stuart, A., & Anandkumar, A. (2020). Fourier neural operator for parametric partial differential equations. arXiv preprint arXiv:2010.08895.

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5. Celledoni, E., Jackaman, J., Murari, D., & Owren, B. (2023). Predictions Based on Pixel Data: Insights from PDEs and Finite Differences, arXiv:2305.00723.

63. On the Backward Behavior of Solutions to Nonlinear Dissipative Systems

Professor E. S. Titi

It is well known that the initial value problem of parabolic equations is ill-posed backward in time, i.e. for certain given initial data the solutions either do not exist backwards in time or they do not depend continuously on the initial data. In [1] Bardos and Tartar proved that solutions of the Navier-Stokes equations are unique backward in time. Using some of the ideas developed in [1] the authors of [2] showed that for the periodic 2D Navier-Stokes equations the set of initial data for which the solution exists for all negative times and has exponential growth is rather rich. They study this set to show that it is dense in the phase space of the Navier-Stokes equations. This yields to a positive answer to a conjecture proposed by Bardos and Tartar in [1]. Moreover, after an appropriate rescaling the large Reynolds limit dynamics on this set is Eulerian. It turns out the Bardos and Tartar conjecture is not valid for general dissipative evolution equations (see [4] and [5])

Interested students can choose one of the following essays/projects:

Essay 1: This essay will be about the paper of Bardos and Tartar [1] and its extension to other nonlinear parabolic (or dissipative) PDEs such as the Kuramoto-Sivashinsky equation (see, e.g., [4] and [5]) and the Navier-Stokes-Voigt$-\alpha$ model (cf. [3], [5] and references therein).

Essay 2: This essay will be about investigating the extension of the results in [2] to the 2D (and possibly 3D) Navier-Stokes-Voigt$-\alpha$ model (cf. [3], [5] and references therein), especially investigating the limit behaviour for large Reynolds numbers and the vanishing limit behaviour of the regularizing parameter α.

Relevant Courses

Essential: Analysis of Partial Differential Equations. Introduction to Nonlinear Analysis.

Useful: Mathematical Analysis of the Incompressible Navier-Stokes Equations. Some knowledge of Fluid Mechanics.

References

1. C. Bardos and L. Tartar, Sur l'unicite rétrograde des équations paraboliques et quelques questions voisines, *Arch. Rational Merh. Anal.* **50** (1973), 10–25.

2. P. Constantin, C. Foias, I. Kukavica, A.J. Majda, Dirichlet quotients and 2D periodic Navier–Stokes equations, *J. Math. Pures Appl.* **76 (9)** (1997) 125–153.

3. V.K. Kalantarov, B. Levant and E.S. Titi, Gevrey regularity of the global attractor of the 3D Navier-Stokes-Voight equations, *Journal of Nonlinear Science* **19** (2009), 133–152.

4. I. Kukavica, M. Malcok, Backward behavior of solutions of the Kuramoto–Sivashinsky equation, *J. Math. Anal. Appl.* **307** (2005) 455–464.

5. Y. Guo and E.S. Titi, On the backward behavior of some dissipative evolution equations, *Physica D* **306** (2015), 34–47.

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7. R. Temam, Infinite-dimensional Dynamical Systems in Mchanics and Physics, Appl. Math. Sci. 2nd edn, vol. **68** (New York: Springer) 1997

64. Double-Diffusive Instabilities in Protoplanetary Discs .... Professor H. Latter

Ordinarily, a negative radial entropy gradient is stabilized by sufficiently strong rotation, as is the case in protoplanetary (PP) discs. But, if cooling is not too strong nor too weak, a form of oscillatory double-diffusive convection, the 'convective overstability' (COS), finds a way around the angular momentum constraint and produces hydrodynamical activity [1,2]. Notable dynamical features that emerge include: coherent nonlinear waves, inertial wave turbulence, zonal flows, and vortices [3-5], each of which may bear on the problem of planet formation [6].

In this essay, students are asked to survey the linear dynamics of the COS, its weakly nonlinear theory, and current numerical simulations. Particular attention should be paid to the formation of coherent structure, such as zonal flows. A comparison could also be made to related instabilities, such as the 'subcritical baroclinic instability' [7], the 'resistive double diffusive instability' [8], and 'semi-convection' in stellar and planetary interiors [9]. Finally, students should outline where in PP discs we might expect the COS to appear and discuss potential applications to dust dynamics and planet formation [2].

Relevant Courses

Essential: Astrophysical Fluid Dynamics, Dynamics of Astrophysical Discs

Useful: Fluid Dynamics of the Environment

References

1. Latter, 2016. MNRAS, 455, 2608.
2. Lesur et al., 2023. In: Inutsuka et al. (Eds), *Protostars and Planets VII*, p. 465.
3. Lyra, 2014. ApJ, 789, 77.
4. Teed and Latter, 2021. MNRAS, 507, 5523.
5. Lehmann and Lin, 2024. ApJ, 970, 15.
6. Raettig, Lyra, and Klahr, 2021. ApJ, 913, 92.
7. Lesur and Papaloizou, 2010. A&A, 513, 60.
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9. Rosenblum et al., 2011. ApJ, 731, 66.

65. Machine Learning for Classification of Astronomical Time Series .... Professor K. Mandel

The night sky is replete with astronomical sources that change in brightness over time, including variable stars, gravitational lensing events, and stellar explosions, such as supernovae and kilonovae. The Vera Rubin Observatory (VRO) is a new 8-meter telescope being constructed in Chile that will begin a 10-year Legacy Survey of Space and Time (LSST) in late 2025. It will regularly scan the sky and record brightness time series (light curves) of millions of time-varying sources in multiple colours of light, revolutionising our understanding of these astrophysical phenomena. Astronomers and data scientists have engineered a variety of machine learning algorithms to automatically sift through the massive data streams and classify the variable and transient sources underlying the time series data. The Photometric LSST Astronomical Time Series Classification Challenge (PLAsTiCC) is an open data challenge to the community to develop and apply new methods to classify simulated astronomical time-series data in preparation for observations from LSST. This data challenge poses the question: how well can we classify objects in the sky that vary in brightness from realistic simulated LSST time-series data, with all its observational and statistical challenges?

This essay is meant to review the relevant scientific motivations, the statistical challenges involved, some of the algorithms that have been developed, their pros and cons, and the classification metrics by which their performance are evaluated. The student will have the opportunity to implement the method(s) of his or her choice or invention on the datasets available from the PLAsTiCC challenge posted on Kaggle (https://www.kaggle.com/c/PLAsTiCC-2018) and evaluate their performance, either for the general challenge, or for more focused scientific goals. Originality and creativity are encouraged.

Relevant Courses

Essential: Astrostatistics

References

1. The PLAsTiCC Team, et al. *The Photometric LSST Astronomical Time-series Classification Challenge (PLAsTiCC): Data set.* 2018, https://arxiv.org/abs/1810.00001.

2. Hložek, R., et al. *Results of the Photometric LSST Astronomical Time-series Classification Challenge (PLAsTiCC).* 2023, The Astrophysical Journal, Supplement, 267, 25.

3. Malz, A., et al. *The Photometric LSST Astronomical Time-Series Classification Challenge (PLAsTiCC): Selection of a Performance Metric for Classifiation Probabilities Balancing Diverse Science Goals.* 2019, The Astronomical Journal, 158, 171.

4. Narayan, G., et al. *Machine-learning-based Brokers for Real-time Classification of the LSST Alert Stream.* 2018, The Astrophysical Journal Supplement, 236, 9.

5. Lochner, M., et al. *Photometric Supernova Classification with Machine Learning.* 2016, The Astrophysical Journal Supplement, 225, 31.

6. Richards, J., et al. *Semi-supervised learning for photometric supernova classification.* 2012, Monthly Notices of the Royal Astronomical Society, 419, 1121.

7. Kessler, R., et al. *Results from the Supernova Photometric Classification Challenge.* 2010, Publications of the Astronomical Society of the Pacific, 122, 1415.

66. Quasinormal Mode Overtones: Fits or Physics? . . . . Dr C. J. Moore

Black holes (BHs) are a crucial topic in many areas of modern physics. Two recent breakthroughs have allowed us to study dynamical BHs with unprecedented detail. Gravitational Wave (GW) astronomy now allows us to observe the mergers of astrophysical BHs for the first time, while Numerical Relativity (NR) enables us to simulate these mergers on computers. Together, these advances have confirmed the basic structure of the GW signal generated by BH mergers: the long inspiral phase (generated by a slowly shrinking binary orbit), the loud merger phase (a highly nonlinear signal produced as the two BHs form a common apparent horizon), and the ringdown phase.

The ringdown portion of the signal is the final stage, associated with the remnant BH settling into its final, stationary state. Often, the ringdown is modeled as a (linear) perturbation of the Kerr metric that describes the remnant BH. Linear perturbation theory predicts the presence of quasinormal modes (QNMs), which are damped oscillations at specific complex frequencies tied to eigenvalues of the perturbation equations. QNMs are widely used in GW astronomy to model the ringdown and for testing general relativity and probing the nature of compact astrophysical objects. For an introduction to QNMs, see [1].

Conventional wisdom suggests that, because the QNM description is based on linear perturbation theory, it can only describe the signal at late times—well after the merger—when the GW has decayed sufficiently to allow it to be considered a perturbation of the remnant BH. However, it has been suggested (see, for example, [2], as well as [3-5]) that the ringdown might actually begin surprisingly early in the merger process, possibly even before the GW strain amplitude reaches its maximum value. These early start times for the ringdown are associated with large amplitudes of QNM overtones (QNMs with high values of the radial quantum number, $n \geq 1$) which then decay more quickly than the fundamental ($n = 0$) modes. If true, this surprising result calls for an explanation: why is the loudest part of the GW signal, presumably associated with the strongest and most dynamical gravitational fields, amenable to the simplified modelling of linear perturbation theory? An investigation in [6] offers a partial explanation by suggesting that much of the nonlinearity may be trapped behind the forming common apparent horizon, never reaching infinity in the form of GWs.

However, the claims in [2-5] are contested. For example, [7] argues that the ringdown does not start early. Instead, it suggests that the seemingly impressive results obtained by fitting QNM models with overtones to NR and GW signals may be the result of overfitting due to the large number of free amplitude parameters associated with numerous QNM overtones.

After reviewing the necessary background, this essay will review the claims made in the literature about QNM overtones and when in the merger process the ringdown can be considered to have started. The essay will go on to discuss what it means to claim that a specific QNM has been detected in either an NR or GW signal. Finally, the essay should address the question implied in the title: does the ringdown really start at early times, with the QNM overtones excited with large initial amplitudes? Or, are the QNM overtones just overfitting the GW signal at early times and obscuring the fact that the true ringdown starts later?

Relevant Courses

Essential: General Relativity (Michaelmas)

Useful: BHs (Lent) and Gravitational Waves and Numerical Relativity (Easter)

References

1. Emanuele Berti, Vitor Cardoso, Andrei O. Starinets, *Quasinormal modes of BHs and black branes*, Class. Quantum Grav. 26 163001 (2009), https://arxiv.org/abs/0905.2975

2. Matthew Giesler, Maximiliano Isi, Mark Scheel, Saul Teukolsky, *Black hole ringdown: the importance of overtones*, Phys. Rev. X 9, 041060 (2019), https://arxiv.org/abs/1903.08284

3. Arnab Dhani, *Importance of mirror modes in binary black hole ringdown waveform*, Phys. Rev. D 103, 104048 (2021), https://arxiv.org/abs/2010.08602

4. Iara Ota, Cecilia Chirenti, *Overtones or higher harmonics? Prospects for testing the no-hair theorem with gravitational wave detections*, Phys. Rev. D 101, 104005 (2020), https://arxiv.org/abs/1911.00440

5. Eliot Finch, Christopher J. Moore, *Modelling the Ringdown from Precessing Black Hole Binaries*, Phys. Rev. D 103, 084048 (2021), https://arxiv.org/abs/2102.07794

6. Maria Okounkova, *Revisiting non-linearity in binary black hole mergers*, preprint (2020), https://arxiv.org/abs/2004.00671

7. Vishal Baibhav, Mark Ho-Yeuk Cheung, Emanuele Berti, Vitor Cardoso, Gregorio Carullo, Roberto Cotesta, Walter Del Pozzo, Francisco Duque, *Agnostic black hole spectroscopy: Quasinormal mode content of numerical relativity waveforms and limits of validity of linear perturbation theory*, Phys. Rev. D 108, 104020 (2023), https://arxiv.org/abs/2302.03050

67. Effects of a Magnetic Field on Stellar Oscillations .... Professor G. I. Ogilvie

Many stars are believed to have a strong magnetic field in their deep interiors, but until recently this has been very difficult to constrain observationally. In the last few years, however, detailed measurements of oscillation modes, together with developments in theory, have provided evidence for strong magnetic fields in some stellar interiors. To explain the suppression of dipole modes in some red giants, a picture has emerged according to which inwardly propagating gravity waves (g modes) are converted into outwardly propagating magnetic waves and dissipated, if they enter a region where the magnetic field exceeds a critical strength [1,2,3]. This concept has also been used to estimate the field strength in a massive main-sequence star [4] and to propose a mechanism of dissipating tidal disturbances [5]. Complementary ideas are that stellar oscillation modes can be dissipated through Alfvén resonances [6] or become trapped or chaotic [7]. Magnetic fields in stellar interiors have also been detected through the asymmetric splitting of oscillation modes and the application of perturbation theory [8].

This essay should give a review of several of these recent developments.

Use of the Astrophysics Data System `ui.adsabs.harvard.edu` is recommended. The theory of MHD waves in a stratified atmosphere is treated at length (but with different applications in mind) in [9].

Relevant Courses

Essential: Astrophysical Fluid Dynamics

Useful: Structure and Evolution of Stars

References

1 Fuller, J. et al. (2015). *Science* **350**, 423

2 Lecoanet, D. et al. (2017). *MNRAS* **466**, 2181

3 Rui, N. Z. and Fuller, J. (2023). *MNRAS* **523**, 582

4 Lecoanet, D. et al. (2022). *MNRAS* **512**, L16

5 Duguid, C. et al. (2024). *Astrophys. J.* **966**, L14

6 Loi, S. T. and Papaloizou, J. C. B. (2017). *MNRAS* **467**, 3212

7 Loi, S. T. and Papaloizou, J. C. B. (2018). *MNRAS* **477**, 5338

8 Li, G. et al. (2022). *Nature* **610**, 43

9 Cally, P. S. and Bogdan, T. J. (2024). arXiv:2408.01591

68. Mathematics of Tidal Coupling in Astrophysical Discs .... Professor R. R. Rafikov

Many astrophysical objects feature discs of gas or particles (or both) orbiting in an axisymmetric potential. Examples include planetary rings, accretion discs around stars and compact objects (white dwarfs, neutron stars, black holes), galaxies, and so on. Very often we find these discs being perturbed by the gravity of some external objects — planetary satellites in the case of planetary rings, stellar companions (donor stars) in the case of accretion discs in binaries, etc. Such gravitational perturbations of discs result in tidal coupling between them and their perturbers, giving rise to various wave-like phenomena and structures that can often be observed.

The goal of the essay will be to explore the mathematical foundations of the disc-perturber coupling. Starting with the classical studies [1,2], the candidate would explore the derivation of the governing equations for the perturbation structure, paying special attention to the roles played by the disc thermodynamics (where applicable [3]), self-gravity, and the characteristics of the perturbing potential [4]. An essay should clearly outline the general approach to obtaining solutions of these equations, explain their connection with the spiral structures observed in many astrophysical disks [5], and highlight the role of various resonances in the disc for wave excitation and propagation.

Relevant Courses

Essential: Dynamics of Astrophysical Disks (H. Latter)

Useful: Astrophysical Fluid Dynamics (R. R. Rafikov)

References

1. Goldreich, P. & Tremaine, S. 1979, Astrophysical Journal, Vol. 233, p. 857-871
2. Goldreich, P. & Tremaine, S. 1980, Astrophysical Journal, Vol. 241, p. 425-441
3. Miranda, R. & Rafikov, R. R. 2020, Astrophysical Journal, Vol. 892, Issue 1, id.65, 20 pp.
4. Sormani, M. C. Sobacchi, E. & Sanders, J. L. 2024, Monthly Notices of the Royal Astronomical Society, 528, p.5742
5. Ogilvie, G. I. & Lubow, S. H. 2002, Monthly Notices of the Royal Astronomical Society, Vol. 330, p. 950

69. Turbulence in Quantum Fluids .... Professor N. G. Berloff

Quantum fluids are systems where quantum effects appear at macroscopic lengthscales. Quantum fluids support quantised vortices that are fundamental topological objects playing an important role in most of the branches of physics from superfluids and superconductors to high energy physics and optics. They exist in classical matter fields described by a classical smooth complex-valued field signifying the points (in 2D) or lines (in 3D) where the amplitude of the field becomes zero and the phase winds around in multiples of h/m, called quanta of circulation. The formation, structure, dynamics, turbulence of the quantised vortices has been a subject of intense research in the last 50 years, but many aspects of the vortex motion even in the simplest settings as an advection of a single vortex of unit charge by a constant superflow are a subject of recent discussions and revisions. In the last two decades the field has undergone an important transformation combining theory with experimental realisations and potential applications.

A key objective will be to describe the turbulence found in superfluid helium, detailing the spectra of various turbulence regimes and drawing comparisons with classical turbulence, focusing on aspects such as energy dissipation, spectra, and vortex interactions. The essay will review the theoretical frameworks currently employed to describe turbulent quantum fluids, assessing their effectiveness in explaining the phenomena observed in experiments. It will also survey recent experimental advances, discussing how they have expanded our understanding of quantum fluid dynamics. A critical comparison between vortex structures and turbulence in quantum fluids versus classical fluids will be undertaken, considering both the similarities and the significant differences, especially in the nature of their turbulent cascades. Furthermore, the essay will discuss the recent theoretical revisions in vortex motion, focusing on how these revisions have reshaped our understanding of interactions between vortices, and the effects of boundaries and obstacles. Finally, the essay will identify unresolved questions in the study of quantum turbulence.

Relevant Courses

Essential: Fluid Dynamics IB and Quantum Mechanics IB

Useful: Fluid Dynamics II, Advanced Quantum Mechanics

References

1. Barenghi, Carlo F., and Nick G. Parker. A primer on quantum fluids. Berlin: Springer, 2016.

2. Donnelly, Russell J. Quantized vortices in helium II. Vol. 2. Cambridge University Press, 1991.

3. Skrbek, Ladislav, et al. "Phenomenology of quantum turbulence in superfluid helium." Proceedings of the National Academy of Sciences 118.16 (2021): e2018406118.

4. Galantucci, Luca, et al. "A new self-consistent approach of quantum turbulence in superfluid helium." The European Physical Journal Plus 135.7 (2020): 1-28.

5. Biferale, L., et al. "Superfluid helium in three-dimensional counterflow differs strongly from classical flows: Anisotropy on small scales." Physical Review Letters 122.14 (2019): 144501.

6. Švančara, Patrik, et al. "Ubiquity of particle–vortex interactions in turbulent counterflow of superfluid helium." Journal of Fluid Mechanics 911 (2021): A8.

7. Salman, Hayder. "Breathers on quantized superfluid vortices." Physical Review Letters 111.16 (2013): 165301.

8. Marchenko, Vladimir Ivanovich, and A. Ya Parshin. "On the energy spectrum of helium II." JETP letters 87 (2008): 326-327.

9. Noble, M. T., et al. "Producing and imaging quantum turbulence via pair-breaking in superfluid He3-B." Physical Review B 105.17 (2022): 174515.

70. Stratified Turbulent Mixing ....
Professor C. P. Caulfield

Understanding how turbulence leads to the enhanced irreversible transport of heat and other scalars (such as salt and pollutants) in density-stratified fluids is a fundamental and central problem in geophysical and environmental fluid dynamics. There is a wide range of highly important applications, not least the description and parameterization of diapycnal transport in the world's oceans, a key area of uncertainty in climate modelling.

Recently, due not least to the proliferation of data obtained through direct observation, numerical simulation and laboratory experimentation, there has been an explosion in research activity directed at improving community understanding, modelling and parameterization of the subtle interplay between energy conversion pathways, turbulence, and irreversible mixing in density-stratified fluids. However, there are still leading order open questions and areas of profound uncertainty concerning this interesting and important research challenge.

For example, since mixing in a stratified fluid inevitably changes the potential energy of the flow, it is of great interest to understand the efficiency of the mixing, i.e. the proportion of the work done on the flow that leads to irreversible mixing, effectively the "taxation rate" of turbulence by stratification. This essay should investigate at least some of the underlying issues of the energetics, instability mechanisms and turbulent processes in high Reynolds number stratified flows leading to the existing uncertainty in this important research area, with enormous and pressing relevance to the changing behaviour of the global climate system.

Relevant Courses

Useful: Fluid Dynamics of Climate, Hydrodynamic Stability

References

1. G. N. Ivey, K. B. Winters & J. R. Koseff (2008): 'Density stratification, turbulence, but how much mixing?' *Annu. Rev. Fluid Mech.* **40** 169-184.
2. M. C. Gregg, E. A. D'Asaro, J. J. Riley & E. Kunze (2018) 'Mixing efficiency in the ocean.' *Annu. Rev. Mar. Sci.* **10** 443-473.
3. C. P. Caulfield (2021): 'Layering, instabilities and mixing in turbulent stratified flows.' *Annu. Fluid Mech.* **53** 113-145.

71. Flying a Canoe: The Fluid Mechanics of a Single Paddle ....
Professor S. B. Dalziel

Paddling solo in a canoe is fundamentally different from rowing or paddling a kayak. A skilled canoe paddler need only use the paddle on one side of the canoe to be able to move the canoe efficiently forwards, backwards or sideways to follow an arbitrary path through the water. Whereas rowing and kayaking rely primarily on the drag force provided by the blade as it moves through the water, canoeing makes use of both 'simple' and 'compound' strokes that may utilise lift as well as drag in an unsteady regime. Many of the elements of the process have been considered before in the context of the flying and swimming of animals [1,2], although relatively little of this understanding has been transferred to paddling [3,4].

An essay on this topic would concentrate on one or two 'simple' strokes where either lift or drag is the dominant mechanism of propulsion. The essay should begin with a literature review of studies that inform an understanding of the unsteady dynamics. The efficiency of the stroke should be addressed for limits such as constant force or constant power provided by the paddler during the stroke. Scope within the essay exists for some simple laboratory experiments providing visualisation of the unsteady flow.

Relevant Courses

Essential: Undergraduate fluid mechanics.

References

1. Taylor, G.I. (1953) Formation of a vortex ring by giving an impulse to a circular disk and then dissolving it away. *J. Appl. Phys.* **24**, 104–105.

2. Linden, P.F. & Turner, J.S. (2001) The formation of 'optimal' vortex rings, and the efficiency of propulsion devices. *J. Fluid Mech.* **427**, 61–72.

3. Jackson, P.S., Locke, N. & Brown, P. (1992) The hydrodynamics of paddle propulsion. *11th Australasian Fluid Mechanics Conference*, Hobart, 1197–1200.

4. Michael, J.S., Smight, R. & Rooney, K.B. (2009) Determinants of kayak paddling performance. *Sports Biomechanics* **8**(2), 167–179.

72. Do Fish Mix the Ocean? .... Professor S. B. Dalziel

Mixing of the density stratification plays an important role in driving the circulation of the oceans [1]. Dominant mechanisms derive their energy from the tides, winds or radiative heating and cooling, but there are other sources. One such mechanism is 'biogenic mixing', the mixing generated by biological activity within the oceans [2].

An essay on this topic would concentrate on the potential for fish to mix a stratified fluid, focusing on the physics rather than the biology of the problem. The essay would begin by reviewing the available literature on mixing due to fish and marine mamals, describing the mechanisms that could contribute to mixing. The essay could then proceed to either draw on the existing literature for the 'mixing efficiency' [3] of different flows to assess the potential for mixing at a pycnocline, or to conduct some simple laboratory experiments using battery-powered fish to visualise and quantify the mixing produced.

Relevant Courses

Essential: Undergraduate fluid mechanics.

Useful: Fluid Dynamics of the Environment, Fluid Dynamics of the Climate.

References

1. Wunsch, C. & Raffaele, F. (2004) Vertical mixing, energy, and the general circulation of the oceans. *Ann. Rev. Fluid Mech.* **36**, 281-314.

2. Dabiri, J.O. (2010) Role of vertical migration in biogenic ocean mixing. *Geophys. Res. Lett.* **37**, doi:10.1029/210GL043556.

3. Winters, K.B., Lombard, P.N., Riley, J.J. D'Asaro, E.A. (1995) Available potential energy and mixing in density-stratified fluids. *J. Fluid Mech.* **289**, 115-128.

73. 'Moisture Modes' and the Tropical Atmosphere .... Professor P. H. Haynes

The tropical atmosphere is driven by radiative heating towards a state which is relatively warm and moist at the surface and relatively cold and dry at altitude and as a result strong convection develops, manifested by tall cumulus clouds. However the entire tropics is not in a state of active convection, but instead there is strong spatial variation at scales ranging from those of individual clouds to scales of hundreds or thousands of kilometres with large regions of active convection adjacent to large regions where convection is rare or even absent altogether. One particular feature of the tropical atmosphere is the so-called 'Madden-Julian oscillation' (MJO). This is actually a quasi-random variation in convection and in dynamical quantities such as wind and temperature, in which a region of active convection appears over the tropical Indian Ocean, drifts eastward into the western Pacific and then diminishes in strength over the eastern Pacific. Many of the global circulation models used for climate prediction give poor simulations of the MJO, suggesting that they poorly represent the physical processes that are responsible for it.

Given the complication of the tropical atmosphere – the range of spatial and temporal scales and the importance of cloud-scale processes including interactions between clouds and radiation – it might seem that simple mathematical models would have limited relevance. However, provided the need for crude but simple representations of cloud-scale processes is accepted, relatively simply models can capture some of the important interactions between these processes and the large-scale dynamics and provide genuine insight into ways in which the representation of tropical circulations in global climate models might be improved.

Some important aspects of the tropical atmosphere can be explained by the standard 'dry' theory of equatorial waves (which will be discussed in the Fluid Dynamics of Climate course). However this theory has failed to explain the MJO in particular. An extended theory that has been studied over the last 15 years or so, and is now being argued to provide a basis for understanding the MJO in particular, includes both simple fluid dynamical equations (e.g. as represented by the 'shallow-water equations') together with a moisture field that is transported with the fluid and affects the fluid dynamics by determining the heating. An interesting limit is when the fluid dynamics is treated as quasi-steady and the entire time evolution is controlled by the moisture field. In this limit simple wave motion is sometimes possible and these waves are described as 'moisture modes'. There is now quite a large literature on 'moisture modes' and their behaviour according to different dynamical formulations (e.g. incorporating different physical processes) and there are also several papers which discuss possible moisture-mode models for the MJO.

An essay on this topic should start by surveying some of the basic papers that have studied moisture modes in different forms, trying to present a unified summary of the important features of the behaviour and how it depends on the physical ingredients incorporated in the model. The essay might then move on to discuss in more detail the extent to which moisture modes provide an explanation for the MJO and how these simple models might be used to advance understanding of the MJO and to improve its representation in climate models. (But a student writing this essay might choose to focus on other topics, such as the way in which convection-scale processes are represented in the simple models, or the extent to which moisture modes, or simple models that allow moisture modes along other sorts of behaviour, are useful to understand other aspects of the tropical atmosphere.)

Relevant papers are listed below. [1] is an early paper that considers a relevant simple model and identifies moisture-mode behaviour. [2] is a recent paper that attempts a systematic analysis that distinguishes clearly between moisture modes and other modes. [3], [4] and [5] are papers that propose moisture-mode models for the MJO. [6] is a paper that argues on the other hand that the physical processes incorporated into the models described in [4] and [5] may not be relevant to the MJO in the real atmosphere and offers an alternative.

Relevant Courses

Essential: An undergraduate course in fluid dynamics.

Useful: Fluid Dynamics of Climate (Not absolutely essential, but any student who is considering choosing this essay and who is NOT taking this course is advised – and welcome – to discuss with the setter.).

References

1. Sobel, A. H., J. Nilsson, and L. M. Polvani, 2001: The weak temperature gradient approximation and balanced tropical moisture waves. J. Atmos. Sci., 58, 3650-3665.

2. Adames, A.F., D. Kim, S. K. Clark, Y. Ming, and K. Inoue, 2019: Scale analysis of moist thermodynamics in a simple model and the relationship between moisture modes and gravity waves. J. Atmos. Sci., 76, 3863-3881,

3. Sugiyama, M., 2009: The moisture mode in the quasi-equilibrium tropical circulation model: Part I: Analysis based on the weak temperature gradient approximation. J. Atmos. Sci., 66, 1507-1523.

4. Sobel, A., and E. Maloney, 2013. Moisture modes and the eastward propagation of the MJO. J. Atmos. Sci., 70, 187-192.

5. Adames, A., and D. Kim, 2016: The MJO as a dispersive, convectively coupled moisture wave: Theory and observations. J. Atmos. Sci., 73, 913-941.

6. Fuchs, Z. and D. J. Raymond, 2017: A simple model of intraseasonal oscillations. J. Adv. Model. Earth Syst., 9, 1195-121.

74. Models for Lava Flows and Lava Tubes ....

Dr D. R. Hewitt

Lava flows occur whenever molten rock erupts onto the Earth's surface. They flow as a gravity current, affected by surface topography and by continuous evolution of their rheology, from cooling and from changing gas and crystal content. Internal tunnels or tubes are a common feature of basaltic lavas: they can form in a lava flow if the surface crusts over, leaving hot lava to flow in a self-regulated conduit below. Such tube systems are much more insulated than an open channel flow, and thus provide a mechanism by which lava is able to travel significantly further than if the whole flow remained fluidized.

This essay will explore modelling approaches to describe the flow of lava, with a specific focus on the formation of lava tubes. The essay should provide a thorough review of current understanding of lava rheology, and standard approaches to describe the shallow, gravity-driven spread of lava. Discussion of experimental evidence for the evolution of lava flows should be included. The essay will go on to investigate models for the formation of lava tubes, mechanisms that might control tube evolution, and experimental evidence of tube formation. Depending on the interests of the candidate, the essay might consider other aspects of lava flows, such as interaction with topography, numerical approaches to describe lava flows, or other aspects of lava tube formation and evolution.

Relevant Courses

Essential: Undergraduate level Fluid Mechanics

Useful: Part III Non-Newtonian Fluid Mechanics

References

1. Griffiths, R.W. (2000). The Dynamics of Lava Flows, Annu. Rev. Fluid Mech. 32
2. Valerio, A. Tallarico, A. Dragoni, M. (2008). Mechanisms of formation of lava tubes, JGR 113
3. Balmforth, N.J., Burbidge, A.S., Craster, R.V. (2001). Shallow Lava Theory. In: Balmforth, N.J., Provenzale, A. (eds) Geomorphological Fluid Mechanics. Lecture Notes in Physics, vol 582. Springer.
4. Hyman, D.M.R., Dietterich, H.R., Patrick, M.R. (2022). Toward Next-Generation Lava Flow Forecasting: Development of a Fast, Physics-Based Lava Propagation Model, JGR, 127.

75. Elasto-Inertial Turbulence and Maximum Drag Reduction .... Professor R. R. Kerswell

The viscoelastic effects introduced by adding long chain polymers to a Newtonian solvent have dramatic consequences for turbulent fluid motion. Perhaps most significant is the substantial reduction in skin friction at high-Reynolds numbers, though viscoelasticity can also seed entirely new chaotic dynamics which are predominantly two-dimensional [1] - so called 'Elasto-Inertial Turbulence' (EIT, [2,3]) - in which the polymer becomes stretched in thin sheet-like structures with attached patches of intense spanwise vorticity. The drag reduction brought about by viscoelasticity has been linked to a stabilisation of near-wall streaks that results in a suppression of high-drag bursting events [4], though it has also been argued that flows with large drag reduction are the weakly-elastic limit of EIT [5]. Recent experimental evidence suggests that EIT is a distinct dynamical regime unrelated to modified Newtonian turbulence [1], though the two behaviours coexist at sufficiently high Reynolds numbers. This essay would explore the features of EIT, its possible origins, where it exists in parameter space and the current evidence that its responsible for the maximum drag reduction regime (the recent review by Xi [6] is a good starting point).

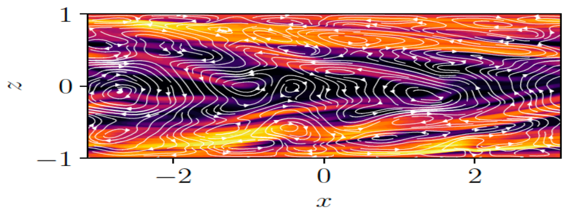

Figure 1: Contours of $Tr(\mathbf{C})/L^2$ (colours) where $\mathbf{C}$ is the conformation tensor describing the orientation of the polymers and the streamlines of the perturbation velocity in 2D Channel flow.

Relevant Courses

Essential: Fluids II, Mathematical Methods.

Useful: Dynamical Systems, Non-Newtonian Flows.

References

1. Sid, S, Terrapon, V. E. & Dubief, Y "Two-dimensional dynamics of elasto-inertial turbulence and its role in polymer drag reduction" *Phys. Rev. Fluids* **3**, 011301, 2018.

2. Samanta, D.S., Dubief, Y., Holzner, C. Schafer, A.N., Wagner, C. and Hof, B. "Elasto-inertial turbulence" *Proc. Nat. Acad. Sci.* **110**, 10557-10562, 2013.

3. Dubief, Y., Terrapon, V. E. and Hof, B. "Elasto-Inertial Turbulence" *Ann. Rev. Fluid Mech.* **55**, 675-705, 2023.

4. Graham, M.D. "Drag reduction and the dynamics of turbulence in simple and complex fluids" *Phys. Fluids* **26**, 101301, 2014.

5. Dubief, Y., Terrapon, V.E. and Soria, J. "On the mechanism of elasto-inertial turbulence" *Phys. Fluids* **25**, 110817, 2013.

6. Xi, L. "Turbulent drag reduction by polymer additives: Fundamentals and recent advances" *Phys. Fluids* **31**, 121302, 2019.

76. Learning Governing Equations from Data .... Dr K. Shah

Emerging techniques in machine learning have leveraged the growing availability of data to extract information about underlying processes in high-dimensional systems. This essay topic focuses on *interpretable* data-driven techniques that learn physics-based governing equations from data, essential for simulating systems for which reliable data is available but governing equations are unknown, e.g., climate physics, neuroscience, ecology. To apply these algorithms to such systems, the key step is to build a library of terms that the governing equations can possibly contain, for which a deep understanding of the system is crucial. The main assumption is that only a few terms describe the leading-order dynamics, making the functional form of the learned equation(s) sparse in the space of possible functions. Using sparsity-promoting techniques, the algorithm identifies the subset of terms that best describe the data.

The essay will begin with a review of two such techniques: Sparse Identification of Nonlinear Dynamics (SINDy, [1]) and PDE Functional Identification of Nonlinear Dynamics (PDE-FIND,

[2]). Code for examples provided in both studies are publicly available on Github which essay writers are encouraged to look through to get a feel for the practical aspects of implementing these algorithms. The essay should also explain differences between the two techniques and their applicability. Possible extensions to the essay include giving an exposition of some published examples that have successfully learnt governing equations from data, based on the essay writer's interests, e.g., ocean subgrid-scale parameterisations [3]; turbulent wakes [4]; oscillator models in biology [5]. The essay should conclude with a discussion of types of problems and types of data to which interpretable data-driven techniques (such as SINDy and PDE-FIND) can be reasonably applied, the challenges of using these techniques on 'real' data (e.g., from observations, numerical simulations, laboratory measurements), and promising areas for future research.

Relevant Courses

Essential:

Undergraduate courses in dynamical systems, linear algebra, differential equations.

Useful:

Familiarity with sparse regression. Depending on examples chosen by the essay writer, useful courses are: fluid dynamics, Fluid Dynamics of Climate, or Biological Physics.

References

1. Brunton et al. *Discovering governing equations from data by sparse identification of non-linear dynamical systems.* Proceedings of the National Academy of Sciences (2016).

2. Rudy et al. *Data-driven discovery of partial differential equations.* Science Advances (2017).

3. Zanna & Bolton. *Data-driven equation discovery of ocean mesoscale closures.* Geophysical Research Letters (2020).

4. Callaham et al. *An empirical mean-field model of symmetry-breaking in a turbulent wake.* Science Advances (2022).

5. Prokop & Gelens. *From biological data to oscillator models using SINDy.* iScience (2024).

77. Extreme bending of elastic rods: Whip cracks and fencing blades . . . . . . Professor Eric Lauga

This essay is devoted to the mathematical description of the bending of elastic filaments. Within the classical continuum field of Elasticity [1] sits a beautiful topic, now known as Kirchhoff's Rod Theory [2], whose purpose is to describe the bending (and twisting) of an elastic rod whose centreline is inextensible and whose planar cross-section remains everywhere normal to the rod centreline. While being linearly elastic from a material point of view, the resulting theory is geometrically nonlinear, and offers an opportunity for rich results in physics, including instabilities and shocks. One extreme example is the use of whips to create the famous whip crack sound [3]. Another one is the use of flicks in epee and foil fencing in order to hit the opponent by bending the blade, instead of pushing directly against the target [4].

In this essay, the student will carry out four tasks increasing in complexity consisting of reviewing past work on the bending of elastic rods, reproducing know computational results and deriving new ones in the context of sport physics:

a. The student will carry out a comprehensive literature review on Kirchhoff's Rod Theory in two dimensions, including its full derivation and some fundamental analytical solutions;

b. The student will then implement a numerical solution to Kirchhoff's Rod Theory in order to reproduce classical results on bending and buckling in a variety of settings;

c. The numerical approach will then be extended to study a more complex problem, namely the propagation of waves in whips [5];

d. The student will finally apply this firm mathematical and numerical basis to derive a model of the flicking of fencing blades.

The ideal candidate will be interested in both mechanics and the mathematics of sports and have a particular taste for computational aspects of applied mathematics.

Relevant Courses

Knowledge of continuum mechanics (including stress tensor, Cauchy equations, Reynolds transport theorem), Vector and Tensor Calculus, Numerical methods, and Asymptotic methods.

References

1. Love, A. (2013) *A Treatise on the Mathematical Theory of Elasticity.* Cambridge University Press, 4th edition.

2. Audoly, B. and Pomeau, Y. (2010) *Elasticity and Geometry: From Hair Curls to the Non-linear Response of Shells.* Oxford University Press.

3. https://www.youtube.com/watch?v=sEA4I433Pfo

4. https://www.youtube.com/watch?v=rXIW6Dsl3Ys

5. McMillen, T. and Goriely, A. (2003) Whip waves, *Physica D* **184**: 192-225.

78. The Shape of a (Viscous) Chocolate Fountain .... Professor J. R. Lister

In a chocolate fountain, molten chocolate flows down over a vertical stack of dome-shaped tiers of increasing size (Try googling images for 'chocolate fountain'). The flow on each tier can be described very simply using lubrication theory. The chocolate flows over the circular edge of each tier to form a thin axisymmetric curtain of falling fluid, which falls until it lands on the next tier below. Observation shows that the curtain contracts inwards as it falls, with radius looking almost linear as a function of height. The theory for an inviscid curtain of fluid is well-established from the study of 'water bells', but molten chocolate is viscous!

This essay would likely take the form of a mini-project to calculate the shape of a falling very viscous Newtonian fluid curtain by using the equations of viscous extensional flow, as adapted to the axisymmetric geometry. The equations of axisymmetric shell theory in elasticity would be a useful comparison. Inertia should (probably) be neglected, but surface tension is relevant. The problem differs from tube-drawing in that radial and axial variations are comparable. Further references and guidance are available on request.

Relevant Courses

Essential: Slow Viscous Flow

References

1. Townsend, A.K. & Wilson, H.J. 2015 The fluid dynamics of the chocolate fountain *Euro. J. Phys.* **37**.

2. Ribe, N.M. 2002 A general theory for the dynamics of thin viscous sheets. *J. Fluid Mech.* **457**, 255.

3. Audoly, B. & Pomeau, Y. 2010 *Elasticity and geometry* OUP

4. Howell, P.D. 1994 *Extensional thin layer flows* PhD thesis, Oxford University (available online).

79. Modelling the Hook of a Swimming Uniflagellar Bacterium .... Professor J.R. Lister

A simple model of a swimming uniflagellar bacterium is an axisymmetric cell body attached to a rigid helical flagellum by a short flexible link called the 'hook'. A molecular motor in the wall of the body rotates the end of the hook, which rotates the flagellum, which provides the driving force that pushes the organism forward. A lot of modelling effort has put into detailed calculation of the resistance matrices for the body and flagellum for biologically realistic geometries, but much less into modelling the bending of the hook or understanding its dynamics.

The essay would likely review the papers below, and try to cut through the complexity and detail to describe the main ingredients in the modelling in each paper, and any similarities and differences between their approaches and results.

Relevant courses

Essential: Slow Viscous Flow

References

1. Nguyen, F.T.M., & Graham, M.D. 2017 Buckling instabilities and complex trajectories in a simple model of uniflagellar bacteria. *Biophys. J.* **112**, 1010–1022.

2. Jabbarzadeh, M. &, Fu, H. C., 2019 Dynamic instability in the hook-flagellum system that triggers bacterial flicks. *Phys. Rev. E.* **97**, 012402.

3. Jabbarzadeh, M. &, Fu, H. C., 2020 Large deformations of the hook affect free-swimming singly flagellated bacteria during flick motility. *Phys. Rev. E.* **97**, 033115.

4. Park, Y., Kim, Y. & Lim, S. 2019 Locomotion of a single-flagellated bacterium. *J. Fluid Mech.* **859**, 586–612.

5. Shimogonya, Y., Sawano, Y., Wakebe, H., Inoue, Y., Ishijima, A. & Ishikawa, T. 2015 Torque-induced precession of bacterial flagella *Scientific Rep.* 5:18488.

80. The Brewer-Dobson Circulation ....

Dr A. Ming

The Brewer-Dobson circulation (BDC) is a mean meridional circulation in the stratosphere that slowly transports air from over the Equator towards the poles across angular momentum contours and downwards over the mid and high latitudes. The strength and variability of this circulation is important in setting the composition of the stratosphere. For example, ozone depleting substances such as clorofluorocarbons are inert in the troposphere but chemically active in the stratosphere. Their atmospheric lifetimes, set by their removal rates from, are determined by the speed of the BDC.

Air entering the stratosphere over the tropical region encounters a cold region on the tropical tropopause layer where the dehydration of the air parcels is important in setting the lower stratospheric moisture. Stratospheric water vapour is a powerful greenhouse gas and a source of hydrogen oxide radicals which control many key chemical reactions. The water vapour signal shows multi-timescale variations from daily to decadal which dominated by temperature variations and the tropical upwelling strength.

This essay will begin by discussing the frameworks used to understand the BDC, including the Transformed Eulerian Mean framework [1] and the downward control principle [2]. The essay will then review the current status of understanding the BDC and challenges in quantifying it. Possible extensions to the essay include discussing the main drivers of variability in the BDC, the predicted changes to the BDC under climate change [3] or the methods of diagnosing the current BDC.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Fluid Dynamics of the Climate

References

1. Butchart, N. (2014), The Brewer-Dobson circulation, Rev. Geophys., 52, doi:10.1002/2013RG000448.

2. Haynes, P. H., M. E. McIntyre, T. G. Shepherd, C. J. Marks, and K. P. Shine, 1991: On the "Downward Control" of Extratropical Diabatic Circulations by Eddy-Induced Mean Zonal Forces. J. Atmos. Sci., 48, 651–678, https://doi.org/10.1175/1520-0469(1991)048<0651:OTCOED>2.0.CO;2.

3. Abalos, M., Calvo, N., Benito-Barca, S., Garny, H., Hardiman, S. C., Lin, P., Andrews, M. B., Butchart, N., Garcia, R., Orbe, C., Saint-Martin, D., Watanabe, S., and Yoshida, K.: The Brewer–Dobson circulation in CMIP6, Atmos. Chem. Phys., 21, 13571–13591, https://doi.org/10.5194/acp-21-13571-2021, 2021.

81. The Quasi-Biennial Oscillation ....

Dr A. Ming

The Quasi-Biennial Oscillation (QBO) manifests as a pattern of alternating descending winds, with a period of about 28 months, and dominates the interannual variability of the tropical stratosphere [1]. It is primarily driven by waves originating in the troposphere. Despite largely being confined to within 10° of the Equator, the impact of the QBO extends to the whole atmosphere [1,2]. The QBO influences polar vortex variability via the Holton-Tan effect [1]. In the tropics, a relationship between the QBO and a major tropical precipitation pattern, the Madden-Julian oscillation, has recently been discovered [3]. Connection between the QBO and other atmospheric phenomenon are a major source of predictability for weather on seasonal to sub-seasonal timescales in the tropics.

Although the basic theoretical framework that describes how wave processes drive the regular QBO is well understood, the quantitative details and interactions between the different processes are not. An increasing number of climate models are able to formulate some kind of QBO but few spontaneously generate one and even then, the phase, strength and downward extent differ significantly from observations.

This essay will begin by discussing the theoretical framework for understanding the QBO, including the Plumb model. The essay will then review the current status of understanding the QBO. Possible extensions to the essay include discussing the mechanisms of influence of the QBO on to other regions, disruptions to the QBO from volcanic eruptions, ozone feedbacks with the QBO, future changes under climate change or even the equivalent oscillation on other planetary bodies.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Fluid Dynamics of the Climate

References

1. Baldwin, M. P., et al. (2001) The quasi-biennial oscillation. Rev. Geos. doi:10.1029/1999RG000073

2. Haynes, P., Hitchcock, P., Hitchman, M., Yoden, S., Hendon, H., Kiladis, G., Kodera, K., Simpson, I., (2021) The influence of the stratosphere on the tropical troposphere. J. Meteor. Soc. Japan, doi:10.2151/jmsj.2021-040.

3. Yoo, C., and S.-W. Son (2016) Modulation of the boreal wintertime Madden-Julian oscillation by the stratospheric quasi-biennial oscillation, Geophys. Res. Lett., doi:10.1002/2016GL067762.

82. Modelling and Design of Artificial Swimmers .... Dr. M. Tătulea-Codrean

Artificial swimmers are widely employed in soft matter physics, engineering, and biomedicine, as well as for educational purposes. A notable example is G. I. Taylor's demonstration of the physics of propulsion at low Reynolds number using mechanically actuated swimmers in syrup. Over the past two decades, there has been a surge of applied mathematical research directed at designing, controlling, and optimizing artificial swimmers for specific tasks and environments. This essay will focus on the use of mathematical modelling to better understand the physical mechanisms behind artificial swimming and to propose optimal designs for artificial swimmers.

The essay will cover some classical calculations on the propulsion of swimmers via surface distortions, and the physics of thrust generation by slender appendages. A strong essay will also explore recent advancements in the modelling and fabrication of either chemically active swimmers or mechanical swimmers with slender appendages. It may include a critical analysis and comparison of these two types of artificial swimmers in various applications.

The following references offer a starting point for further reading.

Relevant Courses

Essential: Undergraduate course in Fluid Dynamics (Part II or equivalent).

Useful: Some content from Part III Slow Viscous Flow and Perturbation Methods.

References

1. H. A. Stone and A. D. Samuel. Propulsion of Microorganisms by Surface Distortions. *Phys. Rev. Lett.* **77**: 4102-4104, 1996.

2. R. Golestanian, T. Liverpool and A. Ajdari. Designing phoretic micro- and nano-swimmers. *New J. Phys.* **9**: 126, 2007.

3. E. Lauga and T. R. Powers. The hydrodynamics of swimming microorganisms. *Rep. Prog. Phys.* **72**: 096601, 2009.

4. A. C. H. Tsang, E. Demir, Y. Ding and O. S. Pak. Roads to Smart Artificial Microswimmers. *Adv. Intell. Syst.* **2**: 1900137, 2020.

5. A. Das, M. Styslinger, D. M. Harris and R. Zenit. Force and torque-free helical tail robot to study low Reynolds number micro-organism swimming. *Rev. Sci. Instrum.* **93**: 044103, 2022.

83. Ocean Biogeochemical Data Assimilation .... Professor J. R. Taylor

The ocean plays a major role in the global carbon cycle. Microscopic free-floating algae known as phytoplankton are resposible for about half of the global primary production, while the deep ocean is the largest non-geological carbon reservoir on the planet. Estimating the state of the biology and chemistry in the ocean, known collectively as ocean biogeochemistry, is important for understanding the present and future climate system. Because of the complexity and nonlinearities associated with the system, computational models are a very valuable tool when studying ocean biogeochemistry. Data assimilation is the process of incorporating measurements into model estimates of the state of a system.

The ocean biogeochemical system has a very large number of degrees of freedom, resulting from a large number of constituents and spatiotemporal variability. Measurements of the system are available from ship-based samples and sensors, autonomous platforms, and satellite-based sensors. However, compared to the number of degrees of freedom, the ocean's biogeochemical system is highly under-sampled. This presents a major challenge for data assimilation. One approach is to approximate the system with a reduced model with fewer degrees of freedom. The objective of this essay will be to explore this approach and to compare the performance of biogeochemical models with varying degrees of complexity.

The essay should start with a literature review of data assimilation techniques, highlighting the challenges associated with estimating the state of the ocean's biogeochemical system. Models of varying complexity and measurement techniques should be briefly discussed. The essay should reflect on the advantages and disadvantages of data assimilation techniques in the context of ocean biogeochemistry.

Next, the essay can use an open source biogeochemical modelling system called OceanBioME.jl to explore the behaviour of biogeochemical models of varying complexities. Data assimilation can be performed using EnsembleKalmanProcess.jl or another existing package. The objective will be to explore the ability of models of varying complexity to reproduce a given timeseries of a biogeochemical state variable. The essay should reflect on the optimal complexity of the biogeochemical model for reproducing data with a given sample size.

Relevant Courses

Useful: Fluid Dynamics of Climate

References

1. Dowd, M., Jones, E. and Parslow, J., 2014. A statistical overview and perspectives on data assimilation for marine biogeochemical models. Environmetrics, 25(4), pp.203-213.

2. Eknes, M. and Evensen, G., 2002. An ensemble Kalman filter with a 1-D marine ecosystem model. Journal of Marine Systems, 36(1-2), pp.75-100.

3. Strong-Wright, J., Chen, S., Constantinou, N.C., Silvestri, S., Wagner, G.L. and Taylor, J.R., 2023. OceanBioME. jl: A flexible environment for modelling the coupled interactions between ocean biogeochemistry and physics. Journal of Open Source Software, 8(90), p.5669.

84. Continuum Descriptions of Dense Granular Flows ....
Dr K. Warburton

Many materials in nature and industry are made up of small solid particles, which when moving together behave much like a fluid. Unlike Newtonian fluids, granular flows display a finite, pressure-dependent yield-stress, and dilate at high shear rates. Describing the behaviour of these granular materials, particularly as they transition from solid- to fluid-like, remains a challenge. Continuum models (averaging the flow over several particle diameters) such as the $\mu(I)$ rheology are attractive compared to grain-scale simulations for their efficiency, and lend themselves well to analogies to fluid dynamics. However, such models begin to break down at low shear rates, when grain-scale dynamics, density fluctuations and non-local effects come into play.

This essay should look at the development of continuum models for both dry and wet granular flows through the inertial regime, the viscously-dominated regime, and non-local behaviour close to static. With these rheological models in hand, example calculations could include creeping flow down a slope, the granular 'viscous' gravity current, or (relevant to sediments beneath glaciers) shear-driven flow at a fluctuating confining pressure.

Relevant Courses

Useful: Non-Newtonian Flows, Slow Viscous Flow, Fluid Dynamics of the Solid Earth

References

1. Jop P, Forterre Y, Pouliquen O. 2006 A constitutive law for dense granular flows. Nature 441, 727–730. (doi:10.1038/nature04801)

2. Boyer F, Guazzelli E, Pouliquen O. 2011 Unifying suspension and granular rheology. Phys. Rev. Lett. 107, 188301. (doi:10.1103/PhysRevLett.107.188301)

3. Baumgarten AS, Kamrin K. 2019 A general fluid-sediment mixture model and constitutive theory validated in many flow regimes. J. Fluid Mech. 861, 721–764. (doi:10.1017/jfm.2018.914)

4. Amarsid L, Delenne JY, Mutabaruka P, Monerie Y, Perales F, Radjai F. 2017 Viscoinertial regime of immersed granular flows. Phys. Rev. E 96, 012901. (doi:10.1103/ PhysRevE.96.012901)

5. Schaeffer D, Barker T, Tsuji D, Gremaud P, Shearer M, Gray J. 2019 Constitutive relations for compressible granular flow in the inertial regime J. Fluid Mech. 874, 926-951 (doi:10.1017/jfm.2019.476)

6. Kamrin, K. 2019 Non-locality in Granular Flow: Phenomenology and Modeling Approaches. Frontiers in Physics 7 (doi:10.3389/fphy.2019.00116)

85. Rising Bubbles Through Volcanic Conduits ....
Dr K. Warburton

Seismicity associated with the resonance between flow and deformation in elastic-walled conduits is widely observed in active volcanoes. One proposed source of this activity is the motion of large gas bubbles rising from the magma chamber, and the pressure perturbations they induce.

The passage of large bubbles through rigid-walled pipes is a classical problem (Bretherton 1961), and the intrusion of a semi-infinite bubble through an elastic-walled tube has been studied in the context of the opening of collapsed airways. This essay would consider the geophysically relevant scenario of a finite length bubble in an elastic medium, rising due to buoyancy. This could include: the impact of gravity and determining the rise speed, looking at the shape of the tail end of the bubble, calculating the pressure distribution over the bubble length.

Relevant Courses

Essential: Slow Viscous Flow

Useful: Perturbation Methods, Fluid Dynamics of the Solid Earth

References

1. Bretherton FP. (1961) The motion of long bubbles in tubes. J. Fluid Mech. 10(2):166-188. doi:10.1017/S0022112061000160

2. Manta, F., Emadzadeh, A., Taisne, B. (2019) New Insight Into a Volcanic System: Analogue Investigation of Bubble-Driven Deformation in an Elastic Conduit. JGR: Solid Earth, 124, 11274–11289. doi:10.1029/2019JB017665

3. Gaver DP, Halpern D, Jensen OE, Grotberg JB. (1996) The steady motion of a semi-infinite bubble through a flexible-walled channel. J. Fluid Mech. 319:25-65. (doi:10.1017/ S0022112096007240)

4. Jensen OE, Horsburgh MK, Halpern D, Gaver DP. (2002) The steady propagation of a bubble in a flexible-walled channel: Asymptotic and computational models. Physics of Fluids 14-2 (doi:10.1063/1.1432694)

5. Magnini, M, Khodaparast, S, Matar, O, Stone, HA., Thome, JR. (2019) Dynamics of long gas bubbles rising in a vertical tube in a cocurrent liquid flow. Phys. Rev. Fluids 4.023601 (doi:10.1103/PhysRevFluids.4.023601)

86. Penetrative Convection .... Professor M. G. Worster

The density of water has a maximum value at a temperature of about 4°C. If the temperature of a horizontal layer of water straddles this temperature then that layer has regions of static stability and of static instability. Convection in the unstable region can 'penetrate' the statically stable region, which affects the heat transfer between the two regions. The essay should review the literature about such convection, including descriptions of stability analyses, laboratory experimentation and numerical simulation. The essay may also include original linear stability analysis of convection in a horizontal layer of fluid with conditions of constant heat flux, zero tangential stress and zero penetration through the horizontal boundaries of the layer.

Relevant Courses

Useful: Fluid Dynamics of the Solid Earth

References

1. Veronis, G (1963) Penetrative convection. *Astrophysical Journal* 137:641–663.

2. Whitehead JA, Chen MM (1970) Thermal instability and convection of a thin fluid layer bounded by a stably stratified region. *J. Fluid Mech.* 40, 549–576.

3. Moore DR, Weiss NO (1973) Nonlinear penetrative onvection. *J. Fluid Mech.* 61(3):553–581.

4. Toppaladoddi S & Wettlaufer JS (2018) Penetrative convection at High Rayleigh numbers. *Phys. Rev. Fluids* 3, 043501.

87. Supergraph Perturbation Theory .... Professor B. C. Allanach

Quantum field theory perturbation theory involving Feynman diagrams can be generalised to "supergraphs" i.e. perturbation theory based upon superfields. You should explain and review supergraph perturbation theory. Choose a related calculation to follow in detail for the second half of the essay: for example supersymmetry preserving regularisation or non-renormalisation theorems using supergraph perturbation theory.

Relevant Courses

Essential: Quantum Field Theory, Supersymmetry, Advanced Quantum Field Theory

References

1. S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry", Front. Phys. **58** (1983) 1 [hep-th/0108200].

2. P. C. West, "A review of non-renormalisation theorems in supersymmetric theories" , Nucl.Phys.Proc.Suppl. 101 (2001) 112-128.

3. I. Jack and D.R.T. Jones, "Regularization of supersymmetric theories", Adv.Ser.Direct.High Energy Phys. 21 (2010) 494-513 [hep-ph/9707278].

88. Aspects of Solvable Models of Gravity: Jackiw–Teitelboim Theory ..... Dr A. Castro

This essay will investigate various classical aspects of Jackiw–Teitelboim (JT) theory. JT is a two-dimensional dilaton gravity model with a linear potential for the dilaton and it is a solvable theory. JT has drawn much attention in recent years for this reason, and its relation to quantum properties of near-extremal black holes.

This essay will mainly focus on the classical properties of JT gravity. The main reference is section 2 of the review [1], and references within as appropriate. Some of the topics the student is expected to cover are:

1. Describing properties of dilaton gravity, such as the action principle, degrees of freedom, equations of motion, gauge-fixing, and construction of classical solutions. Constructing and describing the solutions to JT gravity.

2. To discuss the role of diffeomorphisms and boundary conditions in JT gravity. One particular outcome is to revisit the action principle.

3. To describe examples where one obtains dilaton gravity from a dimensional reduction, and discuss what physics of the higher dimensional theory it describes.

4. Connection of JT to physics near extremal of the Reissner-Nordström black holes in AdS_4 and $Mink_4$.

The above list is indicative, and the specific contents can be modified upon discussion with the essay setter.

Relevant Courses

Essential: Quantum Field Theory and General Relativity

Useful: String Theory and Black Holes

References

1. T. G. Mertens and G. J. Turiaci, "Solvable models of quantum black holes: a review on Jackiw–Teitelboim gravity," Living Rev. Rel. **26** (2023) no.1, 4.
arXiv:2210.10846 [hep-th].

89. Anyons and Topological Quantum Computing .... Professor N. Dorey

In the familiar setting of $(3+1)$-dimensional relativistic QFT particles must obey either Fermi or Bose statistics. In $(2+1)$-dimensions a much richer set of possibilities exists. In particular, Chern-Simons theories give rise to *anyons* [1,2]: particles with exotic statistics related to the representation theory of the braid group. In recent years, the existence of anyons has been substantiated by experiment and they have also found a surprising new application to the seemingly unrelated problem of building a quantum computer.

In the traditional approach to quantum computing, information is stored in two-level subsystems known as qubits. Computation is achieved by applying a sequence of unitary operators representing logical gates to states in a finite dimensional Hilbert space corresponding to a tensor product of elementary qubits. However the fragile nature of quantum coherence makes this impossible to realise in practice without the extra complication of quantum error correction. In seminal work, Kitaev et al [3,4], proposed a quite different model of quantum computation in which the Hilbert space is associated to the groundstate of a system of anyonic particles and quantum gates are realised by braiding the anyon worldlines (see the review article [5] and references therein). Importantly, quantum information is stored in a way which is robust against local perturbations providing a new route to fault-tolerant quantum computation.

The essay will explore the properties of anyons in $(2+1)$-dimensional QFT including those relevant to topological quantum computation.

Relevant Courses

Essential: Quantum Field Theory; Symmetries, Fields and Particles

Useful: Topological Quantum Matter, Quantum Computation, Advanced Quantum Field Theory.

References

1. Leinaas, J. M., & Myrheim, J. (1977). On the theory of identical particles. Il nuovo cimento, 37, 132.
2. Wilczek, F. (1982). Quantum mechanics of fractional-spin particles. Physical review letters, 49(14), 957.
3. Kitaev, A. Y. (2003). Fault-tolerant quantum computation by anyons. Annals of physics, 303(1), 2-30.
4. Freedman, M., Kitaev, A., Larsen, M., & Wang, Z. (2003). Topological quantum computation. Bulletin of the American Mathematical Society, 40(1), 31-38.
5. Nayak, C., Simon, S. H., Stern, A., Freedman, M., & Das Sarma, S. (2008). Non-Abelian anyons and topological quantum computation. Reviews of Modern Physics, 80(3), 1083-1159.

90. Physics of the Riemann Zeta Function ....
Professor S. A. Hartnoll

The Riemann zeta function plays a central role in the theory of the distribution of prime numbers. Remarkably, aspects of the theory of the Riemann zeta function have deep connections to physics. This is an open-ended essay in which candidates are invited to explore one or two connections of the Riemann zeta function to physics in some depth.

An overview of some connections can be found in [1] and [2]. Two possible directions, mentioned in these overviews, are the theory of the primon gas [3] and quantum chaos in the distribution of Riemann zeros as manifest in periodic orbit theory [4] and/or connections to random matrices [5]. However, other directions are also possible.

Relevant Courses

A good understanding of undergraduate quantum mechanics, classical dynamics, statistical physics and mathematical methods should be enough to get started on this essay. Background in number theory is not expected.

References

1. D. Schumayer and D. A. W. Hutchinson, *Colloquium: Physics of the Riemann hypothesis*, Rev. Mod. Phys. 83, 307 (2011).
2. M. Wolf, *Will a physicist prove the Riemann hypothesis?*, Rep. Prog. Phys. 83, 036001 (2020).
3. B. Julia, *Statistical theory of numbers*, in "Number Theory and Physics" (Springer Proceedings in Physics vol 47) ed J. M. Luck et al, pp 276–93 (1990).
4. M. Berry, *Some quantum-to-classical asymptotics*, in "Chaos and Quantum Physics," (Les Houches, Session LII, 1989) ed M-J. Giannoni et al, pp 252-303 (1991).
5. P. Bourgade and J. P. Keating, *Quantum chaos, random matrix theory, and the Riemann ζ-function*, in "Chaos: Poincare Seminar 2010", pp. 125-168 (2013).

91. Quantum Field Theory at Finite Temperature ....
Dr R. A. Reid-Edwards

The first time you come into contact with quantum field theory it will almost certainly have been with zero temperature in mind. However, there are many important applications of quantum field theory where finite temperature effects are important. These range from phase transitions in the early universe to studying the thermodynamics of black holes.

The purpose of this essay is to introduce and contrast three main approaches to QFT at finite temperature; the Imaginary Time (or Matsubara) Formalism, The Real Time Formalism and Thermofield Dynamics. Although some consideration should be given to the imaginary time formalism, it is expected that the focus of the essay will be on the Real Time formalism and Thermofield Dynamics. The KMS condition and the origin doubling required in both of the latter formalisms should be explained clearly. The essay might also give a concise account of one (or more) applications of one (or more) of the above formalisms to a problem of interest.

Relevant Courses

Essential: Quantum Field Theory

Useful: Advanced Quantum Field Theory, Statistical Field Theory

References

1. *Finite Temperature Field Theory*, A. K. Das, ISBN: 9789810228569, World Scientific
2. *Topics in finite temperature field theory*, A. K. Das, e-Print: hep-ph/0004125 [hep-ph]
3. *Real and Imaginary Time Field Theory at Finite Temperature and Density*, N.P. Landsman & C.G. van Weert, Phys.Rept. 145 (1987) 141
4. *Quantum Field Theory in Curved Space-Time and Black Hole Thermodynamics*, R. M. Wald, ISBN: 9780226870274, University of Chicago Press

92. The BFSS and BMN Models and Their Holographic Duals .... Professor J. E. Santos

The holographic principle, also known as gauge/gravity duality, establishes a connection between certain quantum field theories and higher-dimensional quantum gravity, serving as a cornerstone of modern high-energy physics. Following Maldacena's seminal conjecture on the duality between string theory in $\mathrm{AdS}_5 \times S^5$ and $\mathcal{N} = 4$ Super Yang-Mills theory, a broader network of holographic dualities was discovered, pairing maximally supersymmetric theories with their gravitational counterparts. These dualities were identified by studying D-branes in string theory and decoupling the brane worldvolume theory from gravity. This essay focuses on the simplest of these dualities, specifically those between the low-energy limits of D0-branes and maximally supersymmetric quantum mechanics. These are known as the BFSS and BMN models, named after their respective authors: Banks, Fischler, Shenker, and Susskind for BFSS, and Berenstein, Maldacena, and Nastase for BMN.

The essay should be divided into three main parts. The first part should explain the BFSS model [1] and how the BMN model [2] arises as a mass deformation of it. The second part should provide a detailed review of D0-branes [3], with a particular focus on their thermodynamic properties. Finally, the third part should discuss the conjectured connection between the BFSS and BMN models and the low-energy limit of D0-branes [4], with special emphasis on the parameter region where the D0-branes can be treated classically.

The essay should be written in a language accessible to other Part III students taking similar courses.

Relevant Courses

Essential: Black Holes and Advanced Quantum Field Theory

Useful: String Theory and Supersymmetry

References

1. T. Banks, W. Fischler, S. H. Shenker, and L. Susskind, *M theory as a matrix model: A conjecture*, Phys. Rev. D 55, 5112 (1997), arXiv:hep-th/9610043.

2. D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, *Strings in flat space and pp waves from $\mathcal{N} = 4$ super Yang Mills*, JHEP 04 (2002) 013, arXiv:hep-th/0202021.

3. *D-branes* by Joe Polchinski. In String Theory: Volume 1 - An Introduction to the Bosonic String and Volume 2 - Superstring Theory and Beyond. Cambridge University Press.

4. N. Itzhaki, J. M. Maldacena, J. Sonnenschein, and S. Yankielowicz, Supergravity and the large N limit of theories with sixteen supercharges, Phys. Rev. D 58, 046004 (1998), arXiv:hep-th/9802042.

93. Fermion-Monopole Scattering .... Professor D. Tong

Strange things happen when fermions scatter off magnetic monopoles. For example, conservation laws can be violated in the scattering process and this leads, among other things, to proton decay, a phenomenon known as the *Callan-Rubakov* effect. Even stranger things happen when chiral fermions scatter off monopoles and the full story is not yet completely understood.

The purpose of this essay is to describe the basics of fermion-monopole scattering. The essay should describe solutions to the Dirac equation in the presence of a monopole and why the scattering reduces to a $d = 1 + 1$ dimensional problem. Much of the physics can then be understood using a methodology known as *bosonization*, in which fermions in $d = 1 + 1$ dimensions can be mapped to bosons, which can then be used to illustrate the Callan-Rubakov effect and related topics.

Relevant Courses

Essential: Quantum Field Theory, The Standard Model.

Useful: Symmetries, Fields and Particles.

References

1. The Callan-Rubakov effect was introduced in C. G. Callan. "*The Proton Decay – Magnetic Monopole Connection*", AIP Conference Proceedings 98.1 (1983), pp. 24-34. and in V. A. Rubakov. "*Monopole catalysis of proton decay*", Reports on Progress in Physics 51.2 (1988), p. 189.

2. For a review of monopoles, including a section on fermion scattering, see John Preskill "*Magnetic Monopoles*", Ann. Rev. Nucl. Part. Sci. **34** (1984), 461-530.

3. For a review of 2d bosonization, see chapter 7 of my *Lectures on Gauge Theory*, https://www.damtp.cam.ac.uk/user/tong/gaugetheory.html.

4. For a recent paper which goes through the details of the Dirac equation in a monopole background, see Y. Hamada, T. Kitahara and Y. Sato, "*Monopole-fermion scattering and varying Fock space,*" JHEP **11** (2022), 116, [arXiv:2208.01052 [hep-th]].

5. Some of the many subtleties that arise when chiral fermions scatter off monopoles were resolved in M. van Beest, P. Boyle Smith, D. Delmastro, Z. Komargodski and D. Tong, "*Monopoles, Scattering, and Generalized Symmetries*," [arXiv:2306.07318 [hep-th]].

94. Deep Neural Networks and Quantum Field Theory .... Professor M. Ubiali

Machine learning techniques are increasingly utilised to gain insight on Quantum Field Theory (QFT) problems. For example in particle physics theory deep neural networks (DNNs) can be applied to detect patterns and symmetry in field-theoretic data, or to extract information on the fundamental parameters of the Standard Model (SM) from the experimental data collected at particle colliders. At the same time the connection between DNNs and QFT has recently gained interest due to similarities in the mathematical structures underlying both fields, and QFT is utilised to understand neural networks and other key ingredients of machine learning.

The purpose of this essay is two-fold. The main goal is to describe the theoretical framework for understanding neural networks through the lens of Wilsonian effective field theory [1]. The approach is based on the observation that DNNs resemble Gaussian processes [2,3], analogous to non-interacting field theories. Deviations from the asymptotic limit leads to non-Gaussian processes, which corresponds to introducing particle interactions. This connection allows one to compute neural network output correlations using Feynman diagrams. Additionally, the theoretical insights gained in the first part of the essay will be tested in the second part on neural networks in the simplest architectures that converge to a Gaussian process in the asymptotic limit.

Relevant Courses

Essential: Quantum Field Theory

Useful: Symmetries, Fields and Particles, Advanced Quantum Field Theory

References

1. James Halverson, Anindita Maiti, and Keegan Stoner, *"Neural Networks and Quantum Field Theory"* [arXiv:2106.10165 [hep-ph]].

2. Carl Edward Rasmussen, Christopher K. I. Williams, *"Gaussian Processes in Machine Learning"*, MIT press [https://doi.org/10.7551/mitpress/3206.001.0001]

3. Daniel A. Roberts, Sho Yaida, Boris Hanin, *"The Principles of Deep Learning Theory,"*, to be published by Cambridge University Press [arXiv:2106.10165 [hep-ph]].

95. Encoding Yang–Mills Theory for Quantum Computation .... Professor M. B. Wingate

Recent developments in building hardware for quantum computation has sparked research into whether near- or distant-future quantum computers could solve problems in quantum field theory (QFT). Let us focus here on digital quantum computers consisting of registers of qubits. An important first issue is how to approximate or "encode" the infinite-dimensional Hilbert space of a QFT using a finite-dimensional Hilbert space spanned by registers of qubits. This essay will look at this encoding problem in the case of $SU(2)$ Yang–Mills theory.

The essay will assume that the reader is familiar with gauge theory in the Lagrangian formulation. It should cover the following points: (1) introduction to the Kogut–Susskind Hamiltonian for $SU(2)$ lattice gauge theory [1], (2) electric and magnetic bases for the Hilbert space upon which the Hamiltonian acts, (3) some truncation schemes for these bases, including discussion of advantages and disadvantages.

Reference [2] would be a good initial starting point for the essay, as it is written pedagogically and has a good list of references. The successful essay will go beyond simply paraphrasing the literature, demonstrating a good understanding of the main concepts. For example, some key ingredients could be explained in more depth, or one could summarize recent efforts to test these approaches via classical simulations of fault-tolerant quantum devices or using existing, noisy quantum devices.

Relevant Courses

Essential: Quantum Field Theory; Symmetries, Fields and Particles

Useful: The following would provide further familiarity with nonabelian gauge theory from different perspectives: Advanced Quantum Field Theory; Standard Model.

References

1. J. B. Kogut and L. Susskind, Phys. Rev. D **11**, 395-408 (1975).
2. I. D'Andrea, C. W. Bauer, D. M. Grabowska and M. Freytsis, Phys. Rev. D **109**, 074501 (2024) [arXiv:2307.11829 [hep-ph]].

96. Dynamical Anyon Automorphisms in Quantum Error Correction .... Prof. B. Béri

Quantum error-correcting codes are often based on topological order (TO)—a phase of many-body quantum matter robust to local perturbations and supporting anyonic excitations [1-5]. While these topological codes typically feature a static TO, a new class of codes, so-called Floquet codes, were recently introduced which evolve in the space of TOs through measurement-induced dynamics [6-8]. The essential mechanism of time-evolution is a sequence of anyon condensations that time-periodically implements an automorphism on the starting TO [9,10]. (The adjective 'Floquet' refers to this time periodicity.) The purpose of this essay is to give a coherent account of Floquet codes, focusing on those based on Abelian TO. The essay should include, but not necessarily be limited to, discussions on: Abelian anyon theories and anyon condensation and autormorphisms therein, illustrated on the colour code and its toric code descendants [9,10]; the robustness of Floquet codes as phases of matter [11]; the essentials of quantum error correction with Floquet codes, including their advantages and challenges compared with static topological codes [6-10].

Relevant Courses

Essential: None.

Useful: Part II Quantum Information and Computation, Part III Quantum Entanglement in Many-body Physics, Part III Topological Quantum Matter.

References

1. P. W. Shor, Scheme for reducing decoherence in quantum computer memory, Phys. Rev. A **52**, R2493 (1995).

2. A. M. Steane, Error Correcting Codes in Quantum Theory, Phys. Rev. Lett. **77**, 793 (1996).

3. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, 10th ed. (Cambridge University Press, Cambridge; New York, 2010).

4. A. Y. Kitaev, Fault-tolerant quantum computation by anyons, Ann. Phys. **303**, 2 (2003).

5. E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, Topological quantum memory, J. Math. Phys. **43**, 4452 (2002).

6. M. B. Hastings and J. Haah, Dynamically Generated Logical Qubits, Quantum **5**, 564 (2021).

7. C. Vuillot, Planar Floquet Codes, arXiv:2110.05348.

8. J. Haah and M. B. Hastings, Boundaries for the Honeycomb Code, Quantum **6**, 693 (2022).

9. M. S. Kesselring, J. C. Magdalena De La Fuente, F. Thomsen, J. Eisert, S. D. Bartlett, and B. J. Brown, Anyon Condensation and the Color Code, PRX Quantum **5**, 010342 (2024).

10. M. Davydova, N. Tantivasadakarn, S. Balasubramanian, and D. Aasen, Quantum computation from dynamic automorphism codes, Quantum **8**, 1448 (2024).

11. D. Vu, A. Lavasani, J. Y. Lee, and M. P. A. Fisher, Stable Measurement-Induced Floquet Enriched Topological Order, Phys. Rev. Lett. **132**, 070401 (2024).

97. Majorization Relations in Quantum Information Theory .... Professor N. Datta

Entropies play a fundamental role in Quantum Information Theory (QIT). Optimal rates of information-theoretic tasks such as storage, transmission and processing of information, and manipulation of entanglement, are characterized by entropic quantities. Hence, studying mathematical properties of entropies is of fundamental importance. These often take the form of inequalities relating entropies of different quantum states. A powerful tool for studying such inequalities is provided by the notion of majorization. Majorization is a preorder on vectors of real numbers. In particular, a majorization relation between two probability vectors determines which of the two is more disordered (or *random*) than the other. This essay will involve understanding the basics of majorization theory and its applications in QIT, not just in the study of entropic inequalities, but also in the context of characterization and inter-convertibility of entanglement. These results span more than two decades of research, starting from the onset of the field to the present. The student can choose to focus on a subset of papers (and references therein) listed below. More recent papers will also be suggested to the interested student.

Relevant Courses

Useful: Quantum Information Theory

References

1. Michael A. Nielsen. An introduction to majorization and its applications to quantum mechanics. 2002. Available online at https://michaelnielsen.org/papers/maj-book-notes.pdf

2. Michael A. Nielsen and Julia Kempe. Separable states are more disordered globally than locally. Physical Review Letters, 86(22):5184–5187, May 2001.

3. Michael A. Nielsen and Guifré Vidal. Majorization and the interconversion of bipartite states. Quantum Information and Computation, 1(1):76–93, 2001.

4. Michael G. Jabbour, Nilanjana Datta. A tight uniform continuity bound for the Arimoto-Rényi conditional entropy and its extension to classical-quantum states. IEEE Trans. Inf. Th., vol. 68, no. 4, p. 2169-2181 (2022)

5. E. P. Hanson and N. Datta, Maximum and minimum entropy states yielding local continuity bounds, J. Math. Phys. 59, no. 4, 042204 (2018).

6. Michael G. Jabbour, Nilanjana Datta. Tightening continuity bounds on entropies and bounds on quantum capacities. *To appear in IEEE Trans. Inf. Theory, 2024.* Available on arXiv:2310.17329

98. Proposals for Table-Top Tests of Quantum Gravity ....
Professor A. Kent

New proposals for experimental tests that probe aspects of quantum gravity have recently been proposed. The first of these proposals [1,2] involve creating superposition states of two mesoscopic systems, for example small metal spheres whose centre-of-mass is in a superposition of two nearby locations. They then bring these systems adjacent to one another, so that their gravitational interactions have a measurable effect on the phases of their quantum states. It is shown that this effect, if it follows standard quantum gravity intuitions (supported by perturbatively quantized general relativity), should lead to create an entangled state from these initially separate systems. Finally, ways of testing for this entanglement are described. Later proposals include [4] tests for non-Gaussianity in Bose-Einstein condensates and [5] tests for nonclassical evolution of systems of quantum harmonic oscillators

It has been argued [1,2,3] that entanglement can only be created if the gravitational force arises from the exchange of quantum systems, i.e. if gravity is genuinely quantum. Reviews (e.g. [6,7]) assess these arguments, some of which remain debated.

An essay on this topic should carefully review some or all of the proposed experiments, and some of the arguments as to what precisely they test about quantum gravity. It should also discuss the practical difficulties that arise in implementing the experiments.

Relevant Courses

Essential: Quantum Information, Foundations and Gravity

Useful: Quantum Information Theory. Quantum Field Theory. Advanced Quantum Field Theory. General Relativity.

References

1. Sougato Bose, Anupam Mazumdar, Gavin W Morley, Hendrik Ulbricht, Marko Toro, Mauro Paternostro, Andrew A Geraci, Peter F Barker, MS Kim, and Gerard Milburn. Spin entanglement witness for quantum gravity. *Physical Review Letters*, 119(24):240401, 2017.

2. Chiara Marletto and Vlatko Vedral. Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. *Physical Review Letters*, 119(24):240402, 2017.

3. Ryan J Marshman, Anupam Mazumdar, and Sougato Bose. Locality and entanglement in table-top testing of the quantum nature of linearized gravity. *Physical Review A*, 101(5):052110, 2020.

4. Non-Gaussianity as a Signature of a Quantum Theory of Gravity Richard Howl, Vlatko Vedral, Devang Naik, Marios Christodoulou, Carlo Rovelli, and Aditya Iyer, PRX Quantum 2, 010325

5. Testing the Quantumness of Gravity without Entanglement, Ludovico Lami, Julen S. Pedernales, and Martin B. Plenio Phys. Rev. X 14, 021022

6. Daniel Carney, Philip CE Stamp, and Jacob M Taylor. Tabletop experiments for quantum gravity: a user's manual. *Classical and Quantum Gravity*, 36(3):034001, 2019.

7. N. Huggett, N. Linnemann, and M. D. Schneider, Quantum Gravity in a Laboratory? Cambridge: Cambridge University Press, 2023.

99. Dualities for Quantum Spin Systems using Bimodule Categories .... Professor F. Verstraete

Quantum spin systems provide an extremely interesting playground for studying interacting quantum many-body systems. Dualities play a central role in the understanding of the associated phase diagrams of their ground states. The goal of this project is to study and characterise the complete "web of dualities" arising from spin systems with global on-site symmetries. Following reference [1], it will be necessary to construct the representation theory of generalised (categorical) symmetries in terms of matrix product operators. The general theory underlying this framework was developed in references [2,3,4,5], and in this project the emphasis will be on the special case of on-site global symmetries.

Relevant Courses

Useful: Part III Quantum Entanglement in Many-body Physics

References

1. Entanglement and the density matrix renormalisation group in the generalised Landau paradigm, Laurens Lootens,Clement Delcamp and Frank Verstraete, arXiv 2408.06334.

2. Gauging quantum states: from global to local symmetries in many-body systems, J Haegeman, K Van Acoleyen, N Schuch, JI Cirac, F Verstraete, Physical Review X 5 (1), 011024

3. On finite symmetries and their gauging in two dimensions, Lakshya Bhardwaj, Yuji Tachikawa, JHEP 03 (2018) 189 • e-Print: 1704.02330

4. Matrix product operator symmetries and intertwiners in string-nets with domain walls, L Lootens, J Fuchs, J Haegeman, C Schweigert, F Verstraete SciPost Physics 10 (3), 053 (2021)

5. Dualities in one-dimensional quantum lattice models: symmetric Hamiltonians and matrix product operator intertwiners, L Lootens, C Delcamp, G Ortiz, F Verstraete, PRX Quantum 4 (2), 020357 (2023)

100. Quantum-to-Classical Transition in the Early Universe .... Dr T. Colas

The standard model of cosmology relies on initial conditions that seed the temperature anisotropies and galaxy overdensities observed in the sky today. Within the inflationary paradigm, these cosmological inhomogeneities arise from quantum fluctuations of the primordial vacuum. Since the theory's formulation, a key question has been whether we can demonstrate the quantum origin of these cosmic inhomogeneities. While current (and arguably future) observations are insufficient to confirm imprints of quantum correlations in the sky, this question has played a pivotal role in bridging quantum information theory and quantum field theory in curved spacetime.

Although focused on primordial cosmology, this essay must address the distinction between quantum and classical theories, as well as the potential quantum-to-classical transition that systems may undergo [1]. It will begin by reviewing the formation of cosmological inhomogeneities during inflation [2], followed by a discussion of the peculiar quantum state in which these inhomogeneities are left at the end of inflation [3]. This so-called *squeezed state* exhibits characteristics that seem 'classical' in some respects [4], while in others, remain distinctly 'quantum' [5] - a point of long-standing debate [6]. The primary aim of the essay is to synthesize and critically evaluate the arguments presented in these references. Depending on the student's interest and available time, the essay may also explore either attempts to detect genuine quantum correlations through Bell tests [7] (whose feasibility is debated in [8]) or the use of higher-order statistics to differentiate between quantum and classical dynamics [9].

Relevant Courses

Essential: Cosmology

Useful: Field Theory in Cosmology, Quantum Information Theory

References

1. W. H. Zurek, "Decoherence and the transition from quantum to classical," Phys. Today **44N10** (1991), 36-44 doi:10.1063/1.881293 [arXiv:quant-ph/0306072 [quant-ph]].

2. L. Senatore, "Lectures on Inflation," doi:10.1142/9789813149441_0008 [arXiv:1609.00716 [hep-th]].

3. A. Albrecht, P. Ferreira, M. Joyce and T. Prokopec, "Inflation and squeezed quantum states," Phys. Rev. D **50** (1994), 4807-4820 doi:10.1103/PhysRevD.50.4807 [arXiv:astro-ph/9303001 [astro-ph]].

4. C. Kiefer and D. Polarski, "Why do cosmological perturbations look classical to us?," Adv. Sci. Lett. **2** (2009), 164-173 doi:10.1166/asl.2009.1023 [arXiv:0810.0087 [astro-ph]].

5. J. Martin and V. Vennin, "Quantum Discord of Cosmic Inflation: Can we Show that CMB Anisotropies are of Quantum-Mechanical Origin?," Phys. Rev. D **93** (2016) no.2, 023505 doi:10.1103/PhysRevD.93.023505 [arXiv:1510.04038 [astro-ph.CO]].

6. D. Sudarsky, "Shortcomings in the Understanding of Why Cosmological Perturbations Look Classical," Int. J. Mod. Phys. D **20** (2011), 509-552 doi:10.1142/S0218271811018937 [arXiv:0906.0315 [gr-qc]].

7. J. Maldacena, "A model with cosmological Bell inequalities," Fortsch. Phys. **64** (2016), 10-23 doi:10.1002/prop.201500097 [arXiv:1508.01082 [hep-th]].

8. J. Martin and V. Vennin, "Obstructions to Bell CMB Experiments," Phys. Rev. D **96** (2017) no.6, 063501 doi:10.1103/PhysRevD.96.063501 [arXiv:1706.05001 [astro-ph.CO]].

9. D. Green and R. A. Porto, "Signals of a Quantum Universe," Phys. Rev. Lett. **124** (2020) no.25, 251302 doi:10.1103/PhysRevLett.124.251302 [arXiv:2001.09149 [hep-th]].

101. The Rise and Fall of the Third Law of Black Hole Thermodynamics .. Professor M. Dafermos

This essay concerns the *third law of black hole thermodynamics* in classical general relativity.

The third law of black hole mechanics or "thermodynamics" was first proposed in 1973 in the seminal paper [1] of Bardeen, Carter and Hawking, in analogy with the unattainability formulation of the third law of ordinary thermodynamics (according to which one cannot cool a system to absolute zero in any finite process). In this thermodynamic analogy, temperature corresponds to the black hole's surface gravity, which is positive in the subextremal case and zero in the extremal case. In Israel's later more precise formulation [2], the third law of black hole thermodynamics states that one cannot transform a sub-extremal black hole to an extremal one in finite advanced time in a regular (i.e. non-singular) process, under reasonable assumptions on the matter model. The paper [2] moreover claimed to give a proof. A standard "text-book" discussion of the third law can be found in Section 5.5.6 of [3].

In a recent paper [4] (see also [5]), the third law was in fact *disproved.*

The purpose of this essay is to give an account of the rise and fall of the third law of black hole thermodynamics, reviewing the orignal formulation of the law, past attempts at a proof, and its final disproof in [4]. (In particular, the essay should describe in detail exactly what was wrong with previous attempts at proof.)

Relevant Courses

Essential: General Relativity, Black Holes

Useful: Analysis of Partial Differential Equations

References

1. J. M. Bardeen, B. Carter, and S. W. Hawking. "The four laws of black hole mechanics". *Comm. Math. Phys.* 31 (1973), pp. 161–170.

2. W. Israel. “Third law of black-hole dynamics: a formulation and proof”. *Phys. Rev. Lett.* 57.4 (1986), pp. 397–399.

3. E. Poisson. *Relativist’s Toolkit: The Mathematics of Black-Hole Mechanics.* Cambridge University Press; 2004.

4. C. Kehle and R. Unger. “Gravitational collapse to extremal black holes and the third law of black hole thermodynamics”. arXiv:2211.15742, to appear in *J. Eur. Math. Soc.*

5. C. Kehle and R. Unger. “Extremal black hole formation as a critical phenomenon”. arXiv:2402.10190

102. Riemannian Black Hole Uniqueness Conjecture ....
Professor M. Dunajski

A combination of results of Hawking, Carter, Israel and D. Robinson implies that, under suitable technical assumptions, any vacuum static or stationary asymptotically flat black hole exterior region must be isometric with the Schwarzschild or the Kerr solution.

An similar result was conjectured to hold in the Riemannian context, with the Schwarzschild and Kerr gravitational instantons being unique asymptotically flat (AF) and Ricci–flat Riemannian metrics. The recent examples of AF Ricci–flat metrics constructed by Chen and Teo disproved this conjecture.

The essay will explore the subject, starting off with a review of asymptotically flat and asymptotically locally flat gravitational instantons, and comparison with asymptotically locally Euclidean gravitational instantons (where the uniqueness theorem is known to hold). In the second part of the essay the Chen–Teo solutions will be explored together with a detailed discussion of either their rod structure.

Relevant Courses

Essential: General Relativity, Black Holes

Useful: Solitons, Instantons, and Geometry

References

1. Aksteiner, S. and Andersson, L. (2021) Gravitational Instantons and special geometry. `arXiv:2112.11863`

2. Chen, Y. and Teo, E. (2011) A new AF gravitational instanton. Physics Letters **B 703** 359–362.

3. Chen, Y. and Teo, E. (2015) Five-parameter class of solutions to the vacuum Einstein equations. Phys. Rev. **D 91**, 124005.

4. Dunajski, M. (2024) Solitons, Instantons, and Twistors (2nd Edition) *Oxford Graduate Texts in Mathematics*, Oxford University Press. (Chapter 10).

5. Lapedes, A. S. (1980) Black-hole uniqueness theorems in Euclidean quantum gravity Phys. Rev. **D 22**, 1837.

103. Open Quantum Systems .... Professor E. Pajer

The study of open quantum systems in physics is a crucial area of research that focuses on understanding quantum systems interacting with their surrounding environment. Unlike isolated systems, open quantum systems exchange energy and information with their surroundings, leading to phenomena such as decoherence, dissipation, and non-Markovian dynamics. These interactions are essential for exploring real-world phenomena, from the foundational aspects of quantum mechanics and quantum field theory to applications like quantum computing, thermodynamics and cosmology. The study of open quantum systems requires new ideas and mathematical constructions beyond those usually encountered in quantum mechanics and field theory: the theory of stochastic processes, the Schwinger-Keldysh or closed-time contour formalism and the master equation for the density matrix are just some examples.

Renewed interest in open quantum systems has appeared in the recent cosmology and high-energy field theory literature, partially because of progress in the path integral approach to the problem, which provides a more familiar language. This essay aims at discussing these recent results. As such it should comprise of two parts. In the first part the student will present an overview of the main ideas and results in the study of open quantum systems, including the Langevin equation, the master equation, Fokker-Planck equation, decoherence and the MSR or Schwinger-Keldysh path integral formalism. Good references for this part of the essay are [1,2,3], but many other can also be found in references therein. In the second part of the essay the student will focus on applications of this formalism. Two different and rich applications are to the description of dissipative fluids [4,5] and to dissipative effects during inflation [6,7,8]. Other formal developments can also be discussed, for example along the lines of [9,10].

Relevant Courses

Essential: Quantum Field Theory.

Useful: General Relativity, Cosmology, Statistical Physics, Field Theory in Cosmology, Advanced Quantum Field Theory.

References

1. Kamenev, Alex. Field theory of non-equilibrium systems. Cambridge University Press, 2023.
   H. P. Breuer and F. Petruccione, "The Theory of Open Quantum Systems," Oxford University Press, 2007

2. H. Liu and P. Glorioso, "Lectures on non-equilibrium effective field theories and fluctuating hydrodynamics," PoS **TASI2017** (2018), 008. [arXiv:1805.09331 [hep-th]].

3. T. Colas, "Open Effective Field Theories for primordial cosmology : dissipation, decoherence and late-time resummation of cosmological inhomogeneities,"

4. M. Crossley, P. Glorioso and H. Liu, "Effective field theory of dissipative fluids," JHEP **09** (2017), 095. [arXiv:1511.03646 [hep-th]].

5. P. Glorioso, M. Crossley and H. Liu, "Effective field theory of dissipative fluids (II): classical limit, dynamical KMS symmetry and entropy current," JHEP **09** (2017), 096. [arXiv:1701.07817 [hep-th]].

6. M. Hongo, S. Kim, T. Noumi and A. Ota, "Effective field theory of time-translational symmetry breaking in nonequilibrium open system," JHEP **02** (2019), 131. [arXiv:1805.06240 [hep-th]]. M. Hongo, S. Kim, T. Noumi and A. Ota, "Effective Lagrangian for Nambu-Goldstone modes in nonequilibrium open systems," Phys. Rev. D **103** (2021) no.5, 056020 [arXiv:1907.08609 [hep-th]].

7. S. A. Salcedo, T. Colas and E. Pajer, "The Open Effective Field Theory of Inflation," [arXiv:2404.15416 [hep-th]].

8. T. Colas, J. Grain and V. Vennin, "Quantum recoherence in the early universe," EPL **142** (2023) no.6, 69002 doi:10.1209/0295-5075/acdd94 [arXiv:2212.09486 [gr-qc]]. C. P. Burgess, T. Colas, R. Holman, G. Kaplanek and V. Vennin, "Cosmic Purity Lost: Perturbative and Resummed Late-Time Inflationary Decoherence," [arXiv:2403.12240 [gr-qc]]. Burgess, Clifford P., Richard Holman, and Gianmassimo Tasinato. "Open EFTs, IR effects & late-time resummations: systematic corrections in stochastic inflation." Journal of High Energy Physics 2016.1 (2016): 1-43.

9. C. O. Akyuz, G. Goon and R. Penco, "The Schwinger-Keldysh coset construction," JHEP **06** (2024), 004. [arXiv:2306.17232 [hep-th]].

10. P. Glorioso, M. Crossley and H. Liu, "A prescription for holographic Schwinger-Keldysh contour in non-equilibrium systems," [arXiv:1812.08785 [hep-th]].

104. Positivity of Bondi Energy in General Relativity .... Professor H. S. Reall

In General Relativity, there is no local definition of gravitational energy. However, for an isolated system one can define different notions of the *total* energy in the spacetime. The ADM energy is defined by a surface integral at spatial infinity. Bondi energy [1] is defined by a surface integral at future null infinity and provides a definition of the total energy at a certain instant of "retarded time". It is monotonically decreasing as a function of time, corresponding to loss of energy through radiation of gravitational waves.

In Newtonian gravity, gravitational energy is negative so one might wonder whether the ADM or Bondi energy could also be negative. Positive energy theorems assert that they cannot, provided that the energy of matter is positive in a suitable sense. These theorems were proved first for the ADM energy [2,3] and subsequently also for Bondi energy [4,5].

The aim of this essay is to explain the proof of the positivity of the Bondi energy. It should start with an introduction to Bondi energy and then go on to explain a proof of its positivity using 2-component spinors [6]. If space permits then the essay might discuss a related development, such as positivity of the Dougan-Mason "quasilocal" energy [7].

The essay should be written at a level that would be understood by another part III student who had taken the relevant courses.

Relevant Courses

Essential: General Relativity, Black Holes

References

1. H. Bondi, M. G. J. Van der Burg and A. W. K. Metzner, Proc. Roy. Soc. A (1962) https://doi.org/10.1098/rspa.1962.0161

2. R. Schoen and S.T. Yau, Commun. Math. Phys. 65, p45 (1979), https://link.springer.com/article/10.1007/BF01940959

3. E. Witten, Commun. Math. Phys. 80, p381 (1981), https://link.springer.com/article/10.1007/BF01208277

4. M. Ludvigsen and J Vickers, J. Phys. A15, L67 (1982), doi 10.1088/0305-4470/15/2/003

5. G. Horowitz and M. Perry, Phys. Rev. Lett. 48, p371 (1982), https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.48.371

6. R. Penrose and W. Rindler, "Spinors and spacetime", Cambridge University Press (1984)

7. A, Dougan and L. Mason, Phys. Rev. Lett. 67, p2119 (1991), https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.67.2119

105. Critical Phenomena in Gravitational Collapse .... Professor U. Sperhake, Dr D. Cors

One of the most striking consequences of the non-linear character of the theory of general relativity (GR) is that gravity sources itself. These non-linearities dominate dynamics in the strong-field regime which, due to the high complexity of the Einstein field equations, can be studied in full only through numerical methods. In the early 1990s, the modern-day numerical relativity techniques were still largely under development and yet, this time saw one of the first dramatic insights into GR gained through a numerical investigation. In 1993 Matthew Choptuik discovered a hidden simplicity within the intricate strong-gravitational-field regime: *critical phenomena in gravitational collapse* which is the subject of this essay.

If we consider the space of spacetime solutions to the Einstein field equations, Choptuik's results demonstrate that the threshold of black hole formation separates spacetimes in which the gravitational field is strong enough to form a black hole and those in which fields disperse to infinity. Near this threshold, at least in spherical symmetry, spacetimes display a set of features that bear resemblance to the phenomena emerging near critical points in other fields of physics: universality, self-similarity and scaling behaviour. As is the case with phase transitions, there are also type I and type II critical phenomena in the context of gravitational collapse depending on the matter model. Type II critical phenomena are particularly interesting from a theoretical perspective as they allow for the formation of arbitrarily small black holes and naked singularities. As is explained in Ref. [2], critical collapse has been studied in many matter models with different symmetry restrictions. The simplicity, promised by the emergence of this area of study, however is not as plain as one might have thought: some matter models allow for the existence of both type I and type II critical phenomena [3, 4] and moving beyond spherical symmetry shows a departure from a straight-forward universality [5, 6].

This essay should describe the main features and theoretical ingredients of critical phenomena in gravitational collapse in spherical symmetry, drawing analogies with the description of dynamical systems. This essay should furthermore study those cases where the threshold of gravitational collapse seems to have both types I and II of critical phenomena, including the reasons behind them and consequences of such cases. A focus on universality classes and approximate self-similarities in the context of gravitational collapse would be a way in which the subject could be investigated in more detail. Optional further work may discuss to what extent one should worry about the existence of naked singularities at the threshold of black-hole formation as discussed, for example, in the introduction of Ref. [7].

Relevant Courses

Essential: Part III General Relativity

Useful: Part III Black Holes, Part II Statistical Physics

References

1. M. W. Choptuik. Universality and scaling in gravitational collapse of a massless scalar field *Phys. Rev. Lett.*, 70 (9), 1993. https://doi.org/10.1103/PhysRevLett.70.9
2. C. Gundlach and J. M. Martín-García. Universality and scaling in gravitational collapse of a massless scalar field *Living Rev. Rel.*, 10 (9), 2007. https://link.springer.com/article/10.12942/lrr-2007-5; https://arxiv.org/abs/0711.4620
3. P. R. Brady, C. M. Chambers, and S. M. C.V. Goncalvez. Phases of massive scalar field collapse *Phys. Rev. D*, 56, 1997. https://journals.aps.org/prd/abstract/10.1103/PhysRevD.56.R6057; https://arxiv.org/abs/gr-qc/9709014.
4. E. Jimenez-Vazquez and M. Alcubierre. Critical gravitational collapse of a massive complex scalar field *Phys. Rev. D*, 106 (4), 2022. https://journals.aps.org/prd/abstract/10.1103/PhysRevD.106.044071; https://arxiv.org/abs/2206.01389.
5. T. W. Baumgarte, B. Brügmann, D. Cors, C. Gundlach, D. Hilditch, A. Khirnov, T. Ledvinka, S. Renkoff and I. Suárez Fernánadez. Critical phenomena in the collapse of gravitational waves *Phys. Rev. D*, 131 (18), 2023. https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.131.181401; https://arxiv.org/abs/2305.17171.
6. K. Marouda, D. Cors, H. R. Rüter, F. Atteneder and D. Hilditch. Twist-free axisymmetric critical collapse of a complex scalar field *Phys. Rev. Lett.*, 109 (12), 2024. https://journals.aps.org/prd/abstract/10.1103/PhysRevD.109.124042; https://arxiv.org/abs/2402.06724.
7. D. Christodoulou. The instability of naked singularities in the gravitational collapse of a scalar field. *Ann. of Math.* (2) 149 (1999), no. 1, 183-217. https://arxiv.org/abs/math/9901147

106. CPT and/or Spin-Statistics .... Professor A. C. Wall

The CPT Theorem states that every quantum field theory satisfying certain axioms is invariant under the combination of charge-conjugation, parity, and time-reversal. The closely related Spin-Statistics Theorem states that integer spin fields must obey bosonic statistics, while half-integer spin fields must obey fermionic statistics. (Although these important results are touched upon superficially in part III courses, there exist far more general and watertight proofs which are not covered by the standard lecture notes.)

Your essay should explain, in a convincing level of detail, at least one general proof (valid for interacting QFTs) of one of these theorems. As a contrastive example, you should also define at least one field theory which (as a result of violating an axiom) violates the conclusion of the theorem. Your essay should discuss the historical context of the theorem, as well as its physical implications.

One option is to discuss both theorems and explain how they are related. It is also permissible to focus on just one of the theorems, but in that case the essay would be expected to go into greater depth regarding the chosen theorem, in order to receive an equally good mark.

Relevant Courses

Essential: Quantum Field Theory

Useful: Advanced QFT, Standard Model

References

1. R.F. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That*, Princeton University Press 2000.
2. A.S. Blum, "From the necessary to the possible: the genesis of the spin-statistics theorem", European Physical Journal H, 39, 543 (2014), and references therein.
3. A.S. Blum and A.M. de Velasco, "The genesis of the CPT theorem", European Physical Journal H, 47, 5 (2022), and references therein.

107. The Gravitational Memory Effect ....

Professor C. Warnick

Gravitational wave detectors, such as LIGO, may be thought of as a collection of test masses whose relative displacements are altered by the passage of a gravitational wave. Measuring these displacements gives rise to the signal. The gravitational memory effect is the name given to phenomenon whereby the test masses do not return to their original positions after the passage of a wave. The memory effect comes in two forms - the linear [1] and nonlinear [2, 3, 4]. This essay will describe both aspects of the memory effect, and may choose either to focus on mathematical aspects or on consequences for detection.

Relevant Courses

Essential: General Relativity

Useful: Black Holes

References

1. Braginsky V, Thorne K "Gravitational-wave bursts with memory and experimental prospects". *Nature* **327**, 123–125 (1987)
2. Christodoulou D "Nonlinear nature of gravitation and gravitational-wave experiments". *Physical Review Letters.* **67** (12): 1486–1489 (1991)

3. Bieri L, Chen PN, Yau, ST "The electromagnetic Christodoulou memory effect and its application to neutron star binary mergers". *Class. Quantum Grav.* **29** 215003 (2012)

4. Thorne, K "Gravitational-wave bursts with memory: The Christodoulou effect" *Phys. Rev. D* **45**, 520 (1992)

108. The Near Horizon Geometry of Black Hole Spacetimes .... Dr Z. Wyatt

A spacetime containing a degenerate Killing horizon, such as an extremal black hole, possesses a well-defined notion of a near-horizon geometry (NHG). The NHG is a geometry in its own right, solving the same field equations. As an example, the NHG of an extremal Reissner–Nordström blackhole involves anti–de Sitter space. Extremal black holes and NHGs have been widely studied due to various applications in AdS/CFT, string theory and classification results for higher dimensional black holes.

The goal of this essay is to survey some part of the work on NHGs [1]. The essay could focus on AdS–structure theorems e.g. [2], classification results and uniqueness theorems for NHGs e.g. [3,4], supersymmetric NHGs e.g. [5, 6], or geometric area inequalities. The essay should include some discussion of at least 2 distinct topics.

Relevant Courses

Essential: General Relativity, Black Holes

References

1. Kunduri & Lucietti, "Classification of near-horizon geometries of extremal black holes", Living Reviews in Relativity (2013)

2. Kunduri, Lucietti & H. S. Reall, CQG 24 (2007) 4169 [arXiv:0705.4214]

3. Chru̇sciel, Reall & Tod, CQG 23, 549–554, (2006). [arXiv:gr-qc/0512041].

4. Kunduri & Lucietti, JMP 50, (2009). [arXiv:0806.2051].

5. Gutowski & Reall, JHEP 02 (2004). [arXiv:hep-th/0401042].

6. Reall, PRD 68, (2003). [arXiv:hep-th/0211290].

109. Vector Charge Theory for Granular Materials – Is Skepticism Warranted? .... Professor M. E. Cates

Vector charge theory (VCT) is a generalization of classical electromagnetism whose excitations include 'fractons' [1]. The latter are objects that cannot move freely but only in a restricted fashion, such as having to move in the direction of their (vector) charge. Recently, a connection was established between the electrostatics sector of VCT and the equations of static equilibrium for certain types of granular packings [2]. From this idea, novel arguments were developed for correlation functions and responses that appear to be consistent with simulations, and speculations offered about what, if anything, other sectors of the VCT (such as electrodynamics) might have to do with granular systems [3]. However, many of these arguments appear questionable in physical motivation and/or mathematical rigour: overall, the value of the VCT/granular analogy is not yet clear. The primary goal of this essay is to assess this work with a critical eye, and reach a reasoned view of how seriously it should be taken by the granular matter community. This will involve surveying the older granular literature to explore earlier models of isostatic packings, such as [4,5]. Following on from (or perhaps *en route* to) the assessment of the VCT/granular analogy, several avenues are possible. In particular, it would be good to also assess some other recent innovative papers on granular elasticity, which range from elegant but formal [6], to somewhat controversial [7]. Along the way, there may be the opportunity for original research... but this will depend on finding some solid ground on which to build!

Relevant Courses

Essential: There are no essential courses for this essay.

Useful: Granular materials, and the relevant versions of VCT, both lie in the classical realm ($\hbar = 0$). In addition, granular systems are nonthermal ($T = 0$), despite which they have strong fluctuations, closely reminiscent of the thermal fluctuations in classical statistical mechanics. Because of this, both Statistical Field Theory and Theoretical Physics of Soft Condensed Matter offer some relevant background, although neither addresses granular systems directly.

References

1. M. Pretko, Generalized electromagnetism of subdimensional particles: A spin liquid story. Phys. Rev. B 96, 035119 (2017)

2. J. N. Nampoothiri, Y. Wang, K. Ramola, J. Zhang, S. Bhattacharjee, and B. Chakraborty, Emergent elasticity in amorphous solids, Phys. Rev. Lett. 125, 118002 (2020)

3. J. N. Nampoothiri, M. D'Eon, K. Ramola, and B. Chakraborty, Tensor electromagnetism and emergent elasticity in jammed solids. Phys. Rev. E 106, 065004 (2022)

4. M. E. Cates, J. P. Wittmer, J.-P. Bouchaud, and P. Claudin, Jamming, force chains, and fragile matter, Phys. Rev. Lett. 81,1841 (1998)

5. R. C. Ball, and R. Blumenfeld, Stress field in granular systems: Loop forces and potential formulation, Phys. Rev. Lett. 115505 (2002)

6. E. Di Giuli, Field theory for amorphous solids, Phys. Rev. Lett. 121, 118001 (2018)

7. H. Charan, M. Moshe, and I. Procaccia, Anomalous elasticity and emergent dipole screening in three-dimensional amorphous solids, Phys. Rev. E 107, 055005 (2023)

110. Retinal Mosaic Formation .... Professor S. J. Eglen

The visual world is translated into neural activity by the retina, a neural structure at the back of the eye. Within the retina, cells of a given class are often spaced regularly, forming what we call *retinal mosaics*. Many theories have been proposed for how these neurons self-organise into mosaics [1]. Many of these theories focus on self-avoidance. More recently, it has been suggested that mosaics might be arranged for efficient coding of natural scenes, e.g. with respect to ON-OFF pairs [2, 3].

The essay would review these different theories of mosaic formation and highlight the advantages and disadvantages of each theory.

Relevant Courses

Essential: None.

Useful: Some background in information theory and spatial statistics.

References

1. Eglen SJ (2012) Cellular Spacing: Analysis and Modelling of Retinal Mosaics. In: Computational Systems Neurobiology (Le Novère N, ed), pp 365–385. Springer Netherlands.https://www.researchgate.net/publication/267248052_Cellular_Spacing_Analysis_and_Modelling_of_Retinal_Mosaics

2. Roy S, Jun NY, Davis EL, Pearson J, Field GD (2021) Inter-mosaic coordination of retinal receptive fields. Nature https://doi.org/10.1038/s41586-021-03317-5

3. Jun NY, Field GD, Pearson JM (2022) Efficient coding, channel capacity, and the emergence of retinal mosaics. Adv Neural Inf Process Syst 35:32311–32324 https://pubmed.ncbi.nlm.nih.gov/37168261/

111. Categorification of Cluster Algebras .... Dr N. J. Williams

Cluster algebras were introduced by Fomin and Zelevinsky in 2002 [1] in the context of Lie theory. They have subsequently found a large array of rich and surprising connections to other parts of mathematics, including geometry, number theory, and even a number of topics in physics.

Cluster algebras are commutative rings with distinguished generators known as *cluster variables*, grouped into overlapping sets known as *clusters*, which are related by a procedure known as *mutation*. Many examples of cluster algebras come from coordinate rings of algebraic varieties.

Buan, Marsh, Reineke, Reiten, and Todorov found a beautiful connection between cluster algebras and the representation theory of quivers (another word for directed graphs) [2]. From the representation theory of a quiver, they constructed a category known as the *cluster category*, which encodes information about a corresponding cluster algebra. For instance, clusters in the cluster algebra correspond to certain objects in the cluster category which are called *cluster-tilting*. The full extent of this picture was only conjectured in [2], and was subsequently proven in the subsequent literature in papers such as [3,4].

The essay should outline the definition of cluster algebras and the construction of the cluster category due to [2], and could then give some of the proofs of the relation between the two. Other subsequent directions for later parts of essay are available, subject to discussion.

Relevant Courses

Essential: Representation Theory at the level of Part II or equivalent

Useful: Part III Lie Algebras and their Representations, Part III Introduction to Geometric Representation Theory

References

1. S. Fomin and A. V. Zelevinsky, Cluster algebras. I. Foundations, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

2. A. B. Buan, B. R. Marsh, M. Reineke, I. Reiten, and G. Todorov, Tilting theory and cluster combinatorics, Adv. Math. **204** (2006), no. 2, 572–618.

3. A. B. Buan, B. R. Marsh and I. Reiten, Cluster mutation via quiver representations, Comment. Math. Helv. **83** (2008), no. 1, 143–177

4. Y. Palu, Cluster characters for 2-Calabi–Yau triangulated categories, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 6, 2221–2248

112. Logarithmic Geometry and Logarithmic Gromov-Witten Theory ...... Professor M. Gross

The theory of logarithmic schemes was developed in the 1980s by Illusie-Fontaine and Kazuya Kato. As described by Kato, a logarithmic structure on a scheme is a "magic powder" which makes relatively nice singular schemes look smooth. A typical example is a normal crossings divisor, which formally looks smooth if viewed as a log scheme.

While the original motivation for introducing log schemes was for its arithmetic applications, more recently log schemes have found powerful applications in mirror symmetry. This essay should cover the fundamentals of log geometry, and then explore applications of interest to the essay writer. A natural direction to explore is the theory of *logarithmic Gromov-Witten invariants*, which give a log version of curve counting. Without developing the full technology of Gromov-Witten invariants, the writer could explore Nishinou and Siebert's proof of Mikhalkin's tropical curve counting result [5]. A more ambitious essay can look at the definitions of logarithmic and punctured invariants of [6] and [7].

The original papers [1], [2] are dense but readable. Chapter 3 of [3] contains a more relaxed introduction to parts of the theory needed for mirror symmetry. [4] is an encyclopedic monograph on the subject. These sources are more than enough to get started. Chapter 4 of [3] contains a proof of the result of Nishinou-Siebert [5] in the two-dimensional case, and [6] develops the theory of logarithmic Gromov-Witten invariants; the most ambitious essay writer would take some of the latter material on board, but this will require some willingness to at least wave ones hands at algebraic stacks.

Relevant Courses

Essential: Part III Algebraic Geometry.

References

1. Luc Illusie, "Logarithmic spaces (according to K. Kato), Barsotti Symposium in Algebraic Geometry (Abano Terme, 1991), Perspec. Math., Vol 15, Academic Press, San Diego, CA 1994. pp. 183–203.

2. Kazuya Kato, "Logarithmic structures of Fontaine-Illusie," Algebraic Analysis, geometry, and number theory (Baltimore, MD 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 191-224.

3. Mark Gross, *Tropical geometry and mirror symmetry*, CBMS Regional Conference Series in Mathematics, **114**. American Mathematical Society, Providence, RI, 2011. xvi+317 pp

4. Arthur Ogus, *Lectures on Logarithmic Algebraic Geometry*, Cambridge University Press.

5. T. Nishinou, B. Siebert, *Toric degenerations of toric varieties and tropical curves.*, Duke Math. J. **135** (2006), 1–51.

6. M. Gross, B. Siebert, *Logarithmic Gromov-Witten invariants*, J. Amer. Math. Soc. **26**, (2013) 451–510.

7. D. Abramovich, Q. Chen, M. Gross, B. Siebert, *Punctured logarithmic maps*, preprint, arXiv:2009.07720.

113. Algebraic Stacks .... Professor M. Gross

Algebraic stacks are a vast generalization of the notion of scheme, developed partly to describe various moduli spaces. For example, $\mathcal{M}_g$, the moduli space of algebraic curves of genus g, cannot be described as a scheme, but is what is known as a Deligne-Mumford stack. Morally, this is a geometric object which is locally a quotient of a scheme by a finite group, but the geometric object remembers something about this local description. If the scheme is smooth, then we obtain the algebraic-geometric equivalent of an orbifold. More generally, an Artin (or algebraic) stack allows quotients by much more complicated equivalence relations. For example, the trivial action of an algebraic group G on the point has a well-defined quotient in the world of algebraic stacks, and this quotient plays the role of the classifying space BG in algebraic geometry. See [2] for a very brief survey, and [3] for a longer survey.

This essay would involve internalizing the (very complicated) definition of stacks, and giving some application(s). The most obvious application is the construction of the moduli space of stable curves [1]. Other possibilities include the construction of the Chow group for Artin stacks [5], and Artin's criterion for algebraicity of stacks [4]. The former will require delving into the theory of algebraic cycles, the latter into deformation theory.

There are several sources for the definitions. The original papers [1] and [4] give concise definitions, and [6] covers these in a more expansive way (but is in French). The main online resource is the immense Stacks Project [8], the latter being a vast compendium of most of algebraic geometry and probably not so useful for a beginner. There is also a good new book on the subject by Martin Olsson, [7].

Relevant Courses

Essential: Part III Algebraic Geometry (Michaelmas term).

References

1. Deligne, P., Mumford, D. *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. No. **36**, 1969, 75–109.

2. Edidin, D., *What is a... stack?*, Notices of the AMS, April 2003, 458–459.

3. Fantechi, B., *Stacks for everybody,* European Congress of Mathematics, Vol. I (Barcelona, 2000), 349–359, Progr. Math., 201, Birkhäuser, Basel, 2001.

4. Artin, M., *Versal deformations and algebraic stacks*, Inventiones Math., **27**, 165–189.

5. Kresch, A., *Cycle groups for Artin stacks,* Invent. Math. **138** (1999), 495–536.

6. Laumon, G.; Moret-Bailly, L., *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics **39**. Springer-Verlag, Berlin, 2000. xii+208 pp.

7. Olsson, M., *Algebraic spaces and stacks*, AMS Colloquium Publications, Volume 62, 2016.

8. http://stacks.math.columbia.edu/

114. The Longest Increasing Subsequence of Random Permutations .... Dr J. Ransford

Given a permutation σ on $\{1, 2, \ldots, n\}$, the length of its longest increasing subsequence is the largest k such that there are $i_1 < \cdots < i_k$ in $\{1, 2, \ldots, n\}$ with $\sigma(i_1) < \cdots < \sigma(i_k)$.

What does the length of the longest increasing subsequence $L(\sigma)$ behave like for a typical permutation σ? The answer is given by the celebrated Baik–Deift–Johansson theorem: If σ_n is a random permutation uniformly distributed in the symmetric group S_n, then as $n \to \infty$,

$$\frac{L(\sigma_n) - 2\sqrt{n}}{n^{1/6}} \xrightarrow{d} F_{\text{GUE}}$$

where F_{GUE} is the Tracy–Widom GUE distribution. This distribution is quite universal and appears in other areas of probability, such as random matrices and Kardar–Parisi–Zhang random growth models. The proof combines several areas of mathematics, including representation theory, combinatorics, the calculus of variations, determinantal point processes and complex analysis. The goal of this essay is to present a summary of the proof of the above result.

Relevant Courses

Essential: Advanced Probability

References

1. Jinho Baik, Percy Deift and Kurt Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.* 12(1999), no.4, 1119–1178.

2. Dan Romik, *The surprising mathematics of longest increasing subsequences.* Vol. 4. Cambridge University Press, New York, 2015.

115. Random Walk on Dynamical Percolation .... Professor P. Sousi

Let $G = (V, E)$ be a finite connected graph and let $p \in (0, 1)$. Assign independent Poisson processes of parameter μ to every edge $e \in E$. At the time of the Poisson process, the state of the edge e refreshes to open with probability p and closed with probability $1 - p$. Let X be a continuous time random walk that moves as follows: it stays at the current vertex for an exponential time of parameter 1 and then chooses one neighbour (in the graph G) equally likely. If the edge connecting the two vertices is open, then the walk makes the jump, otherwise it stays in place.

How long does it take for the walk to reach equilibrium? What is the maximum expected hitting time and the cover time? In recent works, estimates on these quantities have been established, but the bounds are not sharp and in the case of $G = \mathbb{Z}_n^d$ they do not cover the whole supercritical regime.

A successful essay will contain background on the problem, give an account of the current state of the art and include proofs of some results.

Relevant Courses

Essential: Advanced Probability, Mixing Times of Markov Chains

References

1. Y. Peres, A. Stauffer and J. Steif. Random walks on dynamical percolation: mixing times, mean squared displacement and hitting times. Probab. Theory Related Fields 162 (2015), no. 3-4, 487-530.

2. Y. Peres, P. Sousi and J. Steif. Mixing time for random walk on supercritical dynamical percolation. Probab. Theory Related Fields 162 (2020), no. 3-4, 809-849.

3. J. Hermon and P. Sousi. A comparison principle for random walk on dynamical percolation. Ann. Probab. 48 (2020), no. 6, 2952-2987.

116. Out of Distribution Generalisation .... Professor R. D. Shah

The assumption that data are i.i.d. is ubiquitous in both the analysis and design of regression methods. However particularly when data have been collected over time, or in different environments, this assumption is rarely satisfied. Instead, the distribution of training and test data may be different and as a consequence, the real world performance of statistical and machine learning methods can be substantially worse than expected. A related problem is that of extrapolation: we may need to make predictions beyond the range of the observed predictors. These are issues of genuine practical importance, and there is now a rapidly growing body literature in both Statistics and Machine Learning attempting to address this [1].

One line of work that has attracted much attention involves exploiting causal connections between training and test data [2, 3, 4]. Other work has involved proposing new models for the data-generating process that can accommodate out of distribution generalisation [5,6].

Other approaches involve looking at either worst case or random [7] perturbations from the data at hand.

This essay could take one of several directions. One approach might compare and contrast some of the different methods introduced relating to this problem, empirically and/or theoretically. Another direction might involve focusing on one or a few approaches and giving a more in depth account of the theory and methods. As the area is still in its infancy, it may be possible to develop small extensions of the methods or theory, though this would not be required.

Relevant Courses

Essential: Modern Statistical Methods

Useful: Causal Inference

References

1. Liu et al., "Towards Out-Of-Distribution Generalization: A Survey" https://arxiv.org/pdf/2108.13624
2. Xinwei Shen, Peter Bühlmann, Armeen Taeb "Causality-oriented robustness: exploiting general additive interventions" https://arxiv.org/pdf/2307.10299
3. Henzi et al. "Invariant Probabilistic Prediction" https://arxiv.org/pdf/2309.10083
4. Rojas-Carulla et al. "Invariant Models for Causal Transfer Learning" JMLR, 19 (2018) 1–34
5. Xinwei Shen and Nicolai Meinshausen, "Engression: Extrapolation through the Lens of Distributional Regression" https://arxiv.org/pdf/2307.00835
6. Pfister and Bühlmann, "Extrapolation-Aware Nonparametric Statistical Inference" https://arxiv.org/abs/2402.09758
7. Dominik Rothenhäusler, Peter Bühlmann, "Distributionally robust and generalizable inference" https://arxiv.org/abs/2209.09352

117. Data-Driven Spectral Problems for Continuous-Time Dynamical Systems .... Dr M. Colbrook

Consider an ODE system:

$$\frac{dx}{dt} = F(x), \quad x \in \mathcal{X} \subset \mathbb{R}^d, t \geq 0,$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is in general non-linear. If we consider a suitable smooth function $g : \mathcal{X} \to \mathbb{C}$, then the chain rule implies that

$$\frac{\partial g}{\partial t}(x) = F \cdot \nabla g(x).$$

This is a *linear* equation for the evolution of g. We have traded something finite-dimensional and non-linear for something infinite-dimensional and linear. We can consider the generator of this semigroup, which is a linear operator known as the Koopman generator. There is a huge amount of interest in computing the spectral properties of this operator from trajectory data, for example, using methods that build finite matrix approximations [1,2]. This is particularly useful for non-linear or high-dimensional systems, with applications in many fields, such as fluid dynamics, biology, and engineering. This spectral data encodes a lot of useful information about the dynamical system.

The aim of this essay is to explore how these ideas can be integrated with recent methods developed for the data-driven analysis of discrete-time dynamical systems. For instance, some methods achieve this with error control [3], while others compute spectral measures of measure-preserving systems [4] (in the continuous-time case, this corresponds to the spectral measures of skew-Hermitian systems). Issues such as the temporal resolution of data needed to approximate this quantities should also be discussed.

The essay is organized into the following stages:

- **Literature Review:** A comprehensive review of the papers [1–4], along with other relevant literature on Koopman operators.
- **Algorithm Development:** The formulation of algorithms that integrate recent advancements in discrete-time systems with methods tailored for continuous-time systems.
- **Numerical Simulations:** The assessment of the new algorithms through numerical simulations on various dynamical systems, including the nonlinear pendulum, the Lorenz system, and other standard benchmark tests.

Relevant Courses

Essential: Linear Analysis Courses

Useful: PDE Courses

References

1. Klus, Stefan, et al. Data-driven approximation of the Koopman generator: Model reduction, system identification, and control. Physica D: Nonlinear Phenomena 406 (2020): 132416.
2. Giannakis, Dimitrios, and Claire Valva. Consistent spectral approximation of Koopman operators using resolvent compactification. Nonlinearity 37.7 (2024): 075021.
3. Matthew J Colbrook and Alex Townsend. Rigorous data-driven computation of spectral properties of koopman operators for dynamical systems. arXiv preprint arXiv:2111.14889, 2021.
4. Colbrook, Matthew J., Catherine Drysdale, and Andrew Horning. Rigged Dynamic Mode Decomposition: Data-Driven Generalized Eigenfunction Decompositions for Koopman Operators. arXiv preprint arXiv:2405.00782 (2024).

118. Analysis and Computer Assisted Proofs .... Professor A. C. Hansen

The problem of proving blow-up of the 3D Euler equation with smooth initial data is considered one of the major open problems in nonlinear PDEs. In connection with that T. Hou poses, in Problem 2 in [7], the question: "*In problems where mathematical analysis precedes a computer-assisted step: How can we ensure that the formulation of the problem is correctly posed so that computability and non-computability of a problem can be determined?*" This question is motivated by the challenge of proving finite-time blow-up of the 3D Euler equation with smooth initial data through a computer-assisted proof – which was recently announced by J. Chen and T. Hou [1,2], thereby solving a century-long open problem. Consider the 3D Euler equations

$$\boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{1}$$

The delicate issue is that a blow-up of the solution to (1) would imply instability in terms of unboundedness [9] of the forward operator taking the initial data to the solution at a given time. One might initially think, as suggested in [9], that this means that 3D Euler blow-up is not computable. Indeed, unboundedness of the solution operator typically yields non-computability of the PDE solution [8]. This complication could hinder the prospects of a computer-assisted proof, where the validity of the computational step needs to be verifiable.

The topic of this essay is to investigate how it is possible to provide a computer assisted proof of blow-up of the 3D Euler equations despite the concerns of the unboundedness of the forward operator. It turns out that this is related to previous issues in computer assisted proofs, where non-computable problems have successfully been used. For example, the computer assisted proof of the Dirac-Schwinger conjecture [4,5,6] (about the asymptotic behaviour of the ground state energy of certain families of Schrödinger operators) by C. Fefferman and L. Seco is based on non-computable problems. The seemingly rather bizarre phenomenon – that non-computable problems can be used in computer assisted proofs – can be explained by the theory of the Solvability Complexity Index hierarchy [3]. The question is: Can this theory help certify the validity of the computer assisted proof of blow-up of 3D Euler?

Relevant Courses

Essential: Part III analysis: either Functional Analysis or PDEs

Useful: Part II: Analysis

References

1. J. Chen and T. Y. Hou. Stable nearly self-similar blowup of the 2d boussinesq and 3d euler equations with smooth data. *arXiv:2210.07191v2*, 2022.

2. J. Chen and T. Y. Hou. Stable nearly self-similar blowup of the 2d boussinesq and 3d euler equations with smooth data ii: Rigorous numerics. *arXiv:2305.05660*, 2023.

3. M. Colbrook and A. C. Hansen. The foundations of spectral computations via the Solvability Complexity Index hierarchy. *J. Eur. Math. Soc.*, (2023).

4. C. Fefferman and L. Seco. On the energy of a large atom. *Bull. Amer. Math. Soc. (N.S.)*, 23(2):525–530, 1990.

5. C. Fefferman and L. Seco. Eigenvalues and eigenfunctions of ordinary differential operators. *Adv. Math.*, 95(2):145–305, 1992.

6. C. Fefferman and L. Seco. On the Dirac and Schwinger corrections to the ground-state energy of an atom. *Adv. Math.*, 107(1):1–185, 1994.

7. C. Fefferman, A. C. Hansen, and S. Jitomirskaya, editors. *Computational mathematics in computer assisted proofs*, American Institute of Mathematics Workshops. American Institute of Mathematics, 2022. Available online at https://aimath.org/pastworkshops/compproofsvrep.pdf.

8. M. B. Pour-El and J. I. Richards. *Computability in analysis and physics.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989.

9. A. F. Vasseur and M. M. Vishik. Blow-up solutions to 3d euler are hydrodynamically unstable. *Communications in Mathematical Physics*, 378:557 – 568, 2019.

119. Limits and Potential in Computational Quantum Mechanics ....
Professor A. C. Hansen

Despite the plethora of papers in both computational and mathematical quantum mechanics, many of the key questions regarding Schrödinger and Dirac operators have remained open. These questions include problems about how to compute both solutions to PDEs, spectra of operators, and the behaviour of the respective solutions and spectra. Interestingly, the foundations of computations have played a key role in solving many of these open problems. This includes the work by W. Arveson [1,2], the recent results on the undecidability of the spectral gap by T. Cubitt, D. Perez-Garcia & M. Wolf [4,5], and the work by R. de la Lave and C. Fefferman [6,7].

This project is about the limits and the potential in computational quantum mechanics – both in pure mathematical questions, as well as in applications. The word 'limits' is to be understood in two ways: (1) limits – as in 'there are limitations' to what can actually be computed and also proved. In particular, recent developments have demonstrated undecidability/non-provability in key questions in quantum mechanics. The second interpretation interpretation of the word 'limits' is as follows (2): as in 'solutions can be obtained by taking limits'. It turns out that both interpretations of the word 'limits' are intimately related and also related to the potential of computational quantum mechanics in both pure and applied mathematics.

The specific task of this project is to investigate some of the results above, for example the undecidability results on the spectral gap. The negative result does not say anything about which spectral gap problems can actually be handled by algorithms, nor does it say anything about the number of limits needed in the general case. Moreover, it is suspected that the proof of the undecidability of the spectral gap can be substantially simplified through recent results [3,8].

The candidate choosing this essay should have a strong background in analysis, and the project is designed for students who really enjoy functional analysis. Some knowledge of computations may be useful.

Relevant Courses

Essential: Part II: Linear Analysis, Part III: Functional Analysis

Useful: Part II: Automata and Formal Languages (this is really not required at all, but it never hurts to have heard the word "computability")

References

1. W. Arveson. Noncommutative spheres and numerical quantum mechanics. In *Operator algebras, mathematical physics, and low-dimensional topology (Istanbul, 1991)*, volume 5 of *Res. Notes Math.*, pages 1–10. A K Peters, Wellesley, MA, 1993.

2. W. Arveson. C^*-algebras and numerical linear algebra. *J. Funct. Anal.*, 122(2):333–360, 1994.

3. M. Colbrook and A. C. Hansen. The foundations of spectral computations via the Solvability Complexity Index hierarchy. *J. Eur. Math. Soc.*, 2023.

4. T. Cubitt, D. Perez-Garcia and M. Wolf. Undecidability of the spectral gap. *Nature*, 528, 207-211, 2015

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7. C. Fefferman and R. de la Llave. Relativistic stability of matter. I. *Rev. Mat. Iberoamericana*, 2(1-2):119–213, 1986.

8. A. C. Hansen. On the Solvability Complexity Index, the n-pseudospectrum and approximations of spectra of operators. *J. Amer. Math. Soc.*, 24(1):81–124, 2011.

120. Interval Arithmetic, Trust and the SCI Hierarchy ....
Professor A. C. Hansen

The Solvability Complexity Index (SCI) hierarchy [1,2,3,6] provides a classification hierarchy in the foundations of computational mathematics, classifying problems in terms of their difficulty. Proofs of classification results in the SCI hierarchy typically have two components: (1) proving the correct lower bound, that is, showing the lower bound of where the problem can lie in the SCI hierarchy; (2) proving the correct upper bound, that is, provide an algorithm that matches the lower bound proved in (1). The second component of a SCI proof often provides new optimal algorithm that – if the problem is sufficiently low in the SCI hierarchy – guarantees a trustworthy computation.

However, implementing algorithms – that provide upper bounds in the SCI hierarchy – to guarantee a trustworthy output, is a rather delicate operation. Indeed, in order to guarantee a 100% reliable calculation the implementation has to be done in interval arithmetic. This is highly non-trivial, both from a theoretical and practical point of view.

This project is about designing interval arithmetic algorithms from the many new algorithms that the SCI research has provided over the last years. This is very similar to the program initiated by C. Fefferman in the 1990s [4] on the use of interval arithmetic in computations in quantum mechanics. Similar ideas can also be found in the computational part of the proof of the Kepler conjecture [5,7].

The project will certainly have a programming aspect to it, however, as in the case of Fefferman's program, there are substantial theoretical challenges to be overcome.

Relevant Courses

Essential: Part II: Linear Analysis

Useful: Part III: Topics in Convex Optimisation, Part III: Functional Analysis (depends on the computer assisted proof chosen)

References

1. J. Ben-Artzi, M. Marletta, and F. Rösler. Computing scattering resonances. *Journal of the European Mathematical Society*, 2023.

2. M. Colbrook and A. C. Hansen. The foundations of spectral computations via the solvability complexity index hierarchy. *J. Eur. Math. Soc.*, 2023.

3. P. Doyle and C. McMullen. Solving the quintic by iteration. *Acta Mathematica*, 163(3-4):151–180, 1989.

4. C. Fefferman and L. Seco. Interval arithmetic in quantum mechanics. In *Applications of interval computations*, pages 145–167. Springer, 1996.

5. T. C. Hales and et al. A formal proof of the kepler conjecture. *Forum of Mathematics, Pi*, 5:e2, 2017.

6. A. C. Hansen. On the solvability complexity index, the n-pseudospectrum and approximations of spectra of operators. *Journal of the American Mathematical Society*, 24(1):81–124, 2011.

7. J. C. Lagarias. *Bounds for Local Density of Sphere Packings and the Kepler Conjecture*, pages 27–57. Springer New York, New York, NY, 2011.

121. The Invariant Subspace Problem and II_1-factors .... Professor A. C. Hansen

In the mid 2000s Haagerup and Schultz provided a great breakthrough for the invariant subspace problem (in the Hilbert space setting), with their von Neumann algebra approach to this classical problem in operator theory. While these techniques are unlikely to solve the invariant subspace problem entirely, they provided a rather complete theory of invariant subspaces for elements in II_1-factors. Indeed, even if the invariant subspace problem is solved in the affirmative, the techniques provided by Haagerup and Schultz provide crucial tools to characterise the projections onto the invariant subspaces.

This project is about understanding how the techniques of Haagerup and Schultz extend the classical techniques of creating invariant subspaces from the functional calculus. In particular, in the classical setting, if the spectrum of the operator can be split into two separate disjoint components, integrating the resolvent along a Jordan curve encapsulating one of the components creates a projection onto a non-trivial invariant subspace. Haagerup and Schultz take this further and show that for operators in a II_1-factor, one can apply a similar technique, even if the spectrum of the operator is connected (but not a singleton).

As a bonus problem, it may also be possible to extend the results of Haagerup and Schultz to go beyond II_1-factors, and demonstrate invariant subspaces in classes of operators outside the classical theory of II_1-factors. The idea is to 'cut through' the spectrum of the operator with a Jordan curve (inspired by Haagerup and Schultz), and integrate the resolvent along this curve. This approach can be extended to operators where one can control the growth of the resolvent close to the spectrum. This idea is related to results on the Brown measure that extends [2] some of the techniques by Haagerup and Schultz.

The candidate choosing this essay should have a strong background in analysis, and the project is designed for students who really enjoy functional analysis.

Relevant Courses

Essential: Part II: Linear Analysis, Part III: Functional Analysis

References

1. U. Haagerup and H. Schultz. Invariant subspaces for operators in a general II_1-factor. *Publications mathematiques de l'IHES*, volume 109, pages 19–111, 2006.

2. A. C. Hansen. On the Solvability Complexity Index, the n-pseudospectrum and approximations of spectra of operators. *J. Amer. Math. Soc.*, 24(1):81–124, 2011.

122. Parity Violation in Cosmological Correlators? ....
Professor A. Challinor

Observations have established that cosmological fluctuations are statistically isotropic and homogeneous. In this essay you should consider whether they are also statistically invariant under reflection, i.e., parity.

Our Universe is currently dominated by dark matter and dark energy, but the underlying physics behind these extensions to the Standard Model of particle physics is not understood. Similarly, the physics behind the inflationary phase, thought to have occurred in the very early Universe, is unknown. Could the physics underlying any of these processes violate parity symmetry, as for the weak interaction? If so, such parity violations should be imprinted in the initial conditions for the perturbations of cosmic fields, or during their subsequent evolution. An example of such physics might be dark energy described by a self-interacting pseudo-scalar field, which couples to electromagnetism through a Chern–Simons interaction [1]. Spacetime variations of the pseudo-scalar field would lead to a rotation of the plane of linear polarization in the cosmic microwave background, leading to parity-violating correlations between its E- and B-modes, e.g., [2]. Intriguingly, there are hints for such correlations in current CMB data, e.g., [3, 4]. Cosmological parity violations could also be imprinted during inflation, and show up in higher-order correlators of the CMB and large-scale structure. There have been recent claims of parity-violating signatures in the connected four-point function (trispectrum) of the galaxy distribution [5, 6], although the robustness of these has been questioned [7]. No signatures of parity violation have been found with measurements of higher-order correlators of the CMB temperature [8] and polarization [9].

Your essay might begin by reviewing which cosmological correlators are sensitive to parity violations in a universe that respects statistical isotropy and homogeneity. You could then discuss the physics of one or two mechanisms for parity violation beyond the Standard Model, such as the Chern–Simons interactions mentioned above, which could impact the evolution (or generation) of cosmological fluctuations. The impact of these on observable correlators of the fluctuations in the CMB or matter distribution should also be reviewed. Finally, you should consider current constraints on such models and some of the difficulties encountered in making such inferences.

Relevant Courses

Essential: Cosmology; Field Theory in Cosmology

Useful: General Relativity

References

1. Carroll S., 1998, Phys. Rev. Lett., 81, 3067 (arXiv:astro-ph/9806099)

2. Komatsu E., 2022, Nature Reviews Physics, 4, 452 (arXiv:2202.13919)

3. Minami Y. & Komatsu E., 2020, Phys. Rev. Lett., 125, 221301 (arXiv:2011.11254)

4. Diego-Palazuelos P., et al., 2022, Phys. Rev. Lett., 128, 091302 (arXiv:2201.07682)

5. Philcox O., 2022, Phys. Rev. D, 106, 063501 (arXiv:2206.04227)

6. Hou J., Slepian Z. & Cahn R., 2022, Mon. Not. R. Astron. Soc., 522, 5701 (arXiv:2206.03625)

7. Philcox O. & Ereza, J., 2024 (arXiv:2401.09523)

8. Philcox O., 2023, Phys. Rev. Lett., 131, 181001 (arXiv:2303.12106)

9. Philcox O. & Shiraishi M., 2024, Phys. Rev. D, 109, 083514 (arXiv:2308.03831)

123. Granular Column Collapse and Critical State Theory $\mu(I)$ Rheological Models ....

Professor J. A. Neufeld

The collapse of a column of granular material is an archetypal experimental test of granular material, and of the rheological models which characterise their deformation and flow. Amongst the key observable parameters are the distance to which the granular pile might deform and slump, the angle which the pile ultimately achieves, and time over which the pile deforms to that ultimate static state. Models of the slumping of the granular column have focused on either using numerical models to simulate the frictional contact between a collection of grains, or on continuum models which aim to encapsulate these frictional granular interactions using an effective rheology, $\mu(I)$, which depends on the inertial number I which can be interpreted as the micro-scale rearrangement time to the large-scale shear rate.

Recently, the original granular slumping experiments have been revisited using a cohesive granular material. Cohesive forces between the grains may support finite stress, and as a result granular columns which start with small enough aspect ratio may remain undeformed even when released. The addition of cohesion between grains presents a non-trivial extension to classical $\mu(I)$ rheologies. This essay will survey the $\mu(I)$ granular rheology, focusing on the granular column collapse as the archetypal setting, and consider how cohesion may be incorporated into the general modelling framework. Students may wish to comment on how this extension is related to the *Cam clay* model of cohesive soils.

Useful: Slow Viscous Flow, Fluid Dynamics of the Solid Earth

References

1. Lajeunesse, E. and Monnier, J. B. and Homsy, G. M. 2005 Granular slumping on a horizontal surface. *Phys. Fluids*, **17**:10, 103302.

2. Gans, A. *et al* 2023 Collapse of a cohesive granular column. *J. Fluid Mech.* **959**, A41.

3. Blatny, L., Gray, J.M.N.T., Gaume, J. 2024 A critical state $\mu(I)$-rheology model for cohesive granular flows. *J. Fluid Mech.* **997**, A67.

124. Correlations of Quantum Spin Systems ....
Dr A. Capel Cuevas

Quantum many-body systems, particularly those involving interacting spins, are at the heart of contemporary research in condensed matter physics, quantum information, and material science. These systems describe a large number of particles interacting following the laws of quantum mechanics, leading to complex collective behaviours that often cannot be predicted by simply analysing individual components. Studying correlations within these systems is essential because they reveal intricate patterns of entanglement, phase transitions, and emergent phenomena, such as magnetism, superconductivity, and topological states, which are crucial for advancing quantum technology. The study of quantum spin systems specifically arises from the need to understand and control these correlations at microscopic scales. Insights from this research not only deepen our understanding of fundamental physics but also inform the development of quantum computers, new materials, and potential applications in quantum communication.

The most standard way to measure correlations of quantum spin systems at equilibrium is through the so-called *operator correlation* or *covariance*. The physical properties of the system are described by a Hamiltonian, and the Gibbs state at any positive temperature encodes the properties of the system at equilibrium. Given a lattice and two spatially separated regions in it, the covariance measures how far the reduced Gibbs state on both regions is from being a tensor product. For reasonable systems, this quantity is expected to decay exponentially with the distance between those regions. However, it generally grows with the size of (part of) those regions. For finite-range Hamiltonians, at high enough temperature this quantity grows linearly with the size of the boundaries of both regions [1]; for exponentially-decaying interactions, with the full size of both regions [2]; and for polynomially-decaying interactions, it grows exponentially with the full size of both regions [3].

The aim of this essay is to understand the main result in [2] and complement its proof. In the formulation that appears in [2], it seems that the covariance scales only linearly with the full size of both regions, but the proof only yields an exponential scaling. In this essay, we will use the techniques of [3] to derive a full proof for the decay of the covariance in the setting of [2], obtaining an exponential scaling. We will explore a possible improvement to a linear scaling with the techniques derived in [1].

Relevant Courses

Essential: Classical Dynamics, Principles of Quantum Mechanics, Statistical Physics.

Useful: Applications of Quantum Mechanics, Quantum Entanglement in Many-Body Physics

References

1. M. Kliesch, C. Gogolin, M. J. Kastoryano, A. Riera, J. Eisert, *Locality of temperature*, Phys. Rev. X 4, 031019 (2014).

2. J. Fröhlich, D. Ueltschi, *Some properties of correlations of quantum lattice systems in thermal equilibrium*, J. Math. Phys. 56, 053302 (2015).

3. D. Kim, T. Kuwahara, K. Saito, *Thermal Area Law in Long-Range Interacting Systems*, arXiv:2404.04172 (2024).

125. Solving the Technical Challenge of Constraining Non-Gaussianity from the Cosmic Microwave Background .... Dr J. Fergusson

Non-Gaussianity is a powerful probe of the physics of the early universe. Specifically, the bispectrum encodes information on any dynamics beyond the free field model of inflation opening a unique window into the physics of the early universe. The equations for creating an estimator for the bispectrum are easy to derive, but naive implementations are computationally intractable, particularly for non-separable shapes. This essay will explore the multitude of technical advances required to make this task possible in practice.

The essay should cover the following topics:

- Motivation for studying non-Gaussianity
- Derivation of the bispectrum estimator
- Discussion of methods for constraining seperable bispectra (KSW,Binned,Wavelets)
- Discussion of methods for constraining non-separable bispectra (Modal/CMBBest) with a discussion of the following challenges:
  - Ensuring the stability and orthogonality of the basis functions and the projections from primordial to observed bispectrum.
  - Accurate tetrapyd integration approaches.
  - Diffusive inpainting of masked regions to reduce mode coupling.

Relevant Courses

Essential: Cosmology

Useful: Field Theory in Cosmology

References

1. Planck 2018 results. IX. Constraints on primordial non-Gaussianity
   Planck Collaboration
   (General review of area)
   https://arxiv.org/abs/1905.05697

2. High-resolution CMB bispectrum estimator with flexible modal basis
   Sohn, W. and Fergusson, J. R. and Shellard, E. P. S.
   (Description of CMBBest and tetrapyd integration techniques)
   https://arxiv.org/abs/2305.14646

3. Using inpainting to construct accurate cut-sky CMB estimators
   Gruetjen, H. F. and Fergusson, J. R. and Liguori, M. and Shellard, E. P. S.
   (Description of inpainting technique)
   https://arxiv.org/abs/1510.03103

4. Efficient optimal non-Gaussian CMB estimators with polarisation
   Fergusson, J. R.
   (Description of Modal estimator)
   https://arxiv.org/abs/1403.7949

126. Efficacy of the Bispectrum as a Cosmological Observable .... Professor P. Shellard

Despite some apparent tensions in cosmological parameters, there is compelling evidence for the inflationary Λ-CDM model, which has become *the standard cosmology.* Nevertheless, there remain a multitude of models for early universe inflation that remain consistent with cosmological observations, even after comparison with CMB data from the Planck satellite. This degeneracy arises because the ability to distinguish between different models using only the power spectrum is largely limited to two parameters, the tensor-to-scalar ratio, r, and the spectral index, n_s. However, there is a good prospect that new insights will come from improving measurements of the bispectrum (or three-point correlation function) and other correlators of primordial fluctuations, especially using next-generation CMB (such as the Simons Observatory) and galaxy surveys (DESI, Euclid satellite, LSST etc). In principle, non-Gaussian correlations encode an enormous amount of information about fundamental physics in the early universe, and provide a powerful tool to discriminate between specific microscopic models (see [1] for an overall review).

The purpose of this essay is to explore the prospects for primordial non-Gaussianity to probe models of inflation. This essay should cover the following three topics, but these need not be equal, with students choosing which parts to emphasise–either on bispectrrum calculations for specific models of primordial non-Gaussianity or on the confrontation between these predictions and observations of the CMB and LSS.

1. A review of key concepts and theoretical underpinnings, why the bispectrum offers the first insight into non-linear interactions and evolutionary effects. Predictions for standard single-field slow-roll inflation models [5, 3]. Brief survey of alternative mechanisms for larger non-Gaussianity (refer to reviews such as [6], [9] and references therein, or see the model section of the 2018 Planck NG paper [10] and/or [1]).

2. A choice of a specific class of non-Gaussian models for for an in-depth discussion, describing how NG is generated and its 'shape'. E.g. a contemporary example could be the so-called "cosmological collider" signal which can reveal the presence of additional massive fields during inflation (see, for example, [4], [7], and [8]).

3. A review of observational techniques for detecting non-Gaussianity. (See e.g. [11] and the Planck NG constraints [10]) The outlook for measuring non-Gaussianity from the chosen models using upcoming survey data of large-scale structure or the CMB. (See [1] and references within)

Relevant Courses

Essential: Cosmology.

Useful: Field Theory in Cosmology, Quantum Field Theory, General Relativity.

References

1. P. Daniel Meerburg and others., "Astro2020 Science White Paper - Primordial Non-Gaussianity", [arXiv:1903.04409]

2. Akrami, Y and others, "Planck 2018 results. X. Constraints on inflation", A & A, 641 (2020) A10 [1807.06211].

3. J. M. Maldacena, “Non-Gaussian features of primordial fluctuations in single field inflationary models,” JHEP **0305**, 013 (2003) [astro-ph/0210603].
4. N. Arkani-Hamed, and J.M. Maldacena, "Cosmological Collider Physics", arXiv:1503.08043 (2015)
5. P Creminelli and M Zaldarriaga. “Single-Field Consistency Relation for the 3-Point Function." JCAP, 0410:006, 2004 [arXiv:astro-ph/0407059].
6. X. Chen, “Primordial Non-Gaussianities from Inflation Models,” Adv. Astron. **2010**, 638979 (2010) [arXiv:1002.1416 [astro-ph.CO]].
7. W. Sohn, D-G. Wang, Fergusson, J.R. and Shellard, E. P. S., "Searching for cosmological collider in the Planck CMB data", (arxiv:2404.07203) JCAP, 09, 016, (1924).
8. G. Cabass, O. Philcox, M. Ivanov, K. Akitsu, S-F. Chen, Shi-Fan, M. Simonović, and M. Zaldarriaga, "BOSS Constraints on Massive Particles during Inflation: The Cosmological Collider in Action", arXiv:2404.01894 (2024).
9. S. Renaux-Petel, “Primordial non-Gaussianities after Planck 2015: an introductory review", [arXiv:1508.06740]
10. Akrami, Y. and others, “Planck 2018 results. IX. Constraints on primordial non-Gaussianity", A & A , 641, A9 (2020) [arXiv:1905.05697] (astro-ph)]; Ade, P. A. R. and others, “Planck 2015 results. XVII. Constraints on primordial non-Gaussianity", [arXiv:1502.01592]; See also the review section of Planck 2013 results.
11. M. Liguori, E. Sefusatti, J. R. Fergusson and E. P. S. Shellard, “Primordial non-Gaussianity and Bispectrum Measurements in the Cosmic Microwave Background and Large-Scale Structure,” Adv. Astron. **2010**, 980523 (2010) [arXiv:1001.4707 [astro-ph.CO]].

127. Neutrinos and Light Relic Particles in Cosmology ....
Professor B. D. Sherwin

Relativistic, weakly interacting particles such as neutrinos are an important component in the early universe, with the cosmic neutrino background making up $\approx 40\%$ of the energy density at high redshifts.

There are several open questions in neutrino physics that cosmology is best positioned to answer. For example, oscillation experiments imply that neutrinos have different masses, but the value of these masses is still unknown. A determination of the unknown neutrino mass would reveal a new scale in physics and could determine the neutrino mass ordering; it could even contribute to understanding the mechanism by which neutrinos obtain their (surprisingly small) mass.

At the same time, cosmological probes have the ability to constrain the effective number of neutrinos in the early universe. Beyond just constraining the well-known number of standard model neutrino species, this implies that cosmology can search for new, weakly interacting relativistic particle species. With clear targets, upcoming experiments will be able to either discover or rule out broad classes of such beyond-the-standard-model particles.

This essay should discuss the insights cosmology can give us about i) the neutrino mass and other properties of neutrinos and ii) the presence of new, as yet undiscovered light relic particles in our Universe. While both topics should be covered, students may choose to emphasize one.

Within topic i), after some background review of the physics of cosmic neutrinos and the neutrino mass, the essay should explain how cosmology can probe the mass of neutrinos. Students should provide some discussion of the observational status of and the prospects for the mass measurement, and may wish to comment briefly on claims of a "negative neutrino mass" tension. Students could consider also briefly discussing the complementarity of cosmological measurements with particle physics experiments.

Within topic ii), the essay should explain how constraints on N_{eff}, the number of effective neutrino species, enable searches for new light particles in the early Universe. The essay should discuss targets for such searches and which kinds of new physics could be discovered or constrained by such measurements. The essay should also explain carefully how light relics affect cosmological observations and comment on observational bounds and prospects for future experiments.

Relevant Courses

Essential: Cosmology

Useful: Advanced Cosmology, Quantum Field Theory, General Relativity

References

1. CMB-S4 Science Book 2016, arXiv:1610.02743 (relevant chapters on neutrino mass and light relics)
2. Lesgourges, J. and Pastor, S., 2012, arxiv:1212.6154 (further details can be found in arXiv:astro-ph/0603494 by the same authors)
3. Green, D. and Meyers, J. arXiv:2407.07878
4. Hou, Z. et al. 2013, Physical Review D, vol. 87, Issue 8, id. 083008
5. Baumann, D., Green, D., and Wallisch, B., 2016, Journal of Cosmology and Astroparticle Physics, Volume 2016, Issue 01, pp.007-007

128. Wall-Crossing and Moduli Spaces ....

Dr F. Rezaee

In algebraic geometry, the notion of the moduli space, which is a space parametrizing geometric objects with given properties, is fundamental. Hilbert scheme is an essential type of moduli space where the geometric objects of interest are embedded in a certain ambient space. Despite its natural definition, Hilbert schemes are highly mysterious to understand: Murphy's law for the Hilbert schemes ([1]) says that there is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme [2].

While classical methods fail to understand complicated Hilber schemes, wall-crossing with respect to Bridgeland stability conditions provides a powerful tool for understanding such subtle spaces.

The goal of this essay is to review the following basic relevant notions such as derived categories ([3,4, 5]) and stability conditions ([6,7,8]). Time permitting, students could also consider some more advanced materials (e.g., [8,9,10,11]).

Relevant Courses

Essential: Part III Algebraic Geometry

References

1. R. Vakil. Murphy's law in algebraic geometry: badly-behaved deformation spaces.Invent. Math., 164(3):569–590, 2006.

2. J. Harris , I. Morrison, Moduli of curves, GMT, Vol 187

3. D. Huybrechts, Fourier-Mukai transforms in algebraic geometry, 2006

4. A. Caldararu, Derived categories of sheaves: a skimming, arXiv:math/0501094

5. R. P. Thomas, Derived categories for the working mathematician. In "Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds", January 1999, eds C. Vafa and S.-T. Yau. AMS/IP Studies in Advanced Mathematics (International Press, Cambridge MA), 2001, 349-361.

6. A. Bayer, A tour to stability conditions on derived categories.

7. E. Macr'ı and B. Schmidt. Lectures on Bridgeland stability. Lect. Notes Unione Mat. Ital., 21:139–211, Springer, Cham, 2017.

8. T. Bridgeland, Stability conditions on triangulated categories, Annals of Mathematics, 166 (2007), 317–345

9. B. Schmidt. Bridgeland stability on threefolds—First wall crossings. J. Algebraic Geom., 29(2):247–283, 2020.

10. P. Gallardo, C. Lozano Huerta, and B. Schmidt. On the Hilbert scheme of elliptic quartics. Michigan Math. J., 67(4):787–813, 2018.

11. E. Macrì and B. Schmidt, Lectures on Bridgeland stability, Lect. Notes Unione Mat. Ital., vol. 21, Springer, Cham, 2017, pp. 139–211.

12. F. Rezaee, Geometry of canonical genus four curves, Proceedings of the LMS, vol 128 (1), 2024.

129. Marton's Conjecture and Entropy Inequalities .... Professor I. Kontoyiannis

The goal of this essay is to explore some exciting recent developments in additive combinatorics, and their connections with information-theoretic ideas and techniques. In two papers that appeared over the past year, several important cases of Marton's conjecture are resolved [1,2]. One of the key elements of the proofs is the use of information-theoretic methods. Specifically, some new so-called sumset entropy bounds are developed and used. These bounds follow a series of works exploring a natural and very fruitful correspondence between information-theoretic inequalities for sums and differences of random variables, and sumset bounds in additive combinatorics. This correspondence was formally introduced by Ruzsa in [4], and an outline of the subsequent development is given in [3].

This essay will first describe the sumset bound correspondence for combinatorics and entropy, along the lines of the discussions in [3,4]. Then it would describe the proof of at least one version of Marton's conjecture from [1,2], emphasising and explaining the role of entropy arguments in this proof (or proofs).

Relevant Courses

Essential: Information Theory, basic probability

Useful: Additive combinatorics, measure-theoretic probability

References

1. W.T. Gowers, B. Green, F. Manners and T. Tao. "On a conjecture of Marton". arXiv: 2311.05762 [math.NT], November 2023.

2. W.T. Gowers, B. Green, F. Manners and T. Tao. "Marton's Conjecture in abelian groups with bounded torsion". arXiv: 2404.02244 [math.NT], April 2024.

3. C.W. Lau and C. Nair. "Information Inequalities via Ideas from Additive Combinatorics". arXiv: 2312.11017 [cs.IT], December 2023.

4. I.Z. Ruzsa. "Sumsets and entropy". *Random Structures & Algorithms*, **34**, no. 1, pp. 1-10, 2009.

130. Mendelian Randomization and Invalid Instrumental Variables .... Professor Q. Zhao

Mendelian randomization is a popular study design (mainly in genetic epidemiology) that uses natural experiments in genetic inheritance as the basis for causal inference [1,2]. Although Mendelian randomization can be used to discover causal genetic variants [3], it primarily refers to the practice of using genetic variants as instrumental variables to infer the causal relationship between heritable traits. A tutorial of instrumental variable methods in biostatistics can be found in [4].

Genetic variants may fail to be valid instrumental variables because

1. They may have weak associations with the exposure/treatment variable; see [5] for a recent survey on weak instruments in econometrics.

2. They may not be exogenous (especially in population-based studies); see [6] for some empirical evidence and [7] for a more theoretical investigation.

3. They may have direct effects on the outcome variable, which is often called "pleiotropy" in the genetics literature; see [8,9] for some statistical remedies.

A successful essay with this title can provide a synthesis of invalid genetic instruments in Mendelian randomization from a statistical perspective and dive deeply into at least one of the three issues listed above. The essay can also compare existing statistical methods for invalid instrumental vairalbes using simulations or real datasets.

Relevant Courses

Useful: Causal Inference. Modern Statistical Methods. Statistics in Medical Practice.

References

1. Davey Smith, George, and Shah Ebrahim (2003). "Mendelian randomization: can genetic epidemiology contribute to understanding environmental determinants of disease?" *International Journal of Epidemiology* 32(1):1-22.

2. Sanderson, Eleanor et al. (2022) "Mendelian randomization." *Nature Reviews Methods Primers* 2(1), 6.

3. Bates, Stephen et al. (2020) "Causal inference in genetic trio studies." *Proceedings of the National Academy of Sciences* 117(39):24117-24126.

4. Baiocchia, Michael et al. (2014) "Tutorial in biostatistics: Instrumental variable methods for causal inference." *Statistics in Medicine* 33(13):2297-2340.

5. Andrews, Isaiah et al. (2019) "Weak instruments in instrumental variables regression: Theory and practice." *Annual Review of Economics* 11:727-753.

6. Brumpton, Ben et al. (2020) "Avoiding dynastic, assortative mating, and population stratification biases in Mendelian randomization through within-family analyses." *Nature Communications* 11:3519.

7. Tudball, Matthew J et al. (2023) "Almost exact Mendelian randomization." arXiv:2208.14035.

8. Kang, Hyunseung et al. (2016) "Instrumental variables estimation with some invalid instruments and its application to Mendelian randomization." *Journal of the American statistical Association* 111(513):132-144.

9. Zhao, Qingyuan et al. (2020) "Statistical inference in two-sample summary-data Mendelian randomization using robust adjusted profile score." *Annals of Statistics* 48(3):1742-1769.

131. Applying Contrastive Learning to Stellar Spectra .... Professor M. Cranmer

The absorption features imprinted on a star's spectrum encode its physical structure, chemical composition, and orbital motion, which in turn provide a fossil record of the host galaxy's chemical and dynamical evolution over cosmic time. The recently launched James Webb Space Telescope (JWST) boasts unique spectroscopic capabilities, enabling efficient high-quality resolved star spectroscopy in neighboring galaxies which have thus far been too distant, faint, and/or crowded for previous observational facilities.

As such, JWST has the potential to shine new light on the evolution of galaxies and their stellar populations. However, due to its novelty and uniqueness, the domain of JWST resolved star spectroscopy is largely unexplored, and there remain many unanswered questions regarding optimal strategies for collecting, reducing, and analysing this valuable data. Whilst traditional machine learning (ML) methods have historically worked well for large surveys in the Milky Way, particularly the APOGEE survey, they are found to not work well for JWST spectra.

Having the need to look beyond traditional ML approaches, one promising new self-supervised approach is that of "contrastive learning", whereby a model learns the mapping between two different observables—in the context of this essay, JWST and APOGEE spectroscopic data.

The essay will begin with a review of the statistical foundations of contrastive learning and discuss the relevant scientific motivations for applying contrastive learning to JWST and APOGEE spectroscopic data. It will then discuss some algorithms developed to implement contrastive learning, their associated challenges, and metrics to evaluate performance. Writers are encouraged to be original and creative, and to look through code for similar examples on GitHub in order to gain intuition for the practical aspects of implementing contrastive learning.

Relevant Courses

Useful: Astrostatistics, Statistical Learning in Practice

References

1. Parker, L., et al. (2024). AstroCLIP: A cross-modal foundation model for galaxies. *Monthly Notices of the Royal Astronomical Society*, 531(4), 4990–5011. https://doi.org/10.1093/mnras/stae1450

2. Fabbro, S., et al. (2018). An application of deep learning in the analysis of stellar spectra. *Monthly Notices of the Royal Astronomical Society*, 475(3), 2978–2993. https://doi.org/10.1093/mnras/stx3298

3. Nidever, D. L., et al. (2023). The prevalence of the α-bimodality: First JWST α-abundance results in M31. *arXiv.* https://arxiv.org/abs/2306.04688

4. Buck, T., & Schwarz, C. (2024). Deep multimodal representation learning for stellar spectra. *arXiv.* https://arxiv.org/abs/2410.16081

5. Sandford, N. R., et al. (2020). Forecasting chemical abundance precision for extragalactic stellar archaeology. *The Astrophysical Journal Supplement Series*, 249(2), 24. https://doi.org/10.3847/1538-4365/ab9cb0

132. Quantum Algorithms for Non-Abelian HSP and StateHSP .... Dr S. Subramanian

The key ingredient of quantum periodicity determination that underlies the success of Shor's quantum algorithm for factorising large integers was subsequently generalised to an efficient quantum algorithm for the Hidden Subgroup Problem over arbitrary finite abelian groups. Despite intense research efforts, no efficient quantum algorithm is known that works for arbitrary finite non-abelian groups. Numerous important problems, including some used in lattice-based cryptography, can be expressed as instances of the non-abelian HSP.

The essay will assume that the reader is familiar with the quantum algorithm for the abelian case. It should begin with a short account of quantum algorithms for non-abelian HSP, such as Refs. [1-4], presenting sketches of their proofs and a clear discussion of the barriers that prevent the techniques in these references from extending to more general groups. A major part of the essay should tackle Ref. [5], which introduces a new State Hidden Subgroup Problem (StateHSP), giving an exposition of the formalism, results, and applications. A good essay will give a coherent account of how these new ideas relate to prior work, and can also discuss the connections between HSP and property testing [6], taking the example of testing entanglement in Ref. [5] as a starting point. An ambitious essay may further explore new avenues of research that use HSP (or StateHSP) to address questions of classical and quantum pseudorandomness.

Any results drawn from representation theory should be should be clearly stated, along with a condensed proof outline for the most important ones.

Relevant Courses

Essential: Quantum Computation

Useful: Any course that covers the representation theory of finite groups is useful but not essential.

References

1. Hallgren, S., Russell, A. and Ta-Shma, A., 2003. The hidden subgroup problem and quantum computation using group representations. SIAM Journal on Computing, 32(4), pp.916-934.

2. Grigni, M., Schulman, L., Vazirani, M. and Vazirani, U., 2001. Quantum mechanical algorithms for the nonabelian hidden subgroup problem. In Proceedings of the thirty-third annual ACM symposium on Theory of computing (pp. 68-74).

3. Ettinger, M., Høyer, P. and Knill, E., 2004. The quantum query complexity of the hidden subgroup problem is polynomial. Information Processing Letters, 91(1), pp.43-48.

4. Kuperberg, G., 2005. A subexponential-time quantum algorithm for the dihedral hidden subgroup problem. SIAM Journal on Computing, 35(1), pp.170-188.

5. Bouland, A., Giurgica-Tiron, T. and Wright, J., 2024. The state hidden subgroup problem and an efficient algorithm for locating unentanglement. arXiv preprint arXiv:2410.12706.

6. Montanaro, A. and de Wolf, R., 2016. A Survey of Quantum Property Testing. Theory of Computing, pp.1-81.

133. Fault-Tolerant Quantum Computation .... Dr S. Subramanian

Arbitrarily long quantum computations can be reliably performed on a noisy quantum computer, as long as the noise is below a constant level, known as a *threshold value* [1,2]. This comes at a cost, however, since the fault-tolerant version of the circuit is generally larger than the initial version and the overheads incurred are polylogarithmic. In 2013, it was shown that polynomial-time computations could be performed with a noisy circuit with only a constant overhead independent of the circuit if we only consider circuits with a length bounded by a polynomial in the width [3]. This was subsequently refined with explicit construction that achieves this [4]. Presently, given a noise model, it is possible to estimate the growth of the overhead as a function of the circuit size [5]. On the other hand, achieving full protection against arbitrary errors comes at a price: It was shown that there exists no quantum error correcting code that is capable of transversely implementing a universal gate set [7,8].

The essay should discuss the threshold theorem [1,2] and constructions that enable fault-tolerant computation [3-6]. An ambitious essay can further explore the importance of transversal gates and the Eastin-Knill theorem [7,8], or recent developments in using quantum expander codes to achieve quantum fault tolerance with constant space overhead.

Relevant Courses

Useful: Quantum Computation

References

1. Aharonov, D., Ben-Or, M. (1997). Fault-tolerant quantum computation with constant error. In Proceedings of the 29$^{\text{th}}$ annual ACM symposium on Theory of computing (pp. 176-188).
2. Knill, E., Laflamme, R., Zurek, W. H. (1998). Resilient quantum computation. Science, 279(5349), 342-345.
3. Gottesman, D. (2013). Fault-tolerant quantum computation with constant overhead. arXiv preprint arXiv:1310.2984.
4. Fawzi, O., Grospellier, A., Leverrier, A. (2020). Constant overhead quantum fault tolerance with quantum expander codes. Communications of the ACM, 64(1), 106-114.
5. Fawzi, O., Müller-Hermes, A., Shayeghi, A. (2022). A lower bound on the space overhead of fault-tolerant quantum computation. arXiv preprint arXiv:2202.00119.
6. Gottesman, D. (2022). Opportunities and Challenges in Fault-Tolerant Quantum Computation. arXiv preprint arXiv:2210.15844.
7. Eastin, B., Knill, E. (2009). Restrictions on Transversal Encoded Quantum Gate Sets. Physical review letters, 102(11), 110502.
8. Kubica, A., Demkowicz-Dobrzański, R. (2021). Using Quantum Metrological Bounds in Quantum Error Correction: A Simple Proof of the Approximate Eastin-Knill Theorem. Physical Review Letters, 126(15), 150503.
9. Fawzi, O., Grospellier, A. and Leverrier, A., 2020. Constant overhead quantum fault tolerance with quantum expander codes. Communications of the ACM, 64(1), pp.106-114.

134. A Case Study in Equivariant Intersection Theory .... Dr V. Arena

Intersection theory constructs a powerful invariant called the *Chow ring* of a variety X, denoted by $A^*(X)$, as a tool to count and study the intersection of subvarieties living inside X. Computing Chow rings of moduli spaces, in general, is a particularly complex problem, especially when working with stacks and integral coefficients.

However, there is a large number of moduli spaces that naturally arise as quotients of smooth schemes, quotiented by the action of some known algebraic group. The Chow ring of such moduli spaces can be studied through the elegant constructions of equivariant intersection theory, developed by Edidin and Graham in [1], which precisely capture the geometry of quotients.

A beautiful showcase of power for equivariant intersection theory is in the computations of Vistoli in [9] and Penev and Vakil in [7], which greatly simplify the computations for the Chow rings of $\mathcal{M}_g$ for small g, previously made by Faber in [2] and [3].

This essay, will be around one computation using equivariant intersection theory. The essay should contain an overview of the background of intersection theory, classical and equivariant, that is needed to execute the computation but focus mainly on the chosen example.

Possible directions of exploration include the following.

- Give a definition of a classifying space BG for an algebraic group G and compute the integral Chow ring $A^*(BG)$ for a particular choice of G;
- Give a definition for the moduli space of curves $\mathcal{M}_g$ and compute the rational Chow ring for a choice of $g \leq 5$;
- Compute the integral Chow ring of weighted projective spaces. As an application, after defining the moduli space of curves $\bar{\mathcal{M}}_{1,1}$, compute its Chow ring;
- Study the T-equivariant Chow rings of a toric variety with torus T;

There are numerous other potential directions, which vary in flavour and in required background. The essay is meant to engage a substantial result or calculation in the subject rather than construct the foundations of intersection theory from the ground up.

Relevant Courses

Essential: Part III Algebraic Geometry, Part II Algebraic Topology.

Useful: Part III Algebraic Topology, Part III Different Geometry, Part II Differential Geometry.

References

1. Dan Edidin and William Graham, *Equivariant intersection theory*, Invent. Math. 131 (1998), no. 3, 595–634. MR1614555
2. Carel Faber, *Chow rings of moduli spaces of curves I: The Chow ring of $\bar{\mathcal{M}}_3$*, Annals of Mathematics 132 (1990), no. 2, 331–419.
3. Carel Faber, *Chow rings of moduli spaces of curves II: Some results on the Chow ring of $\bar{\mathcal{M}}_4$*, Annals of Mathematics 132 (1990), no. 2, 421–449.
4. William Fulton, *Intersection theory*, Second, Springer-Verlag, Berlin, 1998. MR1644323
5. Andrew Kresch, *Cycle groups for Artin stacks*, Inventiones mathematicae 138 (December 1999), no. 3, 495–536
6. Sam Payne, *Equivariant Chow cohomology of toric varieties*, Mathematical Research Letters 13 (2005).
7. Nikola Penev and Ravi Vakil, *The Chow ring of the moduli space of curves of genus 6*, Algebraic Geometry 2 (2013).
8. Angelo Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Inventiones mathematicae 97 (1989), 613–670.
9. Angelo Vistoli, *Appendix the Chow ring of $\mathcal{M}_2$*, Invent. Math. 131 (1998), no. 3, 365–644

135. High Frequency Analysis of the Magnetic Harmonic Oscillator .... Professor P. Raphael

Weak turbulence is drift to high frequencies of trapped non linear waves. This purely non linear phenomenon is very badly understood and the heart of an intense research activity. The aim of this essay is to explore one possible scenario of such a drift through solitary waves interaction in the continuation of [1].

The essay will more specifically focus onto deriving pointwise bounds at high frequency for the magnetic harmonic oscillator $H - 2L$ in two dimensions.

No special knowlegde is required except basic PDE's.

(1) Use WKB to compute the eigenbasis.

(2) Understand the functional framework of soliton interaction.

(3) Derive resolvent estimates possibly adapted to the non linear problem.

Relevant Courses

Essential: Analysis of Partial Differential Equations

References

1. Faou, E; Raphaël, P.; Faou, E.; Raphaël, P., *On weakly turbulent solutions to the perturbed linear Harmonic oscillator*, Amer. Math. Journal. 2023.

136. Reducing the Effect of Time Trends on the Performance of Response-Adaptive Clinical Trial Designs .... Dr S. Baas, Dr S. S. Villar

Clinical trial designs using response-adaptive randomization (RAR) base the allocation of treatment to future participants on interim analyses of the clinical trial data. Goals of RAR are, for example, to have a high amount of trial participants allocated to the best performing treatment, to have high statistical power, to have high estimation precision, or a combination of these goals. Despite its long history, growing interest, and wide range of theoretical advancements, the number of practical applications of RAR designs is relatively small. Robertson et al. (2023) list a number of potential reasons for this, in particular they state: "The occurrence of time trends caused by changes in the standard of care or by patient drift (i.e. changes in the characteristics of recruited patients over time) is seen as a major barrier to the use of RAR in practice". Next to patient drift there can also be a time trend in the treatment effect (e.g., when physicians or surgeons improve their treatment method over time). Under no time trends, there is already a risk of underestimating the expected outcome for inferior treatments in RAR designs, yielding an overestimate of the treatment effect. Under time trends, this problem is exacerbated, as inferior treatments will tend to be chosen less often as time progresses. Hence, for RAR designs under time trends there is an increased risk of bias and type I error rate inflation using standard inference approaches. Villar et al. (2018) evaluate the effect of time trends on type I error rate and power, along with possible ways of overcoming the type I error rate inflation while ensuring high power, however debate about the use of RAR under potential time trend persists, with others advocating for a fixed design with equal allocation to each treatment.

This project will evaluate different approaches for dealing with time trends in RAR designs. The project will compare an extension of the approach in Section 4 of Villar et al. (2018), using other basis functions, with an approach based on the power prior, introduced in Ibrahim and Chen (2000), another comparator can be based on stratified analyses (Jennison 2023). Preferably, the essay also considers the probability of having an imbalance in the wrong direction due, e.g., the treatment effect switching signs, which is a currently unexplored topic in the literature. The main RAR design under consideration should be a blocked Bayesian RAR design, which is most often implemented in practice. The essay should incorporate a literature review of existing work. Challenges of the project include ways of efficiently determining the power parameter of the power prior, determining the comparators, and setting up the simulation study comparison.

Relevant Courses

Essential: Statistics in Medical Practice (Statistics in Medicine)

References

1. J. G. Ibrahim and M. H. Chen (2000). Power prior distributions for regression models. Statistical Science, 46-60. URL: https://www.jstor.org/stable/2676676

2. C. Jennison (2023). Comment: Group Sequential Designs with Response-Adaptive Randomisation. Statistical Science, 38(2), 219-223. doi: 10.1214/23-STS865D

3. D. S. Robertson, K. M. Lee, B. C. López-Kolkovska, and S. S. Villar. Response-Adaptive Randomization in Clinical Trials: From Myths to Practical Considerations. Statistical Science, 38(2), 7 2023. doi: 10.1214/22-STS865

4. S. S. Villar, J. Bowden, and J. Wason (2018). Response-adaptive designs for binary responses: How to offer patient benefit while being robust to time trends? Pharmaceutical statistics, 17(2), 182-197. doi: 10.1002/pst.1845

137. Sheaves and Curvature in Graph Representation Learning .... Dr P. Lio

Extensions of deep learning techniques to graphs and their generalizations have recently achieved widespread success. Characterizing the expressivity and representational power of message-passing graph neural networks remains challenging, however, due e.g. to interactions with long-standing problems in graph theory and to tradeoffs imposed by inductive biases. One case of the latter is the so-called *oversmoothing phenomenon*, wherein adding network layers leads to indistinguishable node features. A recent line of work approaches oversmoothing from a geometric perspective, viewing graphs and their relevant extensions as (discretizations of) objects to which notions from Hodge theory, sheaf theory, curvature, etc. may be applied to analyze oversmoothing.

The goal of this essay is to present an integrated perspective on a selection of results in this area. The essay should begin with a brief introduction to oversmoothing in graph representation learning [5] before providing a exposition — likely accounting for about half the text — of one or both of the following topics, emphasizing the interplay between each construction and its non-discrete counterpart:

- Cellular sheaves and their spectral theory [4]
- Combinatorial notions of Ricci curvature [7, 6]

The essay will then apply this exposition toward proving a subset of results in [10, 9, 12, 8], contextualizing each within the others and within the broader literature. Numerical confirmations of the theory are encouraged.

Relevant Courses

Useful: Part III Differential Geometry, Part III Algebraic Topology. Prior exposure to machine learning would be helpful.

References

1. Hansen, Jakob, *Laplacians of Cellular Sheaves: Theory and Applications*, University of Pennsylvania (2020)
2. Curry, Justin Michael, *Sheaves, cosheaves and applications*, University of Pennsylvania (2014)
3. Bodnar, Cristian, *Topological deep learning: graphs, complexes, sheaves*, University of Cambridge (2023)
4. Hansen, Jakob and Ghrist, Robert, *Toward a spectral theory of cellular sheaves*, Journal of Applied and Computational Topology Vol. 3, pp.315–358 (2019)
5. Hamilton, William L., *Graph representation learning*, Morgan & Claypool Publishers (2020)
6. Sreejith, RP and Mohanraj, Karthikeyan and Jost, Jürgen and Saucan, Emil and Samal, Areejit, *Forman curvature for complex networks*, Journal of Statistical Mechanics: Theory and Experiment Vol. 2016 063206 (2016)
7. Ollivier, Yann, *Ricci curvature of metric spaces*, Comptes Rendus Mathematique Vol. 345, pp.643–646 (2007)
8. Fesser, Lukas and Weber, Melanie, *Mitigating over-smoothing and over-squashing using augmentations of Forman-Ricci curvature*, Learning on Graphs Conference (2024)
9. Nguyen, Khang and Hieu, Nong Minh and Nguyen, Vinh Duc and Ho, Nhat and Osher, Stanley and Nguyen, Tan Minh, *Revisiting over-smoothing and over-squashing using Ollivier-Ricci curvature*, International Conference on Machine Learning, pp.25956–25979 (2023)
10. Bodnar, Cristian and Di Giovanni, Francesco and Chamberlain, Benjamin and Lio, Pietro and Bronstein, Michael, *Neural sheaf diffusion: A topological perspective on heterophily and oversmoothing in GNNs*, Advances in Neural Information Processing Systems, Vol. 35, pp.18527–18541 (2022)
11. Leus, Geert and Marques, Antonio G and Moura, José MF and Ortega, Antonio and Shuman, David I, *Graph Signal Processing: History, development, impact, and outlook*, IEEE Signal Processing Magazine, Vol. 40(4), pp.49–60 (2023)

12. Duta, Iulia and Cassarà, Giulia and Silvestri, Fabrizio and Liò, Pietro, *Sheaf hypergraph networks*, Advances in Neural Information Processing Systems Vol. 36 (2024)

13. Barbero, Federico and Bodnar, Cristian and de Ocáriz Borde, Haitz Sáez and Bronstein, Michael and Veličković, Petar and Liò, Pietro, *Sheaf neural networks with connection laplacians*, Topological, Algebraic and Geometric Learning Workshops 2022, pp.28–36 (2022)

14. Lim, Lek-Heng, *Hodge Laplacians on graphs*, SIAM Review Vol. 62(3), pp.685–715 (2020)

15. He, Yu and Bodnar, Cristian and Lio, Pietro, *Sheaf-based Positional Encodings for Graph Neural Networks*, NeurIPS 2023 Workshop on Symmetry and Geometry in Neural Representations (2023)

138. Renormalization Group and Deep Learning ....
Professor F. Verstraete

Renormalization group (RG) methods provide a theoretical framework for understanding multi-scale phenomena, including neural network architectures such as Restricted Boltzmann machines (RBMs). The essay will examine the Renormalization Group principles and apply them to Ising-like RBMs. It will discuss existing exact mappings of Variational RG and Deep Learning [1] in RBM architecture and verify the ML-RG dictionaries [2][3], by numerically visualizing RG-like parameter flows[4][5]. Finally, the essay will also explore extensions to multiscale entanglement renormalization ansatz (MERA) which is seen in quantum many-body physics to establish a cohesive link between RG principle and deep learning[6].

Relevant Courses

Useful: Statistical Field Theory

References

1. Pankaj Mehta and David J. Schwab. An exact mapping between the variational renormalization group and deep learning, [arxiv:1410.3831], 2014

2. Lin, H.W., Tegmark, M. & Rolnick, D. Why Does Deep and Cheap Learning Work So Well?. J Stat Phys 168, 1223–1247 (2017). [doi.org/10.1007/s10955-017-1836-5]

3. Daniel A. Roberts, Sho Yaida, Boris Hanin, "The Principles of Deep Learning Theory," Cambridge University Press [arXiv:2106.10165].

4. Satoshi Iso, Shotaro Shiba, and Sumito Yokoo. Scale-invariant feature extraction of neural network and renormalization group flow. [Phys. Rev. E, 97:053304, May 2018]

5. Ellen De Mello Koch, Robert De Mello Koch, and Ling Cheng. Is deep learning a renormalization group flow? [IEEE Access, 8:106487–106505], 2020.

6. C. Bény. Deep learning and the renormalization group, [arxiv:1301.3124] 2013.

7. Cardy J. Scaling and Renormalization in Statistical Physics. Cambridge University Press; 1996.