Faculty of Mathematics
Part III Essays: 2025-26

Titles 1 – 65

Department of Pure Mathematics
& Mathematical Statistics

Titles 66 – 103

Department of Applied Mathematics
& Theoretical Physics

Titles 104 – 126

Additional Essays

Introductory Notes

Overview

As explained in the Part III Handbook, students are expected to submit an essay (written during the year) by a deadline early in the Easter Term for 3 units of examination credit (equivalent to the credit available from a 3-hour written paper for a 24-lecture course). This Essay Booklet contains details of the approved essay titles, together with general guidelines and instructions for writing an essay. A timetable of relevant events and deadlines is included on page (vi) of this document.

The essay is a key component of Part III. Experience has shown that the great majority of students find that working on their essay is an enjoyable change from learning from lectures and is valuable training for research as well as a range of other careers.

Essay Marks and Examination Credit

Each essay is awarded a numerical mark out of a maximum of 100 and a 'quality mark' (just as with written papers). Further details can be found in Appendix IV of the Part III Handbook and descriptors for the broad grade ranges of quality marks for essays are also reproduced below, in Appendix 2. The Faculty Board does not necessarily expect the mark distribution for essays to be the same as that for written examinations. Indeed, in recent years for many students their essay mark has been amongst their higher marks across all examination papers, both because of the typical amount of effort devoted to the essay and the different skill set tested (compared to a time-limited written examination). The Faculty Board wishes that hard work and talent thus exhibited should be properly rewarded.

Candidates should note that, since there is a maximum of 16 units of credit that can be obtained from examination papers for lecture courses, it is essential to submit an essay in order to achieve a total amount of examination credit in the range 17-19 units. See *Section 9 – Examinations and Assessment* in the Part III Handbook for further details.

Essay Titles

The titles of essays in this booklet have been approved by the Part III Examiners.[1] Additional titles may be approved by the Part III Examiners and will be added to this booklet not later than 1 March. Essay titles **cannot** be approved informally: the only allowed essay titles are those which appear in the final version of this document, available on the Faculty website.[2]

Requesting an Additional Title

If you wish to write an essay on a topic not covered in this booklet you should approach your Part III Subject Advisor or Departmental Contact or another member of the academic staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board (email: undergrad-office@maths.cam.ac.uk) **not later than 1 February** requesting that an essay on that topic be approved.[3] The new essay title will require the approval of the Part III Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. If you request an essay title you are under

[1] The titles are also published in the University's journal of record, the Cambridge University Reporter.
[2] All additional titles will also be published in the Cambridge University Reporter.
[3] See Regulation 17 of the Regulations for the Mathematical Tripos.

no obligation to write the corresponding essay. Once announced, an essay title is open to any student, subject to the guidance below.

Interaction with the Essay Setter

Normally candidates may consult the setter up to three times before the essay is submitted. This includes the first meeting described below, which may take the form of a group meeting at which the setter describes the essay topic and answers general questions. There is a range of practices across the Faculty for the other two meetings depending on the nature of the essay and whether, say, there is a need for further references and/or advice about technical questions. It is also possible for the setter to suggest a slightly different pattern of meetings, provided that at least three are offered. The setter may comment on an outline of the essay (for example in the second meeting), and may offer general feedback (for example, on mathematical style in general terms, or on whether clearer references to other sources are required) on a draft of the essay in the final meeting. The setter is not allowed to give students an expected grade for their essay.

Essay Title Choice

In order to ensure that individual setters are not oversubscribed and **all** students receive adequate guidance, students will be asked to nominate three preferred essay titles **by noon on Friday, 28 November 2025** via a dedicated Part III Essay Moodle. You must nominate three titles by at least two distinct essay setters and can expect to be notified of your allocated title during the following week. The allocation process will aim to assign first-preference titles to students wherever possible, subject to capacity of the essay setter. Where the number of students who name a given setter's title as their first preference exceeds the declared capacity of the setter, the selection of students up to the declared capacity will be made at random.

Before deciding on your preferred titles, you are strongly advised to attend a first meeting (either in person or online) for each essay title of interest. Such meetings will be arranged by the essay setter. They will normally take place during the fifth and seventh week of lectures,[4] and will be advertised via the Part III Essay Moodle. Some essay setters may ask you to contact them to arrange a meeting. It is the responsibility of each student to ensure that they are aware of relevant meeting dates and other arrangements.

Changing Title after the Title-Choice Deadline

It is hoped that the vast majority of students will be allocated their preferred title, and that, with rare exceptions, students remain satisfied with their assigned title as they begin to work on the essay in earnest. However, some students may discover that their assigned title is not suitable for them after all. A subsequent change of title is therefore possible, subject to capacity of the relevant essay setter.

Students wishing to change title following the initial allocation must observe the following steps:

1) Consult the Part III Essay Moodle to identify essay setters with spare capacity.
2) Meet with the setters of any titles that appeal to establish suitability of the title.
3) Email partiii-essays@maths.cam.ac.uk with your current title, your desired title, with copy to **both** the setters of the current and the desired title. You should use the subject line "Part III Essay Title Change".

[4] Essay setters are discouraged from arranging essay meetings during the sixth week of lectures to avoid clashes with the Part III Progress Interviews.

Title change requests will be processed and approved on a first-come-first-served basis, using the time stamp of your email, normally within two working days.[5] Title change requests not respecting the above format will be returned.

It is not advisable to request a change of title if you have already invested significant time and effort in your essay. Title changes are possible until the end of the seventh week of lectures of the Lent term. Change requests received after this date will be noted, but the new essay setter will not be expected to provide the normal amount of guidance.

Choosing an Essay Title after the Title-Choice Deadline

Students who did not nominate three preferred titles but wish to choose a title after the deadline for essay-title choices on 28 November must observe the following steps:

1) Consult the Part III Essay Moodle to identify essay setters with spare capacity.
2) Meet with the setters of any titles that appeal to establish suitability of the title.
3) Email partiii-essays@maths.cam.ac.uk with your desired title, with copy to the setter of the desired title. You should use the subject line "Part III Essay Late Title Choice".

Title allocation requests will be processed and approved on a first-come-first-served basis, using the time stamp of your email, normally within two working days.[5] Title allocation requests not respecting the above format will be returned.

Late title choices are possible until the end of the seventh week of lectures of the Lent term. Allocation requests received after this date will be noted, but the essay setter will not be expected to provide the normal amount of guidance.

Dropping an Essay Title after the Title-Choice Deadline

In order for all students to have fair access to one of their preferred titles, it is essential that you let us know if you are no longer interested in pursuing an allocated essay by emailing partiii-essays@maths.cam.ac.uk with your name and allocated title. You may request a new title at a later stage - see Choosing a Title after the Title-Choice Deadline above.

Content of the Essay and Originality

The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but it is not unusual for candidates to find novel approaches. All sources and references used should be carefully listed in a bibliography. Candidates are reminded that mathematical content is more important than style.

Presentation of the Essay

There is no prescribed length for the essay in the University Ordinances, but the Faculty Board Advice to the Part III Examiners suggests that 5,000-8,000 words is a normal length, and exceptionally long essays (i.e. more than twice this maximum) are strongly discouraged (see Appendix 2 in this booklet). In order to provide greater clarity, the Faculty Board now requires that all candidates use a standard LaTeX template, which will be provided. The length of an essay should normally then be between 20 and 30 pages and should only exceptionally be more than 35 pages. Any exceptions to this standard guidance will be noted

[5] The administrative offices of the University will close for a set period over the Christmas vacation. The closure dates will be clearly advertised nearer the time and you should not expect a response during this period.

below, alongside the description of the essay concerned. If you are in any doubt as to the length of your essay, please consult either the essay setter or your Part III Subject Advisor or Departmental Contact.

Academic Misconduct and Plagiarism

Before starting your essay, you **must** read both

- the University's statement on the Definition of Academic Misconduct
- the Faculty Guidelines on Plagiarism and Academic Misconduct, which are reproduced in Appendix 1 of this document.

The University takes a very serious view of academic misconduct in University examinations. The powers of the University Disciplinary Panels extend to the amendment of academic results or the temporary or permanent removal of academic awards, and the temporary or permanent exclusion from membership of the University. Fortunately, incidents of this kind are very rare.

Use of Artificial Intelligence (AI)

Content produced by AI platforms, such as ChatGPT, is not original work and will be considered a form of academic misconduct to be dealt with under the University's disciplinary procedures.

Output generated by AI tools should not be presented as your own work. This applies both to the detailed content and to the outline or overall plan of your essay.

Signed Declaration

The essay submission process includes signing the following declaration. It is important that you read and understand this before starting your essay.

> I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the *Definition of Academic Misconduct* and the *Faculty Guidelines on Plagiarism and Academic Misconduct* and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

If you are in any doubt as to whether you will be able to sign the above declaration, you should consult the member of staff who set the essay in the first instance. If the setter is unsure about your situation, you should consult the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) as soon as possible.

Viva Voce Examination

The Part III Examiners have power, at their discretion, to examine a candidate *viva voce* (i.e. to give an oral examination) on the subject of their essay.

Time Management

It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.

Confirming your Essay Submission

At the beginning of the Easter Term, you must state which written papers you have chosen, and confirm which essay you will be submitting for examination. At that point, you will be sent the appropriate *Paper Choice Form* to complete, to confirm both your paper choices and your essay title. Your Director of Studies must counter-sign this form, and you should then send it to the Chair of Part III Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 7 May 2026**. **This deadline must be strictly adhered to.**

Submitting your Essay

You should submit your essay via the Part III Essay Moodle **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 7 May 2026**. Alongside your essay you will need to submit a completed and signed Essay Submission Form as found on page (vii) of this document. Your Director of Studies must counter-sign this form, and you should then send it to the Chair of Part III Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 7 May 2026**. **This deadline (or any extension authorized by the University) will be strictly adhered to – see below.**

- The title page of your essay should bear **only** the essay title. Please **do not** include your name or any other personal details on the title page or anywhere else on your essay.
- Essays will need to be submitted in pdf format using the LaTeX template (no hard copies will be accepted). Students who believe they will be unable to use the LaTeX template should contact the UGO as soon as possible.

More detailed submission instructions will be provided closer to the submission date.

Extension of Submission Deadline

The essay submission deadline can be extended **only** if authorization is obtained by following the appropriate procedures set out by the University. According to a policy approved by the General Board's Education Committee for managing extensions for dissertations and coursework, candidates are now permitted to self-certify for a short extension of the essay submission deadline, up to seven days, for any reason (medical or non-medical). Candidates must request such an extension via the Faculty's Part III Essay Deadline Extension Request Form prior to the submission date, and must include evidence that they have informed their College Tutor of their request. While there is no requirement for any other supporting evidence to be provided, candidates are reminded that it is unlikely to be in their interest to spend excessive time on their essay at the expense of revision for their written papers. Candidates considering requesting an extension are strongly encouraged to discuss the matter with their College Director of Studies well in advance of the deadline. Applications for extensions beyond seven days must be made to the University's Examination Access and Mitigation Committee by the College on the student's behalf.

Consequences of Missing the Essay Submission Deadline

The submission deadline, including any authorised extension, will be strictly adhered to; candidates who miss the deadline are liable to receive no examination credit for their essay. Any candidate who thinks they may be at risk of missing the deadline, due to e.g. illness or other grave cause, should contact their College Tutor and Director of Studies as soon as possible.

Assessment of the Essay

The essay is marked by the setter, who is appointed as an Assessor for Part III, and marking is carried out in accordance with the Essay Descriptors included as Appendix 2 in this booklet. Each essay mark is checked by a “standard checker” and may subsequently be moderated by the Part III Examiners for consistency within and across subject areas. Essay marks are expected to be released alongside the marks on written papers by the end of **Wednesday, 24 June 2026**.

Return of Essays

It is not possible to return essays to candidates. You are therefore advised to retain your own copy when submitting your essay.

Further Guidance

Advice on writing an essay is provided in two Wednesday afternoon talks listed below. Slides from these talks will subsequently be made available on the Part III Academic Support Moodle.

Feedback

If you have suggestions as to how these notes might be improved, please write to the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk).

Timetable of Relevant Events and Deadlines

Wednesday 5 November 4:15pm	Talk: *Planning your essay: reading, understanding, structuring*
Friday 28 November noon	Deadline for essay title choices
Sunday 1 February	Deadline for Candidates to request additional essays
Wednesday 28 January 4:15pm	Talk: *Writing your essay: from outline to final product*
Thursday 7 May noon	Deadline for Candidates to return form stating choice of papers and title of essay
Thursday 7 May noon	Deadline for Candidates to submit essays
Thursday 4 June	Part III Examinations expected to begin

Mathematical Tripos, Part III 2026
Essay Submission Form

To the Chair of Examiners for Part III of the Mathematical Tripos

I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University's statement on the Definition of Academic Misconduct and the Faculty Guidelines on Plagiarism and Academic Misconduct and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed (Candidate): .... Date:..

Title of Essay: ……

……

Essay Number: ....

Candidate Name: ……………………………………………………………………………. College: …………………………………………

Signed (Director of Studies): ……………………………………………………………. Date: ……………………………………………

Assessor Comments

The Assessor may provide comments on your essay which are intended to support your academic development. Any comments received by the Undergraduate Office will be sent to you by email as soon as possible following the publication of results. Please note that it is not mandatory for the Assessor to offer such comments, and that, where provided, comments do not represent a formal justification of the final mark.

If you would like to receive your comments, please provide a personal email address below (i.e. an email address other than CRSid@cam.ac.uk). Comments will not be sent to CRSid email addresses.

Email Address: ……

Appendix 1: Faculty of Mathematics: Guidelines on Plagiarism and Academic Misconduct

For the latest version of these guidelines please see:

https://www.maths.cam.ac.uk/internal/faculty/plagiarism

University Resources

The University publishes information on Plagiarism and Academic Misconduct, including

- the University's definition of academic misconduct
- information for students, covering
    - Why does academic integrity matter?
    - Students' responsibilities
    - Collusion (including proofreading)
    - Artificial Intelligence
- information about Referencing and Study skills
- information about further online Resources and local sources of support
- Plagiarism FAQs.

There are references to the University's definition:

- in the Part IB and Part II Computational Project Manuals
- in the Part III Essay Booklet (linked from the Part III Essays page)
- in the Computational Biology Handbook (linked from the Computational Biology Course page).

Please read the University's definition of academic misconduct carefully.

The University has outlined Rules of Behaviour for both current and former registered students (Statutes and Ordinances 2022, Chapter II; p.186). All registered students and formerly registered students are responsible for following the Rules of Behaviour.

Not knowing or forgetting about the rules or their consequences is not a justification for not following them.

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However, neither the University Statement nor the Faculty Guidelines supersede the University's Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

The scope of academic misconduct

Academic misconduct may be due to

- **plagiarism**
  this refers to using another person's language and/or ideas as if they are your own

- **collusion**
  this refers to collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed
- **contract cheating**
  this refers to contracting a third party to provide work, which is then used or submitted as part of a formal assessment as though it is the student's own work
- **use of artificial intelligence**
  this refers to the use of ChatGPT and other tools that can generate essay-style text, computer code, presentations, outlines and other content.

What is plagiarism?

Plagiarism can be defined as **the unacknowledged use of the work of others as if this were your own original work**. In the context of any University examination, this amounts to **passing off the work of others as your own to gain unfair advantage**.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.

Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to Turnitin, for example).

Where plagiarism or another form of academic misconduct is suspected, the Examiners of the relevant part of the Tripos may, at their discretion, examine a candidate *viva voce*.

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.) are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a 'ghost writing service'). Furthermore, you should not deliberately reproduce someone else's work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition, you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essay, project or coursework will be based on what you have read and learned from other sources, and it is important that in your essay, project or other coursework that you show exactly where, and how, your work is indebted to these other sources. The golden rule is that **the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else**.

A good guideline for avoiding plagiarism is not to repeat or reproduce other people's words, diagrams or computer programs. If you need to describe other people's ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay, project or other coursework, it is not sufficient merely to repeat or paraphrase someone else's view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

Quoting

A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:

- short quotations should be in inverted commas, and a reference given to the source
- longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).

Paraphrasing

Paraphrasing means putting someone else's work into your own words. Paraphrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. "Smith (2001) goes on to argue that ..." or "Smith (2001) provides further proof that ..."). As with quotation, the full details of the source should be given in the bibliography or reference list.

General indebtedness

When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

Use of web sources

You should use web sources as if you were using a book or journal article. The above rules for quoting (including 'cutting and pasting'), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

Collaboration

Unless it is expressly allowed, collaboration is collusion and is considered academic misconduct. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

Use of artificial intelligence

Content produced by AI platforms, such as ChatGPT, is not original work and will be considered a form of academic misconduct to be dealt with under the University's disciplinary procedures. Several methods for detecting AI-generated text are already available and may be employed by the Examiners.

In addition to issues of academic integrity, students should be aware of several issues that have been reported:

- possible inaccuracy of the content generated, ranging from not being up-to-date to being entirely fictitious
- possibility of bias introduced and prejudicial views being perpetuated, based on existing online content
- ethical concerns around the gathering and use of user data, due to questionable consent and privacy practices of the platforms in question.

Appendix 2: Part III Essays and Grade Descriptors

The Assessor for each essay awards a numerical mark out of a maximum of 100 to each essay and in addition assigns a 'quality mark' (see Appendix IV of the Part III Handbook). The Faculty Board has specified that, just as with written papers, the minimum performance deserving of a distinction on a paper or an essay is associated with α-, while the minimum performance deserving of a pass is associated with β-.

The Faculty Board does not necessarily expect the mark distribution for essays to be the same as that for written examinations. Indeed, in recent years for many students the essay mark has been amongst their highest marks across all examination papers, both because of the typical amount of effort they have devoted to the essay and the different skill set being tested (compared to a time-limited written examination). The Faculty Board wishes that the hard work and talent thus exhibited should be properly rewarded.

There is no prescribed length for the essay in the University Ordinances and the Faculty Board recognises that the length of an essay is only a weak reflection of the quantity of work involved and bears no relation to the quality of the work done. However, it is anxious to prevent the essay absorbing too much of the candidate's time. It is therefore perfectly content if a topic is set for which an excellent essay requires about 5000 words and would normally be unhappy if a topic were set for which an excellent essay required more than about 8000 words.

In order to provide greater clarity, the Faculty Board now requires that all candidates use a standard LaTeX template and has agreed that the expected length of an essay should normally then be between 20 and 30 pages and should only exceptionally be more than 35 pages.

Where an Assessor feels that a relaxation of the upper limit of 35 pages is justified for a particular essay title, they should specify an appropriate page range and upper limit, to be communicated to candidates in the essay booklet, and provide the Examiners with a clear explanation of why this is necessary, to be considered when the essay title is approved. The justification should explain why the increased upper limit does not represent an unreasonable increase in the time and effort that a candidate is expected to devote to this particular essay topic compared to others. Valid reasons might include the requirement to include an unusual number of diagrams, or the necessity to include data or code with the core of the essay.

In light of remarks above, as well as the comments of both internal and external Examiners over the years, the Faculty Board considers the following descriptors of the broad grade ranges for an essay to be appropriate. The Board trusts that these guidelines prove useful in guiding the judgement of the inevitably large number of Assessors marking essays, and thereby strengthen the mechanisms by which all essays are assessed uniformly. They are intended to be neither prescriptive nor comprehensive, but rather general guidance consistent with long-standing practice within the Faculty.

An Essay of α-Grade Standard (α-, α, α+)

Typical characteristics expected of an essay of α-grade standard include:

- Demonstration of clear mastery of the underlying mathematical content of the essay.
- Demonstration of thorough understanding and cogent synthesis of advanced mathematical concepts.
- A well-structured and well-written essay of appropriate length (5000-8000 words) with
    - few grammatical or presentational issues

    - a clear introduction demonstrating an appreciation of the context of the central topic of the essay
    - a coherent presentation of that central topic
    - a final section which draws the essay to a clear and comprehensible end, summarising well the key points while suggesting possible future work.

An essay of α-grade standard would be consistent with the quality expected of an introductory chapter of a PhD thesis from a leading mathematics department. A more elegant presentation and synthesis than that presented in the underlying papers, perhaps in the form of a shorter or more efficient proof of some mathematical result would be one possible characteristic of an essay of α-grade standard. Furthermore, it would be expected that an essay containing publishable results would be of α+ standard, but, for the avoidance of doubt, publishable results are **not necessary** for an essay to be of α+ standard. A mark in the α+ range should be justified by an explicit additional statement from the Assessor highlighting precisely which aspects of the essay are of particularly distinguished quality.

An Essay of β-Grade Standard (β-, β, β+)

Essays of β-grade standard encompass a wide range, but all should demonstrate understanding and synthesis of mathematical concepts at the level expected for a pass mark in a Part III lecture course.

Typical characteristics expected of an essay of β+ standard include:

- Demonstration of good mastery of most of the underlying mathematical content of the essay.
- A largely well-structured essay of appropriate length (5000-8000 words) with
    - some minor, grammatical or presentational issues
    - an introduction demonstrating an appreciation of at least some context of the central topic of the essay
    - a reasonable presentation of that central topic
    - a final section which draws the entire essay to a comprehensible end, summarising the key points.

Such essays would not typically exhibit extensive reading beyond the suggested material in the essay description, or original content.

Typical positive characteristics of an essay of β- (pass) standard include:

- Demonstration of understanding of some of the underlying mathematical content of the essay
- An essay consistent with the quality expected of an upper-second-class final-year project from a leading mathematics department

while negative characteristics might include some non-trivial flaws in presentation, for example:

- An inappropriate length
- Repetition or lack of clarity
- Lack of a coherent structure
- The absence of either an introduction or conclusion.

For the avoidance of doubt, a key aspect of the essay is that the important mathematical content is presented clearly in (at least close to) the suggested length. An excessively long essay is likely to be of (at best) pass standard.

Contents

1. Localising Lie Algebra Representations: Around the Beilinson-Bernstein Theorem ....

Professor I. Grojnowski

Let $\mathfrak{g}$ be a semisimple complex Lie algebra; this is the tangent space at 1 of an algebraic group G.

In an introductory Lie algebra course we learn that the finite dimensional representations of $\mathfrak{g}$ are parameterised by highest weights $\lambda \in P^+$ and that their characters are given by the Weyl character formula.

What about other representations of $\mathfrak{g}$? For example, what are the characters of the unique quotient of a Verma module with highest weight λ, when λ is not a dominant weight?

This problem was solved in a remarkable 3 page paper by Beilinson and Bernstein that completely changed how we think about representation theory. In this paper they show how any irreducible representation, or more generally any representation on which the center of the enveloping algebra acts as a scalar, can be 'localised' to a sheaf of D-modules on the flag variety; and conversely.

For this essay, which requires algebraic geometry at the Part III level, you should study this theorem and its sequels, especially [BMR]. To begin you should learn the definition of the sheaf D_X of differential operators on a smooth algebraic variety X, that this is a non-commutative deformation of functions on the cotangent bundle T^*X, and (Bernstein's) theorem that such modules are big; indeed they have coistropic support in T^*X.

Then understand the statement of the BB theorem, and prove it by hand for $\mathbf{P}^1$ — and in that way construct every single irreducible infinite dimensional representation of $\mathfrak{sl}_2$. To understand the proof in general you will want to understand the geometry of the flag variety $\mathcal{B}$ of G – a smooth projective variety on which G acts transitively – and perhaps also of the moment map $\mu : T^*\mathcal{B} \to \mathfrak{g}^*$, as well as various old theorems of Harish-Chandra describing the center of the enveloping algebra, and Kostant, showing normality of the nilpotent cone.

An ambitious essay would then understand what happens in characteristic p. This is the theorem of Bezrukavnikov, Mirokovic, Rumynin.

If you have a good technical understanding of these two theorems there are many things that you can now understand, which we can discuss individually depending on your interests!

References

1. A Beilinson, J. Bernstein, "Localisation de g-modules", Comptes Rendus de l'Academie des Sciences, Serie I, 292 (1): 15–18, 1981, MR 0610137

2. Bezrukavnikov, Mirkovic, Rumynin, "Localization of Modules for a Semisimple Lie Algebra in prime characteristic"', arXiv:math/0205144, Annals of Math,

Many notes and text book expositions can be found on D-modules, the basic geometry of the flag variety, and the BB-theorem. You will want to start with one of these, and not the original paper.

2. The Representation Theory of the Symmetric Groups .... Professor S. Martin

The problem with the symmetric groups is that there is too much symmetry and not enough group (a quotation attributed to G.D. James).

The representation theory of the symmetric groups goes back to pioneering work of Frobenius and his greatest student Schur (who proved rather more than a lemma), who generalised Frobenius' work to the general linear group over $\mathbb{C}$. Schur's methods were then developed independently by Weyl and by Rev Alfred Young (a collaborator of J.H. Grace) in Cambridge and at the Rectory of Birdbrook, Essex, where Young was a parish priest.

Young's methods now look somewhat antiquated and the approach adopted in the modern treatment in the lecture notes of James is both characteristic-free and diagrammatic in nature, except in those places, such as the construction of character tables of symmetric groups, where the results themselves depend upon the base field.

One should consider the construction of the irreducibles as well as various results for modules, such as the Branching Theorem and the Littlewood–Richardson rule and the lecture notes by James is highly recommended for this. There should be one major application of the theory. Possible applications include the combinatorial algorithms contained in Sagan's book (also material in Fulton is relevant), applications with Deligne's categorical approach in Snowden's notes, applications to the McKay conjecture in Law's notes, representations of other diagrammatic algebras such as the Temperley-Lieb or partition algebras in the book of Bowman.

Relevant Courses

Essential: Any undergraduate course in representation theory; basic combinatorics

Useful: the Part III course on modular representation theory

References

1. C. Bowman, Diagrammatic Algebra, Springer 2025
2. W. Fulton, Young tableaux, with applications to representation theory and geometry, LMS Student Texts, vol. 35, CUP, Cambridge, (1997).
3. W. Fulton and J. Harris, Representation theory, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991.
4. G. James The representation theory of the symmetric groups, Lecture Notes in Mathematics, vol. 682, Springer, Berlin, 1978
5. S. Law. Notes taken by Leonard Tomczak from a former Part III course available at

   `https://sites.google.com/view/staceylawmaths/home`
6. B. E. Sagan, The symmetric group: representations, combinatorial algorithms and symmetric functions, GTM 203 (2nd edn), Springer–Verlag, New York (2001).
7. A. Snowden lecture notes:

   `people.maths.ox.ac.uk/horawa/math711.pdf`

   (Notes taken by Aleksander Horawa.)

3. Blocks with a Cyclic Defect Group .... Professor S. Martin

This topic is perhaps the highlight of the classical theory of modular representation theory as initiated and developed by Richard Brauer, and then refined further by Sandy Green and Jon Alperin. In this theory a central role is played by the p-blocks of the modular group algebra and by certain p-subgroups of G called defect groups. As such, the more complicated the defect group, the more representation theoretically complicated is the (p-)block. For example, blocks of defect zero - where the defect group is trivial - are matrix algebras wherein everything is semisimple. Given a block of a finite group with a cyclic defect group, there is combinatorial gadget called a Brauer tree which describes the structure of the indecomposable projective modules completely - in the sense that we know their 'module diagram' [4,5] (this is much more than just knowing say, their composition series) . One could adopt Green's original approach [2] and [1, Ch 5] or use some methods from the representation theory of algebras as in [3, Chapter 6] to give a construction of these trees. At least one non-trivial example should be included for illustration of the theory, e.g. symmetric groups or $\mathrm{SL}_2(p)$ or a friendly sporadic group [6].

Relevant Courses

First courses in ordinary and modular representation theory

References

1. J.L. Alperin, Local representation theory, CUP (1986).
2. J.A. Green, Walking around the Brauer tree, *J. Australian Math. Soc.* **17** (1974), 197–213.
3. D.J. Benson, Representations and cohomology vol I. (Basic representation theory of finite groups and associative algebras), 2nd edn., Cambridge Studies in Advanced Mathematics **30**. CUP (1998).
4. D.J. Benson and J.F Carlson, Diagrammatic methods for modular representations and cohomology, Commun. in Algebra **15**, 53–121 (1987).
5. D.J. Benson and J.H. Conway, Diagrams for modular lattices, J. Pure Appl. Algebra 37 (1985), 111-116.
6. G. Hiss & K. Lux, K, Brauer trees of sporadic groups, Oxford Science Publications, OUP (1989).

4. The Zamolodchikov Periodicity Conjecture .... Dr N. J. Williams

A *Y-system* is a particular set of algebraic recurrence equations determined by a pair of Dynkin diagrams (a particular class of graphs). They were introduced by Zamolodchikov in his study of the thermodynamic Bethe ansatz equations [1]. Zamolodchikov noticed that solutions to the Y-system were periodic, with the period determined by the choice of Dynkin diagrams, and conjectured that this was true in general.

Different cases of this conjecture were proven by several authors, but the general case was proven by Keller in 2008 [2]. Keller's proof is quite remarkable. It first builds upon the pioneering work of Fomin and Zelevinsky, who showed how to interpret the Zamolodchikov periodicity conjecture as the periodicity of a certain transformation coming from the theory of cluster algebras [3]. Keller then uses the "categorification" of cluster algebras to interpret this transformation as a functor on a particular category. Zamolodchikov periodicity then becomes equivalent to the periodicity of this functor, which follows from classical results about the functors on the derived categories of quiver representations.

The essay will explore the Zamolodchikov periodicity conjecture and its proof. It will start off by describing the Y-system and the original statement of the conjecture. It will then move on to show how the conjecture may be reformulated in the language of cluster algebras, giving the requisite background on these algebras. The essay should then give some exposition of Keller's proof of the conjecture.

Relevant Courses

Essential: Part II Representation Theory or equivalent, Part III Homological Algebra

Useful: Part III Category Theory

References

1. A. B. Zamolodchikov, On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories, Phys. Lett. B **253** (1991), no. 3-4, 391–394

2. B. Keller, The periodicity conjecture for pairs of Dynkin diagrams, Ann. of Math. (2) **177** (2013), no. 1, 111–170

3. S. Fomin and A. V. Zelevinsky, Y-systems and generalized associahedra, Ann. of Math. (2) **158** (2003), no. 3, 977–1018

5. Habiro Cohomology, After Scholze, Wagner, ... Professor I. Grojnowski

q-series such as

$$\sum_{1}^{n}(1-q)(1-q^2)\cdots(1-q^n)$$

occur often in mathematics arising from physics. Examples like the series above have some remarkable properties - they do not converge to analytic functions anywhere, but they can be evaluated at any finite root of unity, as the sum above just terminates. Precisely, they are elements of the ring $\lim_{n,m} \mathbb{Z}[q]/(q^n-1)^m$, the "Habiro ring" [Habiro], and can be thought of as "functions with compatible power series expansions around any root of unity".

Recently - within the last year! - Scholze and his students and collaborators have intepreted these q-series as classes in a new cohomology theory they have discovered, called Habiro cohomology. This is supposed to be a cohomology theory for algebraic varieties defined over the integers, generalising almost every other cohomology theory known (de Rham, etale, crystalline, prismatic,...) It is very much an area of active research, but it is also surprisingly accessible.

An essay could begin with the classical story - the definition of the de Rham complex of a smooth algebraic variety (expositons are in [Deligne], [Voisin], and many other places). This is a filtered complex, its associated graded is the 'Hodge cohomology'; and the relationship between the two on a smooth projective variety is the Hodge theorem. The de Rham cohomology of a family of smooth algebraic varieties aquires a connection, the Gauss-Manin connection, which interacts nicely with the Hodge filtration.

You should understand this well enough to compute it explicitly for the Legendre family of elliptic curves $X = \{y^2 = x(x-1)(x-\lambda)\} \to S = \{\lambda \in \mathbb{C}\}$.

An ambitious essay would be to understand the analogues of this picture for the Legendre family in Habiro cohomology [Shirai,GW].

Before this, begin with the definition of the q-de Rham cohomology — for affine spaces, and then for 'framed manifolds', and the question of dependence of framing of the q-completed such. (Scholze's first attempt at this is explained in [Scholze 2016].)

Then learn the basic properties of the Habiro ring, such as its description as compatible families of power series expansions. Now understand the definition of Habiro cohomology - either the concrete one in Scholze's 2025 lectures, or the topological one of Wagner (this last requires *much* more background and is probably inaccessible in Part III). Finally, the still mysterious q-deformation of Gauss-Manin.

Alternately, if you rather like explicit q-difference equations and playing with functions, you could understand Nahm sums and elements of the Habiro ring of a number field F attached to elements of $K_3(F)$, as in [GSWZ].

Or you could learn about the formal properties of complex perturbative Chern-Simons theory, knot theory invariants, and why they are defined in the Habiro ring - which again requires understanding properties of solutions of q-difference equations. The sum above is an example of such; it is the Kashaev invariant of a trefoil knot.

References

1. Unusually for an essay, you will want to watch Scholze's video 2025 lecture series – though results and definitions may be written down before the essay is due, they are not yet in the literature.

2. Deligne, Travaux de Griffiths, Seminar Bourbaki 376, Lecture Notes in Math 180, Springer Verlag 1970, 213–237

3. Habiro, Cyclotomic completions of polynomial rings, arxiv:math/0209324

4. Garoufalidis, Scholze, Wheeler, Zagier, The Habiro ring of a number field, arxiv:2412.04241

5. Garoufalidis, Wheeler, Explicit classes in Habiro cohomology, arxiv:2505.19885

6. Shirai, q-deformation with (ϕ, Γ) structure of the de Rham cohomology of the Legendre family of elliptic curves, arXiv:2006.12310

7. Scholze, courses at MPIM 2024, 2025, online.

8. Scholze 2016, Canonical q-deformations in arithmetic geometry, arXiv:1606.01796

9. Wagner, (i) Bonn PhD Thesis, 2025, (ii) q-deRham cohomology and topological Hochschild cohomology over ku, arxiv:2510.06057,

10. Voisin, Hodge theory and complex algebraic geometry I, Cambridge Studies in advanced math, 2002.

6. Coherent Sheaves and Braid Group Actions ....
Professor A. M. Keating

The braid group on n strands, Br_n, captures symmetries in a range of settings across geometry and topology. Arguably its most natural incarnation is as the group of homeomorphisms of a disc with n marked points, taken up to a natural (homotopy) equivalence; this yields many further actions as transformations of topological or smooth manifolds, by homeomorphisms or diffeomorphisms. In contrast, suppose we're given a complex manifold, or variety, say X; in this setting, the transformations of X are biholomorphic maps, which in general are far too rigid to realise braid group actions. Following beautiful work of Seidel and Thomas, we now understand that in many cases, we instead get natural braid group actions on an important invariant of X: its bounded derived category of coherent sheaves, $D^b\mathrm{Coh}(X)$.

The essay should introduce $D^b\mathrm{Coh}(X)$, spherical objects and spherical twists, and then explain how to get braid group actions as autoequivalences of $D^b\mathrm{Coh}(X)$. It should state Seidel–Thomas' theorem about the faithfulness of these braid group actions, and then give a careful account of a proof, for which several strategies are now available. *N.B. Any classical properties of the braid group that are used should be clearly stated, but the essay is not expected to include proofs of these.*

Relevant Courses

Essential: Algebraic geometry, commutative algebra

Useful: Homological algebra

References

1. Daniel Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Oxford Math. Monogr., The Clarendon Press, Oxford University Press, Oxford, 2006, viii+307 pp.

2. Paul Seidel and Richard Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001), no. 1, 37–108.

3. Christopher Brav and Hugh Thomas, *Braid groups and Kleinian singularities*, Math. Ann. 351 (2011), no. 4, 1005–1017.

4. Yu Qiu and Jon Woolf, *Contractible stability spaces and faithful braid group actions*, Geom. Topol. 22 (2018), no. 6, 3701–3760.

7. Stability in Complex Dynamics ....
Professor H. Krieger

For any $c \in \mathbb{C}$ we may associate a degree 2 polynomial map $f_c(z) = z^2 + c$ which is an algebraic self-map of the Riemann sphere; this is an example of a *rational map*, and $\mathbb{C}$ parametrizes a family of rational maps in this way. More generally, an *algebraic family of rational maps* is a morphism $V \to \mathrm{Rat}_d$ of an irreducible quasi-projective variety V into the space Rat_d of rational maps of a fixed degree $d \geq 2$. Such a family is said to be *stable* if there is a uniform bound on the period of the attracting cycles of the rational maps occurring in the family. This notion of stability in a family of rational maps has many strong consequences, like continuity of the chaotic ('Julia') set as one moves within the family. As an example, in the quadratic polynomial family given above, the family fails to be stable precisely along the boundary of the Mandelbrot set.

This essay will provide an introduction to complex dynamics via the stability and rigidity theorems of Mañé-Sad-Sullivan and McMullen. The essay should provide a brief background on the basic objects of complex dynamics [1], present the λ-lemma of MSS [3], and exposit McMullen's seminal theorem [4] that a stable algebraic family of rational maps is either trivial or arises from addition on complex tori. Depending on the writer's interests, the essay can go on to discuss one of McMullen's two applications of the rigidity theorem [4] or DeMarco's work [2] on the bifurcation locus (this requires Part III Complex Manifolds in addition to the requirements listed below).

Relevant Courses

Essential: Complex analysis, Riemann surfaces.

Useful: (Undergraduate) algebraic geometry.

References

1. P. Blanchard. "Complex analytic dynamics on the Riemann sphere." Bull. Amer. Math. Soc. (N.S.) 11(1): 85-141 (July 1984).
2. L. DeMarco. "Dynamics of rational maps: Lyapunov exponents, bifurcations, and capacity." Math. Ann. 326, 43–73 (2003).
3. R. Mañé, P. Sad, and D. Sullivan. "On the dynamics of rational maps." Annales scientifiques de l'École Normale Supérieure 16.2 (1983): 193-217.
4. C. McMullen. "Families of Rational Maps and Iterative Root-Finding Algorithms." Annals of Mathematics 125, no. 3 (1987): 467–93.

8. Pure Motives ....

Dr J. Laga

The Chow groups $\mathrm{CH}^i(X)$ of algebraic cycles of a variety are the algebro-geometric analogues of singular homology of a topological space. They appear prominently in central conjectures in algebraic and arithmetic geometry, such as the Hodge, Tate, Beilinson–Bloch and standard conjectures. In contrast to homology, Chow groups are difficult to compute, especially when $i \geq 2$. Nevertheless, they carry a lot of structure, and the most systematic way to study this structure is through Grothendieck's theory of Chow motives, also known as pure motives.

The goal of this essay is to learn about algebraic cycles through the lens of Chow motives. The essay should be structured as follows:

1. Survey the theory of Chow motives, following [2,5,6].
2. Compute a few examples of motives, such as those of curves, projective space and a blow-up, as described in the above references.
3. Discuss one (or more) of the following topics: The 'infinite-dimensionality' of $\mathrm{CH}^2(S)$ for a surface S with $p_g(S) > 0$ [3]; the decomposition of the motive of a surface [4]; or the notion of finite-dimensionality of motives/Chow groups due to Kimura [1,5].

Relevant Courses

Essential: Part III Algebraic Geometry.

Useful: Part III Complex Manifolds (this may provide a useful Hodge-theoretic perspective, but is not necessary)

References

1. Kimura, *Chow groups are finite dimensional, in some sense*, Math. Ann., Vol 331, 2005
2. Milne, *Motives—Grothendieck's dream*, Surv. Mod. Math., Vol 6, 2013
3. Mumford, *Rational equivalence of* 0*-cycles on surfaces*, J. Math. Kyoto Univ., Vol 9, 1968
4. Murre, *On the motive of an algebraic surface*, J. Reine Angew. Math., 409, 1990
5. Murre, Nagels and Peters, *Lectures on the theory of pure motives*, American Mathematical Society, Providence, RI, 2013
6. Scholl, *Classical Motives*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math.

9. Moduli of Higher Dimensional Varieties .... Professor D. Ranganathan

The goal of this essay is to study the construction and properties of moduli spaces of varieties of general type. In the 1960s, Deligne and Mumford constructed a proper moduli space of stable curves, and Mumford initiated a detailed study of its geometry and topology. The foundations of the higher dimensional theory were established by Kollár, Shepherd-Barron, and Alexeev. The KSBA theory aims to exhibit a higher dimensional analogue of the moduli space of stable curves and the minimal model program is the key technical input into the theory.

The essays aims to give the writer a chance to explore the KSBA theory through examples. A good essay will give a quick introduction, through examples, to the minimal model program, including semi-log canonical singularities and their behaviour under one-parameter degenerations. After this, you should choose one concrete example and discuss it in detail. Potential examples could include:

(i) Moduli spaces of polarized abelian and toric varieties, following Alexeev,
(ii) Moduli of hyperplane arrangement pairs, following Hacking–Keel–Tevelev, or
(iii) Spaces of elliptically fibered surfaces, following Miranda and Abramovich–Vistoli.

These are just three of many potential avenues.

The essay should include an overview of how the moduli spaces are constructed, as well as a detailed geometric description of the different spaces that can (or cannot) appear in the moduli spaces, and comment on the irreducibility of the space (or lack thereof).

Relevant Courses

Essential: Part III Algebraic Geometry, Part III Toric Geometry

References

1. Abramovich, Dan, and Angelo Vistoli. "Complete moduli for fibered surfaces." Recent progress in intersection theory. Boston, MA: Birkhäuser Boston, 1997. 1-31.

2. Alexeev, Valery."Complete moduli in the presence of semiabelian group action." Annals of mathematics (2002): 611-708.

3. Hacking, Paul, Sean Keel, and Jenia Tevelev. "Compactification of the moduli space of hyperplane arrangements." Journal of Algebraic Geometry 15.4 (2006): 657-680.

4. Miranda, Rick. "The moduli of Weierstrass fibrations over $\mathbb{P}^1$." Mathematische Annalen 255.3 (1981): 379-394.

10. Log Concavity Conjectures for Matroids ....
Professor D. Ranganathan

A matroid is a structure from combinatorics that generalizes a number of elementary discrete mathematical objects, including configurations of vectors in a vector space and finite graphs. They were first discovered in an attempt to extend the notion of duality from planar graphs to arbitrary graphs.

There are a number of conjectures concerning the structure of matroids, broadly termed "log concavity" statements. These concern things like the sizes of coefficients of the chromatic polynomial of graphs. The conjectures remained open for decades and were finally resolved in groundbreaking work of J. Huh and collaborators a few years ago. The breakthrough came in two steps. First, it was observed that some matroids have a connection to algebraic geometry, and the properties of the cohomology rings of these varieties, specifically their Hodge theory, led to resolutions of the log concavity conjectures in these cases. Second, it was realized that the Hodge theory package could be proved directly for all matroids, using algebraic geometry as guidance but without actually using any results from algebraic geometry.

The goal of this essay is to discuss a proof of the log concavity conjectures in a range of cases that at least include all graphs. There are a number of approaches: Huh's original proof for graphs via a study of Milnor numbers, Huh–Katz's proof for matroids realizable over some field using toric intsersection theory, Adiprasito–Huh-Katz's proof in general by developing combinatorial Hodge theory, or the recent proof by Berget–Eur–Spink–Tseng using equivariant K-theory.

The essay should give an introduction to the theory of matroids, enough to define and describe the characteristic polynomial. It should then introduce the relevant geometric ingredients in the chosen approach to the conjectures, and overview the proof. A good essay will include a large number of examples exhibiting the relevant aspects of Hodge theory, toric intersection theory, or K-theory, as appropriate.

Relevant Courses

Essential: Part III Algebraic Geometry, Part III Algebraic Topology, Part III Toric Geometry

References

1. Adiprasito, Karim, June Huh, and Eric Katz. "Hodge theory for combinatorial geometries." Annals of Mathematics 188.2 (2018): 381-452.

2. Berget, Andrew, Eur, Chris, Spink, Hunter, and Tseng, Dennis . "Tautological classes of matroids." Inventiones mathematicae 233.2 (2023): 951-1039.

3. Huh, June, and Eric Katz. "Log-concavity of characteristic polynomials and the Bergman fan of matroids." Mathematische Annalen 354.3 (2012): 1103-1116.

4. Huh, June. "Milnor numbers of projective hypersurfaces and the chromatic polynomial of graphs." Journal of the American Mathematical Society 25.3 (2012): 907-927.

11. Wall-Crossing and Moduli Spaces ....

Dr F. Rezaee

In algebraic geometry, the notion of moduli space, which is a space parametrizing geometric objects with given properties, is fundamental. The Hilbert scheme is an essential type of moduli space where the geometric objects of interest are embedded in a certain ambient space. Despite their natural definition, Hilbert schemes are highly mysterious to understand: Vakil's Murphy's law for Hilbert schemes says that there is no geometric possibility so horrible that it cannot be found generically on some component of some Hilbert scheme.

While classical methods fail to fully understand complicated Hilbert schemes, wall-crossing with respect to Bridgeland stability conditions on derived categories (and more recently, the stability scattering diagram) provides a powerful tool for understanding such subtle spaces.

The goal of this essay is to review some basic relevant notions such as derived categories ([1,2]) and stability conditions ([3,4,5]). Furthermore, time permitting, students could also consider some more advanced materials considering the applications of wall-crossing in the geometry of moduli spaces (e.g., [6-9]).

Relevant Courses

Essential: Part III Algebraic Geometry, Part III Homological Algebra.

Useful: Part III Commutative Algebra.

References

1. D. Hyubrechts, Fourier-Mukai transforms in algebraic geometry, 2006.

2. A. Caldararu, Derived categories of sheaves: a skimming, arXiv:math/0501094.

3. A. Bayer, A tour to stability conditions on derived categories.

4. E.Macrì and B. Schmidt. Lectures on Bridgeland stability. Lect. Notes Unione Mat. Ital., 21:139–211, Springer, Cham, 2017.

5. T. Bridgeland, Stability conditions on triangulated categories, Annals of Mathematics, 166 (2007), 317–345.

6. B. Schmidt. Bridgeland stability on threefolds—First wall crossings. J. Algebraic Geom., 29(2):247–283, 2020.

7. P. Gallardo, C. Lozano Huerta, and B. Schmidt. On the Hilbert scheme of elliptic quartics. Michigan Math. J., 67(4):787–813, 2018.

8. F. Rezaee, An interesting wall-crossing: Failure of the wall-crossing/MMP correspondence, Selecta Mathematica New Series, Volume 30, Number 102, 2024.

9. F. Rezaee, Geometry of canonical genus four curves, Proceedings of the LMS, vol 128 (1), 2024.

12. Markov Numbers ....

Dr F. Rezaee

Markov triples (x, y, z) are integer solutions to the Diophantine equation $x^2 + y^2 + z^2 = 3xyz$. Any integer appearing in a Markov triple is a Markov number. For example, every other number in the Fibonacci sequence (i.e., $1, 2, 5, 13, 34, \ldots$) is a Markov number. All of the Markov triples can be obtained from $(1, 1, 1)$ by a recurrent procedure. Markov numbers have very beautiful properties and patterns and they are ubiquitous across mathematics from combinatorics to graph theory to number theory and arithmetic geometry to approximation theory to cluster algebra to hyperbolic, symplectic and algebraic geometry, to name a few.

Despite their simple definition and helping to resolve conjectures, Markov numbers are still mysterious and there are several challenging open problems regarding them, e.g., there is a conjecture due to Frobenius which has been open for more than a century (suggested in 1913): Each Markov number is the maximal number in exactly one Markov triple.

Since the mathematics of Markov numbers is rich, depending on students' interests, there are several directions for an essay. The goal of this essay is to review and summarize some papers on interesting results in which the Markov numbers (or Fibonacci numbers) and their properties have played a crucial role. Some of the references are listed (not all of them have to be considered in one essay), but there are more that can be used in the essay.

Depending on the particular topic, the essay could be considered as the first step toward a more serious and challenging research project that students could consider in their future studies.

Regarding the Frobenius conjecture mentioned above, after reviewing the history and latest developments on its resolution (e.g.,[1,2,7,9]), a highly ambitious essay may also investigate the conjecture using, for example, the connection to the scattering diagram and the properties of exceptional objects ([4,14]), or using some combinatorial approaches.

Relevant Courses

Useful: There is no specific prerequisite, but basic algebra, number theory, or combinatorics will be helpful. Also, more advanced courses such as algebraic geometry and commutative algebra will be useful.

References

1. M. Aigner, Markov's Theorem and 100 Years of the Uniqueness Conjecture: A Mathematical Journey from Irrational Numbers to Perfect Matchings / Martin Aigner.Cham, Springer, 2013. Print.

2. A. Baragar, On the unicity conjecture for Markoff numbers, Canad. Math. Bull. Vol. 39 (1), 1996 pp. 3-9

3. H. Cohn, Growth Types of Fibonacci and Markoff. The Fibonacci Quarterly, 17(2), 178–183. 1979

4. M. Gross and F. Rezaee, Geometry of the stability scattering diagram for P^2 and applications, arXiv:2510.26770, 2025.

5. A. Itsara, G. Musiker, J. Propp, and R. Viana, Combinatorial Interpretations for the Markoff Numbers.

6. C. Lagisquet, E. Pelantová, S. Tavenas, L. Vuillon, On the Markov numbers: Fixed numerator, denominator, and sum conjectures, Advances in Applied Mathematics, Volume 130, 2021, 102227.

7. K.Lee, L. Li, M. Rabideau, R. Schiffler, On the ordering of the Markov numbers, Advances in Applied Mathematics,Volume 143, 2023,102453.

8. J. G. Propp, The Combinatorics of Frieze Patterns and Markoff Numbers, Integers, Vol 20, 2005.

9. D. Rosen and G. S. Patterson, Jr., Some Numerical Evidence Concerning the Uniqueness of the Markov Numbers, Mathematics of computation, volume 25, number 116, 1971.

10. C. Series, The geometry of markoff numbers.The Mathematical Intelligencer 7, 20–29 (1985)

11. B. Springborn,The hyperbolic geometry of Markov's theorem on Diophantine approximation and quadratic forms,L'Enseignement Mathématique (2) 63 (2017), 333–373.

12. B. Springborn, The worst approximable rational numbers, Journal of Number Theory,Volume 263, 2024, Pages 153-205.

13. G. Urzúa and J. P. Zúñiga,The Birational Geometry of Markov Numbers. Moscow Mathematical Journal, 25(2), 197–248. (2025)

14. A. P. Veselov, Markov fractions and the slopes of the exceptional bundles on P2, 2025.

13. The Existence of Minimal Immersed 2-Spheres .... Dr A. Guerra

An old theorem due to Birkhoff asserts that every Riemannian manifold M diffeomorphic to a sphere admits one closed geodesic. A closed geodesic is simply a map $u\colon S^1 \to M$ which is a critical point for the energy functional

$$E[u] = \int_{S^1} |du|^2.$$

Birkhoff's theorem is remarkable in that it uses global topological methods to find non-minimizing critical points of the energy functional.

The purpose of this essay is to address the 2-dimensional counterpart of Birkhoff's theorem, which is a famous theorem due to Sacks and Uhlenbeck. It asserts that if M is diffeomorphic to a sphere (or, more generally, if its universal cover is not contractible), then there is a map $u\colon S^2 \to M$ which is a critical point of the energy functional. Such a map is said to be a *harmonic map* and it parametrizes an immersed minimal surface, which is the natural two-dimensional generalization of geodesics.

The essay should begin with a basic discussion about the connection between harmonic maps and minimal surfaces in two dimensions. It should then move on to present a complete proof of the theorem of Sacks and Uhlenbeck, following either the original presentation in [3] or the ones in [1,2]. In particular, a proof of the removable singularity theorem from [1] should be presented in full detail.

Relevant Courses

Essential: Analysis of PDEs.

Useful: Elliptic PDEs, Differential Geometry, Riemannian Geometry.

References

1. T. Colding, and W. Minicozzi (2011). A Course in Minimal Surfaces (Vol. 121). American Mathematical Society.
2. M. Grüter (1984). Conformally invariant variational integrals and the removability of isolated singularities. Manuscripta Mathematica, 47(1–3), 85–104.
3. J. Sacks, and K. Uhlenbeck (1981). The Existence of Minimal Immersions of 2-Spheres. The Annals of Mathematics, 113(1), 1.

14. Blow Up for the Defocusing Wave Equation ....

Professor P. Raphael

The possibility of singularity formation for the non linear defocusing NLS equation in super critical regimes has been a long standing open problem which found its resolution in [1]. A slightly different method is implemented in [2] to treat the case of the defocusing complex wave equations.

This essay aims at comparing the two methods. A very good essay will explain the key steps in the proof [2] by in particular clarifying which part of the analysis relies on numerical computations. An outstanding essay will design an implementation of the finite codimensional stability method developed in [1]. The case of the real valued defocusing equation is a major open problem in the field.

References

1. Merle, F.; Raphaël, P.; Rodnianski, I.; Szeftel, J., *On blow up for the energy super critical defocusing non linear Schrödinger equations*, Invent. Math. **227** (2022), no. 1, 247–413.
2. F. Shao, Wei D., Zhang Z.; On blow-up for the supercritical defocusing nonlinear wave equation, arXiv:2405.19674.

15. The (Riemannian) Penrose Inequality ....

Dr R. Teixeira da Costa

The Penrose inequality is a conjecture [6] which asserts that, in an asymptotically flat $(1+3)$-dimensional Lorentzian manifold satisfying the Einstein vacuum equations, a black hole region whose horizon has area A contributes $\sqrt{A/(16\pi)}$ (in appropriate units) to the total mass. Every such spacetime arises from the evolution of initial data for the Einstein equations, which is an asymptotically Euclidean 3–dimensional Riemannian manifold of prescribed first and second fundamental forms satisfying some constraint equations. Thus, one can formulate an analogous conjecture to the Penrose inequality at the level of initial data sets in General Relativity.

The Riemannian version of the Penrose inequality was first shown in [4,5] using geometric flows. Recently, a more elementary proof was given in [1], based on the analysis of a suitable boundary value elliptic problem. These techniques are quite powerful, see e.g. [2,3]. The goal of this essay is to present some of the key ideas of the proof in a self-contained manner.

Relevant Courses

Essential: Part III Analysis of PDE, Part III Differential Geometry

Useful: Part III Elliptic PDE

References

1. Agostiniani, V., Mantegazza, C., Mazzieri, L., & Oronzio, F. (2022). Riemannian Penrose inequality via Nonlinear Potential Theory. 1–27. http://arxiv.org/abs/2205.11642
2. Agostiniani, V., & Mazzieri, L. (2017). On the Geometry of the Level Sets of Bounded Static Potentials. Communications in Mathematical Physics, 355(1), 261–301. https://doi.org/10.1007/s00220-017-2922-x
3. Agostiniani, V., Mazzieri, L., & Oronzio, F. (2021). A green's function proof of the positive mass theorem. 2, 1–22.https://arxiv.org/pdf/2108.08402
4. Bray, H. L. (2001). Proof of the Riemannian Penrose inequality using the positive mass theorem, Journal of Differential Geometry 59, 177–267.
5. Huisken, G., & Ilmanen T. (2001). The inverse mean curvature flow and the Riemannian Penrose inequality, Journal of Differential Geometry 59, 353–437.
6. Penrose, R. (1973). Naked singularities. Annals of the New York Academy of Sciences, 224(1), 125–134. https://doi.org/10.1111/j.1749-6632.1973.tb41447.x

16. Advances in the Ribe Programme ....

Dr A. Zsák

In 1976 Ribe proved his remarkable rigidity theorem: uniformly homeomorphic Banach spaces have the same local structure meaning that any finite-dimensional subspace of either of the two spaces is isomorphic with a uniform constant to a subspace of the other space. The implication of this is that one should be able to express any local property of Banach spaces, that is properties depending only on the finite-dimensional subspaces of a space, purely in terms of the metric structure without reference to the linear structure. In view of this, Bourgain proposed an ambitious programme that became known as 'The Ribe Programme'. The goal is to find purely metric descriptions of well-known local properties of Banach spaces and to extend the study of these properties from Banach spaces to general metric spaces. The aim of this essay is to present some of the advances of this programme: the metric theory of type and cotype, and the nonlinear Dvoretzky theorem.

The correct definition of metric cotype was finally found by Mendel and Naor decades after the linear version [**1**]. In the same paper, Mendel and Naor used metric cotype to settle a number of open problems. The essay should describe metric cotype and focus on some of its applications, specifically on identifying all pairs (p, q) for which L_p embeds into L_q uniformly or coarsely.

The notion of metric type is much older and was introduced by Enflo long before the linear version. However, it remained a major open problem for a long time whether metric type and linear type coincide in Banach spaces. This was solved fairly recently by Ivanisvili, van Handel and Volberg [**2**]. The essay should include a proof of this result.

Dvoretzky's famous theorem states that any n-dimensional space contains a subspace of dimension roughly $\log n$ that is almost Euclidean. The $\log n$ dependence on the dimension is essentially best possible. Later Bourgain, Figiel and Milman proved a nonlinear analogue for metric spaces: any n-point metric space contains a subset of size about $\log n$ that almost isometrically embeds into Euclidean space. It then came as a major surprise when Bartal, Linial, Mendel and Naor discovered a remarkable threshold phenomenon: provided one is prepared to relax almost isometric embedding to an embedding of distortion $D > 2$, one can find a much bigger subset of size polynomial in n. Naor and Tao later gave a much simpler probabilistic proof of this in [**3**]. The essay should present this proof.

The survey articles by Naor [**4**] and Ball [**5**] provide very useful backround for the essay.

Relevant Courses

Essential: Functional Analysis

References

1. Mendel, Manor; Naor, Assaf Metric cotype. *Ann. of Math. (2)* 168 (2008), no. 1, 247–298.

2. Ivanisvili, Paata; van Handel, Ramon; Volberg, Alexander Rademacher type and Enflo type coincide. *Ann. of Math. (2)* 192 (2020), no. 2, 665–678.

3. Naor, Assaf; Tao, Terence Scale-oblivious metric fragmentation and the nonlinear Dvoretzky theorem. *Israel J. Math.* 192 (2012), no. 1, 489–504.

4. Naor, Assaf An introduction to the Ribe program. *Jpn. J. Math.* 7 (2012), no. 2, 167–233.

5. Ball, Keith The Ribe programme. Séminaire Bourbaki. Vol. 2011/2012. Exposés 1043–1058. *Astérisque* No. 352 (2013), Exp. No. 1047, viii, 147–159.

17. Chowla's Cosine Problem .... Professor W. T. Gowers

Chowla's cosine problem is the following question. Let A be a set of n integers. How large can the minimum be of the sum $s_A(x) = \sum_{a \in A} \cos(ax)$ over $x \in [0, 2\pi)$ (as a function of n)? Since the average of $s_A(x)$ is zero, the minimum must be negative. Also, the average of the $s_A(x)$ is easily checked to be $n/2$, so the magnitude of the sum must exceed $\sqrt{n/2}$, which led Chowla to conjecture in 1965 that there is a constant $c > 0$ such that for every A there exists x with $s_A(x) \leq -c\sqrt{|A|}$. Until recently, the best known bound in this direction, due to Ruzsa (building on work of Bourgain), was of the form $-\exp(c\sqrt{\log n})$. But earlier this year two papers independently obtained power-type bounds – that is, bounds of the form $-n^\gamma$ for an absolute constant $\gamma > 0$ – one by Zhihan Jin, Aleksa Milojevič, István Tomon and Shengtong Zhang and the other by Benjamin Bedert. The aim of this essay is to present at least one proof of this breakthrough and to discuss some of the ideas from earlier papers on the problem.

Relevant Courses

Essential: None of this year's Part III courses is essential.

Useful: It might be worth obtaining notes from an additive combinatorics course – such courses have been offered several times at Part III, but notes can also be found online from many courses round the world.

References

1. Zhihan Jin, Aleksa Milojevič, István Tomon and Shengong Zhang, *From small eigenvalues to large cuts, and Chowla's cosine problem*, arXiv 2509.03490
2. Benjamin Bedert, *Polynomial bounds for the Chowla cosine problem*, arxiv 2509.05260
3. Jean Bourgain, *Sur le minimum d'une somme de cosinus*, Acta Arithmetica **45** (1986), 381-389.
4. Imre Ruzsa, *Negative values of cosine sums*, Acta Arithmetica **111** (2004), 179-186.

18. Lagrangians of Hypergraphs .... Professor I. Leader

The Lagrangian of a hypergraph is a function that in some sense seems to measure how 'tightly packed' a subset of the hypergraph one can find. Lagrangians have beautiful properties and are of great interest, both in their own right and because they have several applications, most notably to the celebrated 'jumping hypergraphs' conjecture.

The main topic would be the way in which Lagrangians influence other properties, ranging from the fact, due to Motzkin, that Lagrangians provide a simple proof of Turan's theorem, right up to the relationship between Lagrangians and 'asymptotic density', with the disproof by Frankl and Rodl of the Erdos conjecture that the set of possible asymptotic densities is discrete. There would also be an examination of the Frankl-Furedi conjecture on maximising the Lagrangian, taking in the proof of this by Tyomkyn in the 'nice' case and the disproof by Gruslys, Letzter and Morrison in the general case.

Relevant Courses

Useful: Probabilistic Combinatorics (just for the basics on set systems)

References

1. T. Motzkin and E. Strauss, Maxima for graphs and a new proof of a theorem of Turan, Canadian Journal of Mathematics, vol 17 (1965), 533-540.
2. P. Frankl and V. Rodl, Hypergraphs do not jump, Combinatorica, vol 4 (1984), 149-159.

3. M. Tyomkyn, Lagrangians of hypergraphs: the Frankl-Furedi conjecture holds almost everywhere (Arxiv 1703.04273)

4. V. Gruslys, S. Letzter and N. Morrison, Hypergraph Lagrangians I: the Frankl-Furedi conjecture is false (Arxiv 1807.00793)

19. Canonical Ramsey Theory ....

Professor I. Leader

Canonical Ramsey theorems extend classical Ramsey theorems, which typically involve colourings involving a specified finite number of colours, to arbitrary colourings. The flavour is often different to that of the classical Ramsey theorems, although there is usually some relationship between a canonical theorem and a higher-order classical theorem.

There would be three themes to the essay. One is the question of bounds for finite canonical Ramsey theorems, dealing with work of Duffus, Lefmann and Rodl. Another is the question of what the actual canonical version of a classical theorem should be, focusing on the canonical Gallai theorem due to Deuber, Graham, Promel and Voigt and the canonical Hindman theorem due to Taylor. And the third is some beautiful work of Kittiassorn and Narayanan about the number of colours that appear in an infinite complete graph when we colour the infinite complete graph.

Relevant Courses

Essential: Ramsey Theory

References

1. Shift graphs and lower bounds on Ramsey numbers $r_k(l;r)$, D.Duffus, H.Lefmann and V.Rodl, Discrete Math, vol 137 (1995), 177-187.

2. On Erdos-Rado numbers, H.Lefmann and V.Rodl, Combinatorica, vol 15 (1995), 85-104.

3. A canonical partition theorem for equivalence relations on Z^t, W.Deuber, R.Graham, H.Promel and B.Voigt, Journal of Combinatorial Theory Series A, vol 34 (1983), 331-339.

4. A canonical partition relation for finite subsets of omega, A.Taylor, Journal of Combinatorial Theory Series A, vol 21 (1976), 137-146.

5. Exactly m-coloured complete infinite subgraphs, B.Narayanan, arxiv1303.2103.

6. A canonical Ramsey theorem for exactly m-coloured complete subgraphs, T.Kittipassorn and B.Narayanan, arxiv1303.2997.

20. The Singularity Probability of a Random iid Matrix ....

Professor J. Sahasrabudhe

Let A_n be a $n \times n$ matrix drawn uniformly at random from all $n \times n$ matrices with entries in $\{-1, 1\}$. This essay concerns the following basic question: what is the probability that the matrix A_n is singular, as n tends to infinity?

Now, a matrix is certainly singular if it has two rows or two columns that are equal, up to sign. Using this, one can fairly easily arrive at the lower bound

$$\mathbb{P}(\det(A_n) = 0) \geq (1 + o(1))2n^2 2^{-n}.$$

What is fascinating is that this easy lower bound is conjectured to be an *equality*, up to the $1+o(1)$ factor. This goes back to the pioneering work of Komlós in the 1960s and has proven to be an extremely difficult problem in the years since. This essay concerns a recent breakthrough of Tikhomirov [5], who has settled this conjecture up to subexponential terms. That is, Tikhomirov showed

$$\mathbb{P}(\det(A_n) = 0) = 2^{-n+o(n)}.$$

The purpose of this essay is to give a clean exposition of Tikhomirov's beautiful proof, highlighting the main ideas. The ambitious might also care to touch on the earlier beautiful work of Kahn Komlós and Szemerédi [2], Tao and Vu [4], Bourgain, Vu and Wood [1], or Rudelson and Vershynin [3].

Relevant Courses

Essential: Undergraduate Probability.

References

1. Bourgain, Jean, Van H. Vu, and Philip Matchett Wood. "On the singularity probability of discrete random matrices." Journal of Functional Analysis 258.2 (2010): 559-603

2. Kahn, Jeff, János Komlós, and Endre Szemerédi. "On the probability that a random ±1-matrix is singular." Journal of the American Mathematical Society 8.1 (1995): 223-240.

3. Rudelson, Mark, and Roman Vershynin. "The Littlewood-Offord problem and invertibility of random matrices." Advances in Mathematics 218.2 (2008): 600-633.

4. Tao, Terence and Van Vu. "On the singularity probability of random Bernoulli matrices." Journal of the American Mathematical Society 20.3 (2007): 603-628.

5. Tikhomirov, Konstantin. "Singularity of random Bernoulli matrices." Annals of Mathematics 191.2 (2020): 593-634.

21. The Spectrum of Random Regular Graphs .... Professor P. P. Varjú

Let G be a finite d-regular graph. Its adjacency matrix is self-adjoint, hence it has real eigenvalues. The largest one is d and corresponds to the all 1's vector. We denote by $\lambda(G)$ the maximal absolute value of the rest of the eigenvalues. (If d is an eigenvalue with multiplicity higher than 1, then $\lambda(G) = d$.) This is an important quantity that is connected to the mixing time of the simple random walk on G and also to its connectivity properties.

Alon and Boppana proved that $\lambda(G)$ cannot be arbitrarily small. They proved $\lambda(G) \geq 2\sqrt{d-1} + o(1)$ as the number of vertices goes to infinity. Friedman [3] proved that this lower bound is realized by a random d-regular graph with high probability. (One can consider several different reasonable models of a random d-regular graph, and the result is valid for each of them.) This result has been revisited by several authors including [4], [1] and [2]. Puder [4] proved only the slightly weaker bound that $\lambda(G) \leq 2\sqrt{d-1}+1$ with high probability, but his method has important applications in related problems.

A successful essay will give an exposition of the proof of Friedman's theorem (or the weaker bound of Puder) following one or more of the references [1], [2], [3] or [4]. The essay may explore further developments of the chosen approach.

Relevant Courses

None. The references are accessible with a strong background at the undergraduate level, and they have different flavours. [1] and [3] are combinatorial and probabilistic, [4] is group theoretic and [2] is analytic.

References

1. C. Bordenave, *A new proof of Friedman's second eigenvalue theorem and its extension to random lifts*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), no. 6, 1393–1439

2. C.-F. Chen, J. Garza-Vargas, J. A. Tropp, R. van Handel, *A new approach to strong convergence*, arXiv:2405.16026

3. J. Friedman, *A proof of Alon's second eigenvalue conjecture and related problems*, Mem. Amer. Math. Soc. **195** (2008), no. 910, viii+100 pp.

4. D. Puder, *Expansion of random graphs: new proofs, new results*, Invent. Math. **201** (2015), no. 3, 845–908

22. Popular Differences .... Professor J. Wolf

Roth's theorem [7] states that any sufficiently dense subsets of $\{1, 2, \ldots, N\}$ contains a non-trivial 3-term arithmetic progression (3-AP), that is, a sequence of the form $a, a+d, a+2d$ with $d \neq 0$. It is well known (by an argument of Varnavides) that Roth's theorem can be rephrased as saying that any subset of $A \subseteq \{1, 2, \ldots, N\}$ of density $\alpha > 0$ contains at least $c(\alpha)N^2$ 3-APs, where the precise shape of the function $c(\alpha)$ continues to be subject to research.

In [3], Green showed that there must be a common difference $d \neq 0$ which is "popular" amongst these 3-APs, in the sense that given any $A \subseteq \{1, 2, \ldots, N\}$ of density $\alpha > 0$, there exists $d \neq 0$ with the property that $|\{a : a, a+d, a+2d \in A\}| \geq (\alpha^3 - o(1))N^2$. The analogous result for 4-APs was proved in [5], but Ruzsa observed that (somewhat surprisingly) this phenomenon does not extend to 5-APs [1].

There is by now a substantial body of literature on generalisations and quantitative aspects of this problem, as well as work in other groups. In particular, this problem has been studied in $\mathbb{F}_p^n$, where $p \geq 3$ is a fixed prime and n is to be thought of as large, which frequently serves as a toy model in additive combinatorics in which certain aspects of the problem simplify [9].

A good essay will include a detailed introduction to the Fourier-analytic techniques leading to a proof of Roth's theorem for 3- and Szemerédi's theorem for 4-term arithmetic progressions in $\mathbb{F}_p^n$ following [4], and will go on to prove the aforementioned results on popular differences for 3-, 4- and 5-APs in this setting. The final section of the essay may be devoted to a discussion of higher-dimensional generalisations [6,2,8] or analogous results in the integers.

Relevant Courses

Useful: Additive Number Theory (Michaelmas, non-examinable)

References

1. V. Bergelson, B. Host, and B. Kra, *Multiple recurrence and nilsequences*, Invent. Math. 160 (2005), 261–303, with an appendix by I. Ruzsa

2. J. Fox, A. Sah, M. Sawhney, D. Stoner, and Y. Zhao, *Triforce and corners*, Math. Proc. Cambridge Philos. Soc. 169 (2020), 209–223

3. B. Green, *A Szemerédi-type regularity lemma in abelian groups, with applications*, Geom. Funct. Anal. 15 (2005), 340–376

4. B. Green, *Montréal notes on quadratic Fourier analysis*, Additive combinatorics (Montréal 2006, ed. Granville et al.), CRM Proceedings 43, 69–102, AMS, 2007

5. B. Green and T. Tao, *An arithmetic regularity lemma, an associated counting lemma, and applications*, In: An irregular mind, Bolyai Soc. Math. Stud., vol. 21, János Bolyai Math. Soc., Budapest, 2010, 261–334

6. M. Mandache, *A variant of the corners theorem*, Math. Proc. Cambridge Philos. Soc. 171, vol. 3 (2021), 607–621

7. K.F. Roth, *On certain sets of integers*, J. London Math. Soc. 28 (1953), 104–109

8. A. Sah, M. Sawhney, and Y. Zhao, *Patterns without a popular difference*, Discrete Analysis 2021:8, 30pp.

9. J. Wolf, *Finite field models in arithmetic combinatorics – ten years on*, Finite Fields and their Applications 32 (2015), 233–274

23. Skein Lasagna Modules Detect Exotic 4-Manifolds .... Dr S. Kang

A pair (W, W') of smooth 4-manifolds is *exotic* if they are homeomorphic but not diffeomorphic; one of the most classical examples is the pair $K3\#\overline{\mathbb{C}P^2}$ and $3\mathbb{C}P^2\#20\overline{\mathbb{C}P^2}$. Exotic smooth 4-manifolds are everywhere; there is even a conjecture saying that every smooth 4-manifold should admit infinitely many exotic smooth structures. However, until recently, the only possible ways to prove exoticness were gauge theory (where one counts instantons or monopoles) and Heegaard Floer theory.

Very recently, a completely new way to prove the exoticness of smooth 4-manifolds was discovered. It uses *skein lasagna module*, which is constructed using *Khovanov homology*, a link homology theory whose construction is purely combinatorial. Ren and Willis then showed that it can be used to detect exotic smooth 4-manifolds, thereby presenting the first purely combinatorial way of detecting them. While their techniques are difficult to apply in general, due to the fact that the skein lasagna module is almost impossible to compute using currently known results, it is still one of the most important recent developments in low-dimensional topology. The goal of this essay is to present some of the key ideas of the proof and try to present the proof, together with some background materials, in a somewhat self-contained manner.

Relevant Courses

Essential: Part III Algebraic Topology, Part III Knots and Knot Concordances

References

1. M. Khovanov, A categorification of the Jones polynomial, Advances in Mathematics, 226 (4): 3216–3281

2. D. Bar-Natan, On Khovanov's categorification of the Jones polynomial, Algebraic & Geometric Topology, 2: 337–370

3. C. Manolescu and I. Neithalath, Skein lasagna modules for 2-handlebodies. Journal für die reine und angewandte Mathematik (Crelles Journal), 2022(788), 37-76.

4. Q. Ren and M. Willis, Khovanov homology and exotic 4-manifolds. arXiv preprint arXiv:2402.10452.

24. Yau's Solution of the Calabi Conjecture ....
Dr A. G. Kovalev

The subject area of this essay is compact Kähler manifolds. Very informally, a Kähler manifold is a complex manifold admitting a metric and a symplectic form, both nicely compatible with the complex structure. The Ricci curvature of a Kähler manifold may be naturally expressed as a differential form which is then necessarily closed. Furthermore, the cohomology class defined by this form depends only on the complex manifold, but not on the choice of a Kähler metric. The Calabi conjecture determines which differential forms on a compact complex manifold can be realized by Ricci forms of some Kähler metric. Substantial progress on the conjecture was made by Aubin and it was eventually proved by Yau. This result gives, among other things, a powerful way to find many examples of Ricci-flat manifolds. The essay could discuss aspects of the proof and possibly consider some applications and examples. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` for further details.

Relevant Courses

Essential: Differential Geometry, Complex Manifolds

Useful: Algebraic Topology, Elliptic Partial Differential Equations

References

1. D. Joyce, *Riemannian holonomy groups and calibrated geometry*, OUP 2007. Chapters 6 and 7.

2. S.-T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. *Comm. Pure Appl. Math.*, **31** (1978), 339–411.

3. A good text on Kähler complex manifolds, e.g. D. Huybrechts, *Complex geometry. An introduction.* Springer 2005.

25. Stable Differential Forms .... Dr A. G. Kovalev

The term 'stable form' was suggested by Hitchin for an alternating multilinear form $\omega \in \Lambda^p(\mathbf{R}^n)^*$ which has an open orbit in the natural action of $GL(n, \mathbb{R})$. Stable forms generalize the notion of non-degenerate bilinear forms to alternating forms of an arbitrary degree p and non-trivial examples include stable forms of degree $p = 3$ in dimensions $n = 6$ and 7. The essay could begin by exploring the work [1] and showing how stable differential 3-forms lead to Ricci-flat metrics on 6- and 7-dimensional manifolds, as critical points of a volume function. In dimension 6 this gives an alternative view on Calabi–Yau manifolds (of complex dimension 3), and in dimension 7 one recovers a geometry associated with the exceptional Lie group G_2. The essay could investigate the significance of stable forms in one of these two geometries and its relation to Ricci-flat metrics with reduced holonomy [2]. Interested candidates are welcome to contact `A.G.Kovalev@dpmms` for further details.

Relevant Courses

Essential: Differential Geometry, Complex Manifolds

Useful: Algebraic Topology, Riemannian Geometry

References

1. N. Hitchin, *The geometry of three-forms in six and seven dimensions*, arXiv:math.DG/0010054 or J. Differential Geom. **55** (2000), 547–576.
2. D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press 2000.
3. A good text on Kähler complex manifolds, e.g. D. Huybrechts, *Complex geometry. An introduction.* Springer 2005.

26. Introduction to the Classification of Manifolds .... Professor O. Randal-Williams

If M and N are compact smooth manifolds of the same dimension, how can we tell whether they are diffeomorphic to each other? We can try to show that they are *not* diffeomorphic using the basic tools of Algebraic Topology like homology and homotopy groups, but how can we show that they *are* diffeomorphic, short of finding an explicit diffeomorphism? This essay will be an introduction to a collection of techniques for answering this question in high dimensions, specialising to the case of simply-connected manifolds.

You should first discuss Smale's h-cobordism theorem, which reduces the problem of showing that M and N are diffeomorphic to finding a special kind of cobordism W interpolating between them. For this you can follow Chapter 1 of [1], specialising to the simply-connected case. It has several quick applications, such as the Poincaré conjecture in high dimensions.

You should now just consider the case of manifolds of even dimension $2n$, and discuss a theorem of Kreck (Theorem C of [2]) which reduces the classification of such manifolds up to stabilisation by connect-sum with $S^n \times S^n$'s to a problem in cobordism theory, which is often tractable.

Finally, you should discuss the problem of destabilising: recognising when a manifold is a connect-sum with $S^n \times S^n$, and recognising when a diffeomorphism $M \# S^n \times S^n \cong N \# S^n \times S^n$

implies a diffeomorphism $M \cong N$. For this you can follow Section 3 of [3], again specialising to the simply-connected case. To finish, you should apply these tools to some interesting example: you should discuss this with me, I can suggest several.

Relevant Courses

Essential: Part III Algebraic Topology

References

1. W. Lück, *A basic introduction to surgery theory*, `https://him-lueck.uni-bonn.de/data/ictp.pdf`
2. M. Kreck, *Surgery and duality*, Ann. of Math. (2) 149 (1999), no. 3, 707–754.
3. M. Kreck, *An extension of results of Browder, Novikov and Wall about surgery on compact manifolds*, `https://www.maths.ed.ac.uk/~v1ranick/papers/kreck0.pdf`

27. Symplectic Structures on Euclidean Space .... Professor I. Smith

There are various constructions of non-standard "exotic" symplectic structures on Euclidean space; some are constructed by hand, but the most interesting arise from contractible affine varieties. Their exotic nature is often tied up with the existence of interesting Lagrangian submanifolds and the behaviour of dynamical systems on the manifold, as probed by holomorphic curve counting invariants (Floer theory, symplectic cohomology); there are many open questions about how much of the underlying affine algebraic geometry is captured by symplectic topology. This essay will construct some examples, prove they are exotic (taking some input from holomorphic curve theory as a black box where necessary), and discuss open questions.

Relevant Courses

Essential: Algebraic Topology, Differential Geometry

Useful: Algebraic Geometry, Complex Manifolds, Symplectic Topology (graduate)

References

1. Paul Seidel and Ivan Smith. *The symplectic topology of Ramanujam's surface.* Comment. Math. Helv. 80 (2005), arXiv:math/0411601.
2. Paul Seidel. *Simple examples of distinct Liouville type symplectic structures.* J. Topol. Anal. 3 (2011), arXiv:1011.1415
3. M. Audin, F. Lalonde, L. Polterovich. *Symplectic rigidity: Lagrangian submanifolds.* Chapter X in "Holomorphic curves in symplectic geometry". Birkhäuser, 1994.
4. Mark McLean. *Computability and the growth rate of symplectic homology* arXiv:1109.4466.
5. Roger Casals. *Overtwisted discs and exotic symplectic structures.* ArXiv: 1402.7099.

28. Rational Homotopy Theory and Arithmeticity .... **Dr R. Stoll**

Homotopy theory is complicated. This is maybe best exemplified by the homotopy groups of spheres $\pi_k(\mathrm{S}^n)$, i.e. the maps $\mathrm{S}^k \to \mathrm{S}^n$ up to deformation. While we have tools to compute these abelian groups in many cases, understanding their general structure is far out of reach. However, by a seminal result of Serre (1951), almost all $\pi_k(\mathrm{S}^n)$ are finite; in particular the picture simplifies drastically when one ignores torsion, for example after *rationalization*, i.e. tensoring with $\mathbb{Q}$. *Rational homotopy theory* is, roughly speaking, the study of spaces after rationalizing their homotopy groups. As suggested by Serre's work, this ought to make things much easier. Indeed Quillen (1969) and Sullivan (1977) proved that the rational homotopy theory of simply connected spaces completely reduces to algebra, thereby enabling a vast amount of computations.

The first goal of this essay is to explain Sullivan's approach, which approximately yields an equivalence of categories between the rational homotopy category of simply connected spaces and the homotopy category of commutative differential graded algebras over $\mathbb{Q}$. Besides the original article [1], a textbook reference is [2].

The second goal is to explain an application to non-rational homotopy theory, namely the result of Wilkerson (1975) and Sullivan (1977) that the group of homotopy self-equivalences of a simply connected finite CW-complex is *arithmetic*. This is a strong structural result; for example an arithmetic group is finitely presented. The main reference is [1]. Working with arithmetic groups requires a basic understanding of algebraic groups; some prior exposure to algebraic geometry would thus be helpful.

Relevant Courses

Essential: Part III Algebraic Topology

Useful: Part III Algebraic Geometry

References

1. Dennis Sullivan. "Infinitesimal computations in topology". Publications mathématiques de l'IHÉS 47.1 (1977), pp. 269–331. doi: 10.1007/bf02684341.
2. Yves Félix, Stephen Halperin, and Jean-Claude Thomas. "Rational Homotopy Theory". Graduate Texts in Mathematics 205. Springer, 2012. doi: 10.1007/978-1-4613-0105-9.

29. Outer Space .... **Professor H. Wilton**

The action of a group Γ on itself by conjugation defines a natural map $\Gamma \to \mathrm{Aut}(\Gamma)$. Its image is the (normal) subgroup of *inner automorphisms*, and the quotient $\mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$ is called the *outer automorphism group* $\mathrm{Out}(\Gamma)$. When $\Gamma = F_n$, the non-abelian free group of rank n, the group $\mathrm{Out}(F_n)$ is especially complicated and interesting.

An important idea in geometric group theory is that one can study interesting groups by constructing nice spaces on which they act. For $\mathrm{Out}(F_n)$, Culler and Vogtmann [4] constructed a certain space of graphs, now known as 'Culler–Vogtmann Outer Space' and denoted by $\mathcal{CV}_n$. Topological properties of $\mathcal{CV}_n$ translate into group-theoretic properties of $\mathrm{Out}(F_n)$. For instance, Culler and Vogtmann showed that $\mathcal{CV}_n$ is contractible, from which it follows that $\mathrm{Out}(F_n)$ has finite cohomological dimension.

The idea of this essay is to describe the construction of $\mathcal{CV}_n$, to give a proof that it is contractible, and to deduce the corresponding results about $\mathrm{Out}(F_n)$. The original Culler–Vogtmann proof of contractibility is combinatorial, but more transparent geometric proofs were given by Hatcher and Skora — see [6], [3] or [5]. The required results about cohomological dimension can be found in [2]. An excellent essay might go on to describe more general deformation spaces (along the lines of [3] or [5]), or to prove the existence of train tracks [1].

Relevant Courses

Essential: Part II Algebraic topology
Useful: Part III Algebraic topology, Part III Geometric group theory

References

1. Mladen Bestvina. A Bers-like proof of the existence of train tracks for free group automorphisms. `arXiv:1001.0325`, 2010.
2. Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.
3. Matt Clay. Contractibility of deformation spaces of G-trees. *Algebr. Geom. Topol.*, 5:1481–1503 (electronic), 2005.
4. Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84(1):91–119, 1986.
5. Vincent Guirardel and Gilbert Levitt. Deformation spaces of trees. *Groups, Geometry, and Dynamics*. Volume 1, Issue 2, 2007, pp. 135–181.
6. Allen Hatcher. Homological stability for automorphism groups of free groups. *Comment. Math. Helv.*. Volume 70, Issue 1, 1995, pp. 39–62.

30. Synthetic Differential Geometry .... Professor P. T. Johnstone

Synthetic Differential Geometry began with the observation, made by F.W. Lawvere in a lecture in 1967 (published twelve years later in [1]), that it should be possible to develop the basic concepts of differential geometry axiomatically, in a suitable (cartesian closed) category which would contain, besides such objects as the 'classical' smooth manifolds — in particular the real line R, which would be an internal ring in the category — also such things as a 'disembodied tangent' D, which he identified equationally as $\{x \in R \mid (x^2 = 0)\}$, and which would have the property that the tangent bundle to a manifold M could simply be constructed as the exponential M^D. Lawvere's axiomatization was incompatible with classical logic; it required the development of the theory of classifying toposes in the next decade before models for it could be constructed. But the development was then rapid: the first book on the subject [2] was published only four years after the first models were found.

An essay on this topic could focus mainly on the axiomatics, which are well covered in the books [2] and [4]; more detailed information on the models is contained in [3], but in order to cover these it would be necessary to develop some of the theory of classifying toposes. The unpublished reference [5] will be made available to anyone expressing an interest in writing on this topic.

Relevant Courses

Essential: Category Theory

Useful: Differential Geometry

References

1. F.W. Lawvere, Categorical dynamics, in *Topos Theoretic Methods in Geometry*, Aarhus Univ. Var. Publ. Ser. 30 (1979), 1–20.
2. A. Kock, *Synthetic Differential Geometry*, LMS Lecture Note Series no. 51 (Cambridge U.P., 1981)
3. I. Moerdijk and G.E. Reyes, *Models for Smooth Infinitesimal Analysis* (Springer–Verlag, 1991).
4. R. Lavendhomme, *Basic Concepts of Synthetic Differential Geometry* (Kluwer, 1996).
5. P.T. Johnstone, Synthetic Differential Geometry (chapter F1 of *Sketches of an Elephant: a Topos Theory Compendium*) (not published yet, but copies will be made available to interested enquirers).

31. Infinite Games in Set Theory .... Professor B. Löwe

One of the most surprising developments in Foundations of Mathematics was that the theory of infinite games was revealed to be intimately and inextricably linked to the deep foundational questions of axiomatic set theory.

Infinite games had been used as a mathematical tool in topology and set theory since the 1920s. In the 1960s, Mycielski and Steinhaus had started the axiomatic investigation of determinacy axioms [5]; this was originally considered a niche and esoteric subfield of set theory since their main axiom, the *Axiom of Determinacy* AD contradicted the Axiom of Choice. Finally, in the 1980s, the famous Martin-Steel theorem established a very close connection between the large cardinal hierarchy and the hierarchy of axioms of determinacy [4].

The basic background for the theory of infinite games is presented, e.g., in [1, pp. 368–382]. The essay would build on that and explore one particular topic of the theory of infinite games.

Particular topics could be from the following list:

(i) *Determinacy of Borel games.* After the 1953 result by Gale and Stewart that all open and closed sets are determined, research in the 1950s and 1960s extended this result through the Borel hierarchy (results by Wolfe, Davis, and Paris); this development culminated in Martin's proof of Borel determinacy [3]. For lower levels of the Borel hierarchy, these results can be proved in weak metamathematical systems, but as one goes up in the Borel hierarchy, the strength of determinacy statements grows until a significant fragment of ZFC is needed.

(ii) *The Axiom of Determinacy.* The arcane world of AD where the Axiom of Choice is false has many curious properties that can be studied: all sets are Lebesgue measurable, there are no uncountable well-ordered sequences of real numbers, and small cardinals such as $\aleph_1$ and $\aleph_2$ have properties that they could not have under the Axiom of Choice. [1, § 28].

(iii) *The Wadge hierarchy.* Using infinite games, Wadge observed that there is a natural complexity hierarchy for sets of reals that forms a wellfounded *semi-linear order* if restricted to determined sets.

Relevant Courses

Essential: Part II *Logic & Set Theory* (or equivalent),

Useful: Part III *Logic & Computability*, Part III *Model Theory*, Part III *Forcing & the Continuum Hypothesis.* The lecture notes of past Part III lecture courses on *Infinite Games* could also be useful [2].

References

1. A. Kanamori. *The higher infinite. Large cardinals in set theory from their beginnings.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1994.

2. B. Löwe. Infinite Games, Part III course in Lent term 2021, University of Cambridge, website.

3. D. A. Martin. Borel determinacy. Ann. Math. 102 (1975), no. 2, 363–371.

4. D. A. Martin, J. R. Steel. A proof of projective determinacy. J. Amer. Math. Soc. 2 (1989), no. 1, 71–125.

5. J. Mycielski, H. Steinhaus. A mathematical axiom contradicting the axiom of choice. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 10 (1962), 1–3.

32. Generic and Virtual Large Cardinals ....

Professor B. Löwe

Generic large cardinals were introduced in Foreman's 1998 invited talk at the *International Congress of Mathematicians* in Berlin: a *generic large cardinal axiom* is the statement that a large cardinal exists in a generic extension [2]. These axioms are generally equiconsistent to the corresponding large cardinal axiom.

Virtual large cardinals were introduced by Gitman and Schindler as a generalisation of Schindler's notion of *remarkable cardinals* (equivalently, "virtually supercompact") [3]. Typically, the consistency strength of a virtual large cardinal is considerably lower than that of the corresponding large cardinal notion. Virtual versions of extendible, huge, and I3 cardinals are discussed in [3]; virtual versions of superstrong, Woodin, and Berkeley cardinals are discussed in [1].

An essay written under this title will give a general introduction into generic and virtual large cardinals and then analyse one particular large cardinal notion and its generic and virtual forms in detail.

Relevant Courses

Essential: Part II *Logic & Set Theory* (or equivalent),

Useful: Part III *Logic & Computability*, Part III *Model Theory*, Part III *Forcing & the Continuum Hypothesis*. The lecture notes of past Part III lecture courses on *Large Cardinals* could also be useful [4].

References

1. S. Dimopoulos, V. Gitman, D. Saattrup Nielsen. The virtual large cardinal hierarchy. Fund. Math. 266 (2024), no. 3, 237–262.
2. M. Foreman. Generic large cardinals: new axioms for mathematics. Doc. Math. 1998, Extra Vol. II, 11–21.
3. V. Gitman & R. Schindler. Virtual large cardinals. Ann. Pure Appl. Log. 169 (2018), no. 12, 1317–1334.
4. B. Löwe. Large Cardinals, Part III course in Lent term 2024, University of Cambridge, website.

33. Coalgebraic Logic and Automata .... Dr J. Siqueira

Algebras describe how to *construct* a new object of a given type from some existing ones and operations. Dually, coalgebras instruct how to *decompose* an object into new ones. A core example of such a coalgebra action is given by the transition function of a non-deterministic automaton on a set Q of states and set Σ of input symbols: a map $\delta\colon Q \to \wp(Q)^{\Sigma}$.

Viewing an arbitrary coalgebra action as a form of transition for a system allows one to uniformly study many types of automata and dynamical systems via coalgebraic methods; this makes the mathematical ideas in universal coalgebra relevant to theoretical computer science. At the same time, coalgebras capture the Kripke models that serve as semantics for modal logic. This Essay explores the coalgebraic point of view for automata and logic. We suggest the following possible directions:

- Describe the basic theory of coalgebraic automata and prove closure properties for the classes of languages they recognise, following [1];
- Discuss coalgebraic logics (as presented in [2]) and the connection to traditional modal logic via predicate liftings;
- Explore the connection between logic and automata theory facilitated by coalgebraic reasoning, as done in [3]. This involves introducing the coalgebraic fixed point logic associated to coalgebras and its game semantics, then showing that there is a sense in which formulas in the logical language are equivalent to coalgebraic automata.

A general reference for the theory of coalgebras is [4].

Relevant Courses

Essential: III Logic and Computability, II Automata & Formal Languages, II Logic & Set Theory

Useful: III Category Theory, III Model Theory

References

1. C. Kupke and Y. Venema, Closure Properties of Coalgebra Automata, Proceedings of the Twentieth IEEE Symposium on Logic in Computer Science (LICS 2005), pp. 199-208.
2. C. Kupke, D. Pattinson, Coalgebraic semantics of modal logics: An overview, Theoretical Computer Science, Volume 412, Issue 38, 2011, pp. 5070-5094.
3. Y. Venema, Automata and Fixed Point Logics for Coalgebras, Electronic Notes in Theoretical Computer Science Volume 106, 2004, pp. 355-375.
4. B. Jacobs, Introduction to Coalgebra. Towards Mathematics of States and Observations, Cambridge University Press 2016, https://doi.org/10.1017/CBO9781316823187.

34. Double Categories ....

Dr J. Siqueira

It is often the case that just having a notion of morphism between objects is not enough to capture all observable nuances of a mathematical structure — we need to consider 2-dimensional maps between morphisms too (e.g., inequalities between maps, or natural transformations between functors). One popular framework for this is given by 2-categories: categories *enriched* over categories, so that we have hom-*categories* rather than hom-sets.

While 2-categories (and bicategories, their weaker counterpart) are common and useful forms of 2-dimensional category theory, there is an alternative: we can consider categories (weakly) *internal* to categories, in much the same was as we can consider groups internal to topological spaces (i.e., topological groups). This amounts to having objects, two notions of morphism (which might not be mutually composable), and some well-behaved way to compare them. Such structures are called (weak) *double categories*. Typical examples arise from situations where a notion of map can be thought of as a kind of object itself (spans, profunctors, relations, etc).

I propose the following two possible directions for an Essay on double categories:

- An investigation of the theory of double categories — such an Essay could discuss double-categorical forms of the Yoneda Lemma (following [2] and [3]), or cartesian closed double categories (as described in [4]);
- Explain the ideas surrounding the "double-operadic theory of systems" of [5]: a double-categorical formalism for the abstract notion of "system", which is broad enough to include mathematical and real-world examples and encompass their forms of compositionality. This is a recent form of applied category theory related to the development of safeguarded AI.

Relevant Courses

Essential: III Category Theory

References

1. M. Grandis, Higher Dimensional Categories: From Double to Multiple Categories, World Scientific, 2019.

2. R. Paré, Yoneda Theory for Double Categories, Theory Appl. Categ. Vol. 25, No. 17, 2011, pp. 436–489.

3. B. Frölich, L. Moser, Yoneda Lemma and Representation Theorem for Double Categories, Theory Appl. Categ. Vol. 41, No. 49, 2024, pp. 1698–1782.

4. S. Niefield, Cartesian closed double categories, Theory Appl. Categ. Vol. 40, No. 3, 2024, pp. 63–79.

5. D. J. Myers and S. Libkind, Towards a double operadic theory of systems, 2025, arXiv.2505.18329.

35. The Grunwald-Wang Theorem .... Professor T. A. Fisher

A natural local-to-global question in number theory is to ask whether an element of a number field k that is locally (i.e. in the completion k_v) an nth power for all but finitely many places v, must be an nth power in k. In general the answer is "no" (there are counter-examples with $k = \mathbf{Q}$ and $n = 8$) but under some extra hypotheses (which are always satisfied for example if n is odd) the answer is "yes". The Grunwald-Wang theorem itself is a closely related local-to-global question about the existence of cyclic extensions of k (i.e. Galois extensions with cyclic Galois group). The essay should include sufficient background material from class field theory and group cohomology to explain the proofs. It could also discuss how the above problem (concerning divisibility in the multiplicative group) extends to other commutative algebraic groups [3], for example elliptic curves.

Relevant Courses

Essential: Local Fields

Useful: Elliptic Curves

References

1. E. Artin and J. Tate, *Class field theory*, (originally published 1967), AMS Chelsea Publishing, Providence, RI, 2009.

2. J.W.S. Cassels and A. Fröhlich (eds.), *Algebraic number theory*, (originally published 1967), LMS, 2010.

3. R. Dvornicich and U. Zannier, Local-global divisibility of rational points in some commutative algebraic groups, *Bull. Soc. Math. France* 129 (2001), no. 3, 317–338.

4. J. Neukirch, *Algebraic number theory*, Springer-Verlag, Berlin, 1999.

5. J. Neukirch, A. Schmidt and K. Wingberg, *Cohomology of number fields*, Springer-Verlag, Berlin, 2000.

36. Classical Invariant Theory and Moduli of Genus 2 Curves .... Professor T. A. Fisher

This essay should begin by reviewing the classical (i.e. 19th century) invariant theory of binary forms of degree n (see for example [3] or [6]) with particular reference to the cases $n = 4$ and $n = 6$. The case $n = 4$ is related to elliptic curves (see [2] or [7]), which are classified up to isomorphism (over an algebraically closed field) by their j-invariant. The case $n = 6$ leads to the definition of the Igusa (or Igusa-Clebsch) invariants that likewise classify genus 2 curves. The main aim of the essay should be to describe the algorithm of Mestre [5] for recovering the equation for a genus 2 curve from its Igusa invariants. If time and space permit, the connection to Siegel modular forms, or extensions such as those in [1], could also be considered.

Relevant Courses

Useful: Elliptic Curves, Algebraic Geometry

References

1. G. Cardona and J. Quer, Field of moduli and field of definition for curves of genus 2, in *Computational aspects of algebraic curves*, World Scientific Publishing Co., 2005.
2. J.E. Cremona, Classical invariants and 2-descent on elliptic curves, *J. Symbolic Comput.* 31 (2001), no. 1-2, 71–87.
3. D. Hilbert, *Theory of algebraic invariants*, CUP, 1993.
4. J. Igusa, Arithmetic variety of moduli for genus two, *Ann. of Math.* (2) 72, (1960), 612–649.
5. J.-F. Mestre, Construction de courbes de genre 2 à partir de leurs modules, in *Effective methods in algebraic geometry*, Birkhäuser, 1991.
6. P.J. Olver, *Classical invariant theory*, CUP, 1999.
7. A. Weil, Remarques sur un mémoire d'Hermite, *Arch. Math.* (Basel) 5, (1954), 197–202.

37. Heuristics for Class Groups and the Negative Pell Equation .... Dr A. J. Morgan

The Cohen–Lenstra heuristics, formulated in 1983 [1], predict, roughly speaking, that class groups of number fields behave like random finite abelian groups. To the present day, refining and proving instances of these heuristics is an active area of research.

While the original heuristics give precise predictions for the behaviour of the p-Sylow subgroups of class groups of quadratic fields for odd primes p, Gauss's genus theory necessitates an adaptation of the heuristics for the 2-Sylow subgroup. Such an adaptation was soon put forward by Gerth [3], and the resulting conjecture is now a (very recent) theorem of Smith [5].

Building on the heuristics of Cohen–Lenstra and Gerth, Stevenhagen [6] conjectured that the density of integers $d \in \mathcal{D}$ for which the *negative Pell equation*

$$x^2 - dy^2 = -1 \tag{1}$$

is soluble in integers x and y is given by an explicit irrational number whose value is approximately 0.581. Here $\mathcal{D}$ is the set of positive squarefree integers all of whose odd prime divisors are congruent to 1 modulo 4; the condition $d \in \mathcal{D}$ is a basic necessary condition for equation (1) to have a solution. Following breakthrough work of Fouvry–Klüners [2], Stevenhagen's conjecture was proven by Koymans–Pagano in 2022 [4].

The essay should begin with a discussion of the Cohen–Lenstra heuristics, including a precise formulation for quadratic fields, and explain why, for the 2-part of the class group of such fields, the heuristics need modifying in order to incorporate Gauss's genus theory. The essay should then discuss Stevenhagen's conjecture on the solubility of the negative Pell equation, before detailing some results in the direction of its resolution, such as the proof by Fouvry and Klüners [2] that equation (1) is soluble for a least 41% of $d \in \mathcal{D}$.

Relevant Courses

Essential: A first course in algebraic number theory, at the level of Part II Number Fields.

Useful: Local Fields and, to a lesser extent, Analytic Number Theory.

References

1. H. Cohen and H.W. Lenstra, *Heuristics on class groups of number fields*, In Number theory Noordwijkerhout, Springer (1983), 33-62.

2. É. Fouvry and J. Klüners, *On the negative Pell equation*, Ann. Math. 172 (2010), 2035-2104.

3. F. Gerth. *The 4-class ranks of quadratic fields*, Invent. Math. 77 (1984), 498-515.

4. P. Koymans and C. Pagano, *On Stevenhagen's conjecture*, Acta. Math., to appear.

5. A. Smith, *The distribution of l^∞-Selmer groups in degree l twist families I*, J. Amer. Math. Soc., to appear,

6. P. Stevenhagen, *The number of real quadratic fields having units of negative norm*, Experiment. Math. 2 (1993), 121-136.

38. Sarnak's Conjecture on Möbius Disjointness ....

Dr J. Teräväinen

A sequence $(a(n))_{n\in\mathbb{N}}$ taking values ± 1 is said to be of low complexity if the number of different length k sign patterns $(a(n+1), \dots, a(n+k))$ that appear in the sequence grows slower than exponentially in k. This notion of low complexity can also be extended to complex-valued sequences using dynamical language.

Sarnak's conjecture states that the Möbius function should be orthogonal to all low complexity sequences $(a(n))_{n\in\mathbb{N}}$, in the sense that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} \mu(n)a(n) = 0.$$

Despite a lot of progress in recent years, this conjecture remains open.

The purpose of this essay is to give an overview of Sarnak's conjecture and discuss some classical results at the intersection of analytic number theory and ergodic theory that relate to this conjecture.

Some possible results to consider include:

1. The Bourgain–Sarnak–Ziegler orthogonality criterion and its applications.
2. Möbius orthogonality of the Thue–Morse sequence.
3. The connection between the Sarnak and Chowla conjectures.

Relevant Courses

Part III Analytic Number Theory.

References

1. T. Tao. The Chowla and Sarnak conjectures. https://terrytao.wordpress.com/2012/10/14/the-chowla-conjecture-and-the-sarnak-conjecture/
2. P. Sarnak. Three lectures on the Möbius function randomness and dynamics. Technical report. Institute for Advanced Study. https://publications.ias.edu/sarnak/works
3. S. Ferenczi, J. Kułaga-Przymus, M. Lemańczyk. Sarnak's Conjecture: What's New. In: Ergodic Theory and Dynamical Systems in their Interactions with Arithmetics and Combinatorics. Lecture Notes in Mathematics, vol 2213. Springer, Cham.
4. J. Bourgain, P. Sarnak, T. Ziegler. Disjointness of Mobius from horocycle flows. In: From Fourier Analysis and Number Theory to Radon Transforms and Geometry. Developments in Mathematics, vol 28. Springer, New York, NY.

39. Abelian Varieties over Finite Fields .... Dr R. Zhou

Abelian varieties are higher dimensional generalizations of elliptic curves and are fundamental objects of study in arithmetic geometry. A theorem of Weil tells us that the action of Frobenius on the ℓ-adic Tate module of a simple abelian A variety over $\mathbb{F}_q$ is given by a Weil q-integer γ_A. Remarkably, Honda and Tate showed that the association $A \mapsto \gamma_A$ gives a bijection between isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and conjugacy classes of Weil q-integers.

The aim of this essay is to understand the proof of the result of Honda and Tate. The essay can begin with some recollections on abelian varieties, and should cover the ingredients needed in the proof such as Tate's isogeny theorem and the theory of CM abelian varieties including the Shimura–Taniyama reciprocity law.

Time permitting, possible additional topics to be covered can include:

- Manin's Problem.
- The Serre–Tate theorem and canonical lifts of ordinary abelian varieties.
- (For the very ambitious) CM liftings on Shimura varieties.

Relevant Courses

Essential: Algebraic Geometry, Local Fields.

Useful: Elliptic Curves

References

1. J. Tate, Endomorphisms of abelian varieties over finite fields, *Inventiones Mathematicae*, 2 (1966), pp. 134-144

2. J. Tate, Classes d'isogénie des variétés abéliennes sur un cops fini, *Séminaire Bourbaki*, vol. 1968/69, pp. 347-363,

3. T. Honda, Isogeny classes of abelian varieties over finite fields, *Journ. Math. Soc. Japan*, 20, 1968, pp. 83-95.

4. J-P. Serre and J. Tate, Good reduction of abelian varieties, *Annals of Mathematics*, Second Series, Vol. 88, No. 3 (Nov., 1968), pp. 492-517

40. Singular Moduli ....

Dr R. Zhou

Singular moduli are the values taken by the modular j-invariant at imaginary quadratic numbers in the upper half plane, and are intimately related to the class field theory of imaginary quadratic fields via the theory of complex multiplication (CM). In their paper *On singular moduli*, Gross and Zagier discovered a remarkable factorization property for the norm of the difference of two singular moduli, and gave two proofs of this result, an algebraic proof and an analytic proof. The computations contained in this paper directly motivated their later breakthrough work relating heights of Heegner points to derivatives of L-series.

The aim of the essay is to give an exposition of the proof of this factorization property. The original paper of Gross–Zagier is probably still the best source to use, but will require you to learn some background on modular and elliptic curves, including reductions of elliptic curves and some CM theory. The essay should begin with some of this background, including a proof of the algebraicity of the j-invariant evaluated at imaginary quadratic numbers. One can then move on to the main part of the essay, where you should cover the proof. You could focus more on the algebraic proof or the analytic proof, or both, depending on your taste.

Relevant Courses

Useful: Local Fields, Elliptic Curves

References

1. B. Gross and D. Zagier, On singular moduli *J. Reine Angew. Math.* (1984).

2. B. Gross and D. Zagier, Heegner points and derivatives of L-series, *Inventiones Mathematicae* (1986).

3. J. Silverman, Advanced Topics in the Arithmetic of Elliptic Curves, *Graduate Texts in Mathematics*, 151.

41. The Gap–Hamming Problem in Communication Complexity .... Professor V. S. Jog

The *Gap-Hamming problem* is a central question in randomised communication complexity. Two parties, each holding an n-bit string, must determine whether the Hamming distance between their strings is greater than $n/2 + \sqrt{n}$ or smaller than $n/2 - \sqrt{n}$, given a promise that one of these cases holds. Naturally, the problem can be trivially solved by communicating $n+1$ bits–one party simply sends their string to the other party, which then declares the answer. Whether it can be solved using significantly fewer than n bits; i.e. $o(n)$ bits, was a key open problem in communication complexity for almost a decade. This was eventually resolved in Chakrabarti and Regev [1] where an $\Omega(n)$ lower bound was proved.

This essay will provide a brief and self-contained introduction to communication complexity and subsequently explain the origin and significance of the Gap–Hamming problem. The core of the essay will focus on understanding and explaining two proofs of the $\Omega(n)$ lower bound for the Gap–Hamming problem: the first, due to Sherstov [3] (which simplifies the proofs of Vidick [2] and Chakrabarti and Regev [1]), uses Talagrand's concentration inequality to establish anti-concentration results; the second, due to Hadar, Liu, Polyanskiy, and Shayevitz [4], relies on strong data-processing inequalities from information theory. Two reference textbooks that will be useful are Kushilevitz and Nisan [5] and Rao and Yehudayoff [6].

Relevant Courses

Essential: Information Theory, Concentration Inequalities

References

1. Amit Chakrabarti, and Oded Regev. "An optimal lower bound on the communication complexity of Gap–Hamming-distance." In Proceedings of the Forty-Third Annual ACM Symposium on Theory of Computing, pp. 51-60. 2011.

2. Thomas Vidick. "A concentration inequality for the overlap of a vector on a large set." Chicago Journal of Theoretical Computer Science 1 (2012): 1-12.

3. Alexander A. Sherstov. "The communication complexity of Gap–Hamming distance." Theory of Computing 8, no. 1 (2012): 197-208.

4. Uri Hadar, Jingbo Liu, Yury Polyanskiy, and Ofer Shayevitz. "Communication complexity of estimating correlations." In Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing, pp. 792-803. 2019.

5. Eyal Kushilevitz and Noam Nisan. "Communication Complexity." (2006).

6. Anup Rao and Amir Yehudayoff. "Communication Complexity and Applications." Cambridge University Press, 2020.

42. Universal Portfolios .... Professor I. Kontoyiannis

This essay will explore and describe the connections between information-theoretic tools and the theory of *universal portfolios* in finance. This is a collection of ideas, mathematical techniques, and algorithms for selecting an optimal portfolio for investment in general discrete-time markets. Various notions of optimality are considered, and the methods introduced lead to very interesting probabilistic analyses. The essay is expected to describe the basic theory as outlined in Chapter 16 of the textbook [1] and developed in the first papers [2–5]. More ambitious essays will explore the more recent literature, in consultation with the essay setter.

Relevant Courses

Essential: Basic probability

Useful: Probability and measure, Coding and cryptography

References

1. Cover, T.M. and Thomas, J.A. *Elements of information theory*, second edition, J. Wiley & Sons, New York, 2012.
2. Bell, R.M., and Cover, T.M. "Competitive optimality of logarithmic investment." *Mathematics of Operations Research*, **5**, no. 2, pp. 161-166, May 1980.
3. Cover, T.M. "An algorithm for maximizing expected log investment return." *IEEE Transactions on Information Theory*, **30**, no. 2, pp. 369-373, 1984.
4. Cover, T.M., and Gluss, D.H. "Empirical Bayes stock market portfolios." *Advances in Applied Mathematics*, **7**, no 2 pp. 170-181, 1986.
5. Bell, R., and Cover, T.M. "Game-theoretic optimal portfolios." *Management Science*, **34**, no. 6, pp. 724-733, 1988.

43. Measures of Risk .... Professor M. R. Tehranchi

A financial company is exposed to the risk of becoming insolvent due to the fluctuating value of its assets and liabilities. Regulatory agencies, therefore, require that such companies keep a certain amount of cash on hand to partially offset this risk. Starting with the influential paper [1] of Artzner et al, there has been an increasing amount of work on developing an axiomatic theory of capital requirements and measures of risk. Early results on risk measures include the following representation theorem of Delbaen [3]. There has been recent interest in robust measures of risk that account for model uncertainty.

The object of this essay is to survey the current literature on measures of risk, and in particular its connection to decision theory.

Relevant Courses

Useful: Advanced Probability, Stochastic Calculus and Applications to Finance

References

1. P. Artzner, F. Delbaen, J.M. Eber, D. Heath. (1999) Coherent measures of risk. Mathematical Finance 9, 203-228.

2. M. Cambou, D. Filipović. (2017) Model uncertainty and scenario aggregation. Mathematical Finance 27, 534-567.

3. F. Delbaen. (2002) Coherent measures of risk on general probability spaces. In Advances in Finance and Stochastics, K. Sandmann and P. J. Schönbucher, eds. Berlin: Springer-Verlag.

4. T. Fadina, Y. Liu, R. Wang. (2024) A framework for measures of risk under uncertainty. Finance and Stochastics 28, 363-390

44. Random Walks on (Random) Planar Graphs .... Dr E. Kammerer

Let G be an infinite (possibly random) planar graph. Is the simple random walk on G transient or recurrent?

Several approaches have been developed to solve this question on different graphs. The most successful one uses circle packings. The main focus of the essay will be the He-Schramm theorem and the recurrence of bounded degree planar graph limits.

If time permits, depending on the interest, one could apply these results to random planar maps or explore a different technique to study random walks on random planar maps: the peeling exploration.

Relevant Courses

Essential: Random discrete structures.

Main reference

1. A. Nachmias, *Planar maps, random walks and circle packing.* École d'Été de Probabilités de Saint-Flour XLVIII – 2018. Cham: Springer (2020; Zbl 1471.60007)– also available on https://arxiv.org/pdf/1812.11224

Additional reference

1. N. Curien, *Peeling random planar maps.* École d'Été de Probabilités de Saint-Flour XLIX – 2019. Cham: Springer (2023; Zbl 1540.60001)– also available on https://www.imo.universite-paris-saclay.fr/∼nicolas.curien/enseignement.html

45. The Gaussian Free Field (in the Continuum) .... Dr E. Kammerer

The Gaussian free field (GFF) in the continuum can be seen as a generalisation of Brownian motion in dimension larger than one. It is a Gaussian process with mean zero and covariance given by the Green function. But, due to the behaviour of the Green function on the neighbourhood of the diagonal, it is a random distribution in the sense of Schwartz.

A successful essay will present different constructions and approximations of the GFF, as well as some of its properties such as its regularity, the Markov property and conformal invariance in dimension two. If time allows, one could explore the question of characterising the GFF by such properties.

Relevant Courses

Essential: Advanced probability

Useful: Stochastic calculus with applications to finance, Random walks and phase transitions, Gaussian processes.

Main reference

1. *Gaussian free field and Liouville quantum gravity* (to appear). Cambridge Studies in Advanced Mathematics 220. Cambridge University Press (ISBN 978-1-00-940550-8/hbk). (2025).– also available on https://arxiv.org/pdf/2404.16642

Additional references

1. W. Werner and E. Powell, *Lecture notes on the Gaussian free field.* Paris: Société Mathématique de France (SMF) (2021; Zbl 1519.60001)– also available on

   https://arxiv.org/abs/2004.04720

2. S. Sheffield, Gaussian free fields for mathematicians. *Probab. Theory Relat. Fields* 139, No. 3–4, 521–541 (2007; Zbl 1132.60072)– also available on

   https://arxiv.org/pdf/math/0312099

3. N. Berestycki, E. Powell and G. Ray, A characterisation of the Gaussian free field. *Probab. Theory Relat. Fields* 176, No. 3–4, 1259–1301 (2020; Zbl 1434.60271)– also available on https://arxiv.org/pdf/1802.01195

4. J. Aru and E. Powell, A characterisation of the continuum Gaussian free field in arbitrary dimensions. J. Éc. Polytech., Math. 9, 1101–1120 (2022; Zbl 1502.60038)– also available on https://arxiv.org/pdf/2103.07273

46. Plaquette Percolation .... Dr F. R. Klausen

Consider the d-dimensional hypercubic lattice and look at Bernoulli-1/2-percolation of k-dimensional cubical simplices. Say that two simplices touch if they share a common p-dimensional face. For example, two plaquettes in $\mathbb{Z}^2$ can share either a point or an edge. For which tuples of positive integers (p, k, d) satisfying $p < k \leq d$ does this process have an infinite component? Take the boundary map on the set of open simplices, for which (p, k, d) will the resulting set of open edges percolate?

A successful essay will contain background on the problem, give proofs of as many cases as possible, and discuss potential generalizations.

Relevant Courses

Useful: Advanced Probability, Random Discrete Structures

References

1. Hansen, Ulrik Thinggaard, Boris Kjær, and Frederik Ravn Klausen. "The Uniform Even Subgraph and Its Connection to Phase Transitions of Graphical Representations of the Ising Model." Communications in Mathematical Physics 406.6 (2025).

2. Hansen, Ulrik Thinggaard, Jianping Jiang, and Frederik Ravn Klausen. "A General Coupling for Ising Models and Beyond." arXiv preprint arXiv:2506.10765 (2025).

47. The Coupling of SLE_4 and the Gaussian Free Field .... Professor J. Miller

The Schramm-Loewner evolution (SLE) is a one parameter ($\kappa > 0$) family of conformally invariant random curves in the upper half-plane $\mathbf{H}$ connecting 0 and ∞. The parameter κ determines the roughness of an SLE curve:

- for $\kappa = 0$, it is a straight line,
- for $\kappa \in (0, 4]$ it is a simple curve,
- for $\kappa \in (4, 8)$ it has self-intersections, and
- for $\kappa \geq 8$ it is space-filling.

The SLE_κ curves are important because they describe the scaling limit of many discrete models from probability theory in two dimensions, in the same way that planar Brownian motion describes the scaling limit of a simple random walk on $\mathbf{Z}^2$. For example, SLE_2, SLE_6, and SLE_8 respectively are the scaling limit of loop-erased random walk, the interfaces in critical percolation, and the uniform spanning tree Peano curve. The purpose of this essay is to focus on SLE_4 and its relationship with the Gaussian free field (GFF). The GFF is a random field on a planar domain which turns out to describe the fluctuations of many different types of random surface models and also arises in random matrices. It was discovered by Schramm and Sheffield that it is possible to define its level sets and they are given by SLE_4 curves and this now underpins a large part of the subject. A successful essay will briefly review the definition and basic properties of SLE, the GFF, describe the level set construction in detail, and then briefly describe the relationship between SLE and the GFF for other values of κ.

Relevant Courses

Essential: Advanced Probability, Stochastic Calculus

Useful: Random Discrete Structures

References

1. Schramm, Oded; Sheffield, Scott. A contour line of the continuum Gaussian free field. Probab. Theory Related Fields 157 (2013), no. 1-2, 47–80.

2. Schramm, Oded; Sheffield, Scott. Contour lines of the two-dimensional discrete Gaussian free field. Acta Math. 202 (2009), no. 1, 21–137.

3. Sheffield, Scott. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. Ann. Probab. 44 (2016), no. 5, 3474–3545.

4. Miller, Jason; Sheffield, Scott. Imaginary geometry I: interacting SLEs. Probab. Theory Related Fields 164 (2016), no. 3-4, 553–705.

48. Percolation on Random Planar Maps .... Professor J. Miller

A planar map is a finite graph together with an embedding in the plane, defined up to continuous deformation. The faces of a planar map are the connected components of the complement of its edges. A planar map is called a triangulation if each face has exactly three adjacent edges (including the unbounded face). Since there are only a finite number of triangulations with n faces, one can talk about picking one uniformly at random and this is what is known as a random planar map. The study of large uniformly random triangulations have been the focus of a considerable amount of literature in recent years. It is also natural to consider random quadrangulations, pentagulations, or maps with mixed face sizes.

One of the most famous models in statistical mechanics is percolation, introduced in the mathematics literature by Broadbent and Hammersley (1957). It is defined by starting with a graph (G, V) and then retaining edges at random independently with some probability p. One is then interested in the connectivity properties of the resulting random graph. Percolation is very interesting when the underlying graph is a random planar map as its large scale geometric structure can be related to random walks with i.i.d. increments and, by some very deep mathematics, to percolation on deterministic planar lattices.

A successful essay will review the computation of percolation thresholds on random planar maps and discuss more recent developments on scaling limits for percolation.

Relevant Courses

Essential: Advanced Probability

Useful: Random Discrete Structures

References

1. O. Angel and O. Schramm (2003). Uniform infinite planar triangulations. Comm. Math. Phys. 241, no. 2-3, 191?213.

2. O. Angel (2003). Growth and percolation on the uniform infinite planar triangulation. Geom. Funct. Anal. 13, no. 5, 935?974.

3. O. Angel and N. Curien. Percolations on random planar maps I: Half-plane models. Ann. Inst. Henri Poincar?e Probab. Stat. 51 (2015), no. 4, 405?431.

4. M. Laurent and P. Nolin. Percolation on uniform infinite planar maps. Electron. J. Probab. 19, no. 79.

5. N. Curien and I. Kortchemski. Percolation on random triangulations and stable looptrees. Probab. Theory and Related Fields 163, no. 1?2, 303?337.

6. Gwynne, Ewain; Miller, Jason. Percolation on uniform quadrangulations and SLE_6 on $\sqrt{8/3}$-Liouville quantum gravity. Asterisque No. 429, (2021)

49. Malliavin Calculus and Wiener Chaos .... Professor J. R. Norris

The set of square-integrable measurable functions of an m-dimensional Brownian motion $B = (B_t : t \in [0,1])$ has an orthogonal decomposition known as the Wiener chaos. Given any stochastic differential equation driven by B, say in d dimensions, with polynomial coefficients

$$dX_t = p(X_t)dB_t, \quad X_0 = x$$

we can write a formal Taylor expansion

$$X_t = \sum_{n=0}^{\infty} X_t^{(n)}$$

where

$$X_t^{(n)} = \frac{1}{n!} \frac{\partial^n}{\partial \varepsilon^n}\bigg|_{\varepsilon=0} X_t^\varepsilon, \quad dX_t^\varepsilon = \varepsilon p(X_t^\varepsilon)dB_t, \quad X_0^\varepsilon = x.$$

This essay will give an introductory account of the Wiener chaos, and of such stochastic Taylor expansions, at least in the nilpotent case where all but finitely many terms are zero. It will then give a careful account of Malliavin calculus as it applies to such Taylor expansions, leading to criteria for the solution X_t to have a smooth density at all positive times t.

While some credit will be given for a well-constructed general account, a significant part of the assessed credit will be reserved for material specific to the case of nilpotent stochastic Taylor expansions, where there are simplifications of the more general notions.

Relevant Courses

Useful: Probability and Measure, Advanced Probability, Stochastic Calculus and Applications to Finance

References

1. M. Hairer, Introduction to Malliavin calculus. Lecture notes online at hairer.org/notes/Malliavin

2. D. Nualart and E. Nualart, *Introduction to Malliavin calculus*, Institute of Mathematical Statistics Textbooks, 9, Cambridge Univ. Press, Cambridge, 2018

50. Scaling Limits for Markov Processes .... Professor J. R. Norris

When a random process has a Markov structure, this allows an economical description of its dynamics. This structure also provides tools to handle limit operations, such as are of interest in stochastic models for large populations or in large random structures.

This essay is intended to offer a flexible opportunity to write an account describing some aspect of the theory and applications of scaling limits for Markov processes.

Part of the essay can present an account of a selection of methods available to establish scaling limits, where the limit objects are either differential equations, or diffusion processes, or continuous-time Markov chains, or some combination of these.

Beyond the simplest case, there are techniques to handle local averaging over inhomogeneous local structure, or variables which oscillate rapidly. It is possible also in some cases to show convergence with limit dynamics in infinite-dimensions or to obtain uniform-in-time convergence, under suitable stability hypotheses.

The methods must be illustrated by a careful exposition of specific examples, to be agreed in consultation with me. A significant part of the assessed credit will be assigned on the basis of this part of the essay.

The examples could have a coherent theme, such as models from mathematical biology, or large random combinatorial structures, or physical particle models.

Relevant Courses

Useful: Markov Chains, Applied Probability, Probability and Measure, Advanced Probability, Stochastic Calculus and Applications to Finance

References

1. R. W. R. Darling and J. R. Norris, Differential equation approximations for Markov chains. Probab. Surv. 5 (2008), 37-79.
2. S. N. Ethier and T. G. Kurtz, Markov Processes: characterization and convergence, Wiley (1986).

51. Long Cycles in the Interchange Process .... Professor P.-F. Rodriguez

A longstanding open problem is to rigorously prove the existence of a phase transition in the quantum Heisenberg ferromagnet in all dimensions $d \geq 3$. This conjecture is related to the following problem. Suppose distinguished particles are placed on the vertices of the lattice $\mathbb{Z}^d$, with independent Poisson clocks on each of the edges. When the clock of an edge rings, the two particles on the two sides of the edge interchange. In this way, a random permutation of $\mathbb{Z}^d$ is formed for any time $\beta > 0$ (which corresponds to the inverse temperature in the Heisenberg model). One of the main objects of study is the cycle structure of the random permutation and the emergence of long cycles for large β, as conjectured by Tóth [1]. The latter was recently proved by Elboim-Sly [2] in dimensions $d \geq 5$.

The purpose of the essay is to study the proof of [2]. A good essay will give a convincing account of the underlying multi-scale argument, by explaining clearly how a certain set of inductive assumptions is bootstrapped from one scale to the next.

Relevant Courses

Essential: Advanced Probability

Useful: Stochastic Calculus

References

1. B. Tóth. Improved lower bound on the thermodynamic pressure of the spin-1/2 Heisenberg ferromagnet. Letters in mathematical physics, 28(1):75–84, 1993.
2. D. Elboim and A. Sly. Infinite cycles in the interchange process in five dimensions. Preprint, arXiv:2211.17023, 2022.

52. Heat Kernel Bounds .... Professor P.-F. Rodriguez

One theme in probability is to derive bounds on the transition density of the random walk, the so-called heat kernel, from geometric information of the underlying graph G. The purpose of the essay is to discuss various techniques for obtaining upper bounds on the kernel, starting from a stand-alone result of Carne and Varopoulos. The essay will also highlight the role of a certain class of Nash-type inequalities, and their use in deriving both diagonal and off-diagonal upper bounds. The main reference for the essay will the monograph [1] by Barlow, with the aim of understanding parts of Chapters 4–6 therein.

Relevant Courses

Essential: Advanced Probability

Useful: Random Discrete Structures, Random walks and phase transitions

References

1. M. Barlow. Random Walks and Heat Kernels on Graphs. London Mathematical Society Lecture Note Series: 438, Cambridge University Press, 2017.

53. Long-Range Percolation .... Dr A. Sarkovic and Professor P. Sousi

In long-range percolation two distinct vertices x, y of $\mathbb{Z}^d$ are connected by an edge with probability $1 - e^{-\beta J(x,y)}$, where $J : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ is a symmetric function satisfying $J(x, y) \sim \|x - y\|^{-d-\alpha}$ as $\|x - y\| \to \infty$ for $\alpha > 0$ and $\beta \geq 0$. The critical point β_c is defined to be

$$\beta_c = \sup\{\beta \geq 0 : \mathbb{P}_\beta(\text{there exists an infinite cluster}) = 0\}.$$

This model undergoes different behaviour at criticality depending on d and α. This essay should focus on the recent breakthrough developments [1, 2, 3] due to Tom Hutchcroft. In several regimes for d and α he obtained precise tail estimates at criticality for the size of the cluster of 0.

A successful essay should contain an exposition on the different qualitative regimes in terms of d and α and a proof of the tail estimates on the cluster size in one of the regimes.

Relevant Courses

Useful: Random discrete structures

References

1. T. Hutchcroft (2025). Critical long-range percolation I: High effective dimension. *available at* arXiv:2508.18807

2. T. Hutchcroft (2025). Critical long-range percolation II: Low effective dimension. *available at* arXiv:2508.18808

3. T. Hutchcroft (2025). Critical long-range percolation III: The upper critical dimension. *available at* arXiv:2508.18809

4. T. Hutchcroft (2025). Dimension dependence of critical phenomena in long-range percolation. *available at* arXiv:2510.03951

5. Hadamard lectures by Tom Hutchcroft (2025). https://www.youtube.com/playlist?list=PLbq-TeAWSXhPQt8MA9_GAdNgtEbvswsfS

54. Improved Cheeger's Inequality ....

Dr A. Sarkovic and Professor P. Sousi

We consider a Markov chain with transition matrix P and invariant distribution π on a finite state space S. The conductance of a set $A \subset S$ is defined to be

$$\Phi(A) = \frac{1}{\pi(A)} \cdot \sum_{x \in A, y \in A^c} P(x,y)\pi(x)$$

and the Cheeger constant (also known as the bottleneck ratio) is defined via

$$\Phi^* = \min_{A:\pi(A) \leq \frac{1}{2}} \Phi(A).$$

If γ is the second smallest eigenvalue of the matrix $I - P$, then the well-known Cheeger's inequality states that for any time reversible chain $(\Phi^*)^2/2 \leq \gamma \leq 2\Phi^*$. Using a generalisation of Cheeger's constant, in which for each k a minimisation is taken over all partitions of S in k sets, [1] establishes a bound on the Cheeger constant in terms of the k-th smallest eigenvalue of $I - P$. Furthermore, in [1] using higher order eigenvalues, the authors improve on spectral algorithms for certain graph partitioning problems. In a closely related work [2], the authors present an algorithm that partitions a graph into components such that on one hand, there are few edges between different components, and on the other hand, each component has a sufficiently large Cheeger constant. In [3], a simpler and more combinatorial proof is provided for the same result in a particular class of graphs.

A successful essay should present the proof of the improvement on Cheeger's inequality and focus on some of its applications or investigate the sharpness of these results.

Relevant Courses

Useful: Mixing Times of Markov Chains

References

1. Tsz Chiu Kwok, Lap Chi Lau, Yin Tat Lee, Shayan Oveis Gharan, Luca Trevisan (2013). Improved Cheeger's Inequality: Analysis of Spectral Partitioning Algorithms through Higher Order Spectral Gap *available at* https://cs.uwaterloo.ca/~lapchi/papers/Cheeger.pdf

2. Shayan Oveis Gharan, Luca Trevisan (2013). Partitioning into expanders *available at* arXiv:1309.3223

3. Federico Vigolo (2021). Remarks on partition into expanders Critical long-range percolation III: The upper critical dimension. *available at* arXiv:2001.01522

55. The Loewner Energy of a Planar Curve .... Professor W. Werner

Consider a simple smooth curve joining two boundary points a and b of a simply connected planar domain. The Loewner energy of this curve is one conformally invariant way to measure how "far" it is from the hyperbolic geodesic between these points. It is defined in a very "directional" way from a to b, as the Dirichlet energy $\int_0^\infty \eta'(t)^2 dt$ of the driving function η that is involved when one codes the curve via its chordal "Loewner chain".

The goal of this essay is to follow [1] to first understand this definition (i.e., the definition, parametrization of Loewner chains – this is purely complex analysis, with conformal maps – background on Loewner chains can be looked up in [2] for instance), and then to use large deviation estimates (so this involves some probability theory, and large deviation principles for one-dimensional Brownian motion) in order to prove the fact that the energy defined from a to b is the same as the energy defined from b to a.

The content is mix of (classical) complex analysis with some probability.

Relevant Courses

Essential: solid Complex Analysis background, Advanced Probability

Main references

1. Yilin Wang, The energy of a deterministic Loewner chain: Reversibility and interpretation via SLE(0+), J. Eur. Math. Soc. 21, 1915-194, 2019 –
also available at https://arxiv.org/abs/1601.05297

2. Walter Hayman: Multivalent functions. Cambridge University Press, 1994

56. Variational Inference for Hierarchical Bayesian Models .... Dr M. Autenrieth, Professor K. Mandel

Hierarchical Bayesian models are widely used across the sciences to describe structured data, share information between related populations, and capture both population-level and individual-level variation. They are particularly powerful in complex scientific domains such as astrophysics, biology, and the social sciences, where data often exhibit nested or multilevel dependence. Fitting such models, however, can be computationally demanding: traditional Markov chain Monte Carlo (MCMC) methods may become prohibitively slow or difficult to scale. Variational inference (VI) provides an alternative approach by turning posterior inference into an optimisation problem over a tractable family of surrogate distributions [1, 2]. While this allows approximate Bayesian computation at scale, the quality of these approximations depends strongly on the choice of variational family and on the structure of the model. In hierarchical settings, simple mean-field approximations often fail to capture the posterior correlations and non-Gaussian dependencies inherent in the hierarchy, resulting in biased or overconfident uncertainty estimates.

This essay will explore recent advances aimed at improving the reliability and flexibility of VI for hierarchical Bayesian models. Students may begin by summarising the foundations of VI, the evidence lower bound, and stochastic optimisation via the reparameterisation trick, then focus on their application to hierarchical models [3]. Possible directions include implementing and benchmarking different VI surrogates (e.g. mean-field, full-rank, or structured) on simplified hierarchical examples inspired by principled scientific models [4, 5], comparing accuracy, robustness, and computational efficiency against MCMC. Students may also investigate recent theoretical work on divergence measures, validation diagnostics, and error bounds [6,7,8]. Originality and creativity are encouraged, particularly for ambitious essays that offer empirical or theoretical perspectives on variational approximations for hierarchical Bayesian models.

Relevant Courses

Useful: Information Theory, Astrostatistics

References

1. Blei, D. M., Kucukelbir, A., & McAuliffe, J. D. (2017). Variational inference: A review for statisticians. Journal of the American statistical Association, 112(518), 859-877.

2. Zhang, C., Bütepage, J., Kjellström, H., & Mandt, S. (2018). Advances in variational inference. IEEE transactions on pattern analysis and machine intelligence, 41(8), 2008-2026.

3. Agrawal, A., & Domke, J. (2021). Amortized variational inference for simple hierarchical models. Advances in Neural Information Processing Systems, 34, 21388-21399.

4. Mandel, K. S., Scolnic, D. M., Shariff, H., Foley, R. J., & Kirshner, R. P. (2017). The Type Ia Supernova color–magnitude relation and host galaxy dust: A simple Hierarchical Bayesian Model. The Astrophysical Journal, 842(2), 93.

5. Mandel, K. S., Thorp, S., Narayan, G., Friedman, A. S., & Avelino, A. (2022). A hierarchical Bayesian SED model for Type Ia supernovae in the optical to near-infrared. Monthly Notices of the Royal Astronomical Society, 510(3), 3939-3966.

6. Margossian, C. C., Pillaud-Vivien, L., & Saul, L. K. (2024). Variational inference for uncertainty quantification: an analysis of trade-offs. In Journal of Machine Learning Research 26 (2025) 1-41

7. Yao, Y., Vehtari, A., Simpson, D., & Gelman, A. (2018, July). Yes, but did it work?: Evaluating variational inference. In International Conference on Machine Learning (pp. 5581-5590). PMLR.

8. Huggins, J., Kasprzak, M., Campbell, T., & Broderick, T. (2020, June). Validated variational inference via practical posterior error bounds. In International Conference on Artificial Intelligence and Statistics (pp. 1792-1802). PMLR.

57. Reducing the Effect of Time Trends on the Performance of Response-Adaptive Clinical Trial Designs ....

Dr S. Baas, Dr S. S. Villar

Clinical trial designs using response-adaptive randomization (RAR) base the allocation of treatment to future participants on interim analyses of the clinical trial data. Often, the goal of RAR is to assign a high number of trial participants to a superior treatment, while it can also be used to increase statistical power, estimation precision, or a combination of inferential and patient-benefit oriented goals. Despite its long history, growing interest, and wide range of theoretical advancements, the number of practical applications of RAR designs is relatively small. Robertson et al. (2023) list a number of potential reasons for this, in particular they state: "The occurrence of time trends caused by changes in the standard of care or by patient drift (i.e. changes in the characteristics of recruited patients over time) is seen as a major barrier to the use of RAR in practice". Next to patient drift there can also be a time trend in the treatment effect (e.g., when physicians or surgeons improve their treatment method over time). Under no time trends, there is already a risk of underestimating the expected outcome for inferior treatments in RAR designs, yielding an overestimate of the treatment effect. Under time trends, this problem is exacerbated, as inferior treatments will tend to be chosen less often as time progresses. Hence, for RAR designs under time trends there is an increased risk of bias and type I error rate inflation using standard inference approaches. Villar et al. (2018) evaluate the effect of time trends on type I error rate and power, along with possible ways of overcoming the type I error rate inflation while ensuring high power, however debate about the use of RAR under potential time trend persists, with others advocating for a fixed design with equal allocation to each treatment.

This project will evaluate different approaches for dealing with time trends in RAR designs. The project will compare an extension of the approach in Section 4 of Villar et al. (2018), using other basis functions, with an approach based on the power prior, introduced in Ibrahim and Chen (2000), other comparators can be based on stratified analyses (Karrison et al., 2003) and the relevance weighted likelihood approach (Hu and Rosenberger, 2000). Preferably, the essay also considers the probability of having an imbalance in the wrong direction or the type III error rate due to, e.g., the treatment effect switching signs, which is a currently unexplored topic in the literature. The main RAR design under consideration should be a blocked Bayesian RAR design, which is most often implemented in practice. The essay should incorporate a literature review of existing work. Challenges of the project include ways of efficiently determining the power parameter of the power prior, determining the comparators, and setting up the simulation study comparison.

Relevant Courses

Essential: Statistics in Medical Practice (Statistics in Medicine)

References

1. J. G. Ibrahim and M. H. Chen (2000). Power prior distributions for regression models. Statistical Science, 46-60. URL: https://www.jstor.org/stable/2676676

2. Karrison TG, Huo D, Chappell R (2003). A group sequential, response-adaptive design for randomized clinical trials. Controlled Clinical Trials 4(5), 506-22. doi: 10.1016/s0197-2456(03)00092-8.

3. D. S. Robertson, K. M. Lee, B. C. López-Kolkovska, and S. S. Villar (2023). Response-Adaptive Randomization in Clinical Trials: From Myths to Practical Considerations. Statistical Science, 38(2), 185-208. doi: 10.1214/22-STS865

4. Hu, F., & Rosenberger, W. F. (2000). Analysis of time trends in adaptive designs with application to a neurophysiology experiment. Statistics in Medicine, 19(15), 2067-2075. doi: 10.1002/1097-0258(20000815)19:15<2067::AID-SIM508>3.0.CO;2-L

5. S. S. Villar, J. Bowden, and J. Wason (2018). Response-adaptive designs for binary responses: How to offer patient benefit while being robust to time trends? Pharmaceutical statistics, 17(2), 182-197. doi: 10.1002/pst.1845

58. Semi-Markov Models for Intermittently-Observed Data .... Dr C. H. Jackson

This essay will suit students interested in applied statistics methodology. In the Statistics in Medical Practice module on multi-state models, it is mentioned that relaxing the Markov assumption is challenging when the data are observed intermittently. This essay will review previous approaches that have been taken to do this. These have included, for example:

- explicitly integrating over unobserved times of transition between states [1, 4, 8],
- algorithms such as Markov Chain Monte Carlo or expectation-maximisation [2, 3, 5],
- "phase-type" sojourn distributions, which express the semi-Markov model as a hidden Markov model, making the likelihood tractable [6, 7].

The essay will describe the essence of each approach and how they differ. Key aspects include:

- assumptions that are necessary to an approach, such as parametric distributions,
- whether the approach works (either in theory or in practice) for any transition structure,
- principles used for estimation (e.g. maximum likelihood, nonparametric methods, Bayesian estimation) and any justifications for these, e.g. properties of estimators,
- computational techniques (e.g. optimisation, MCMC) and their relative expense,
- accessibility of software for using the method in practice.

A final discussion will summarise the relative merits of different approaches and identify any open challenges and priorities for further research in the area.

A successful essay will present the information in an organised manner, and show critical thinking. For example, if quoting a claim about a method made by an author, it should be clear if this is a quote, and if there is evidence for the claim.

An outstanding essay will think beyond the approaches suggested by authors, e.g. by identifying if there may be benefits in combining aspects of different approaches.

Practical computational work to demonstrate the methods is not necessary, though for students interested in working with data, experimenting with using one or two methods in practice may help to gain understanding of them.

Relevant Courses

Essential: Statistics in Medicine (Statistics in Medical Practice)

References

1. M. Aastveit, C. Cunen, and N. Hjort. A new framework for semi-Markovian parametric multi-state models with interval censoring. *Statistical Methods in Medical Research*, 32(6):1100–1123, 2023.

2. H. Aralis and R. Brookmeyer. A stochastic estimation procedure for intermittently-observed semi-Markov multistate models with back transitions. *Statistical Methods in Medical Research*, 28(3):770–787, 2019.

3. R. Barone and A. Tancredi. Bayesian inference for discretely observed continuous time multi-state models. *Statistics in Medicine*, 41(19):3789–3803, 2022.

4. Y. Foucher, M. Giral, J. P. Soulillou, and J. P. Daures. A flexible semi-Markov model for interval-censored data and goodness-of-fit testing. *Statistical Methods in Medical Research*, 19(2):127–145, 2010.

5. Y. Gu, D. Zeng, G. Heiss, and D. Y. Lin. Maximum likelihood estimation for semiparametric regression models with interval-censored multistate data. *Biometrika*, 111(3):971–988, 2024.

6. C. Jackson. Stable and practical semi-Markov modelling of intermittently-observed data. *arXiv preprint arXiv:2508.20949*, 2025.

7. A. C. Titman and L. D. Sharples. Semi-Markov models with phase-type sojourn distributions. *Biometrics*, 66(3):742–752, 2010.

8. S. Wei and R. J. Kryscio. Semi-Markov models for interval censored transient cognitive states with back transitions and a competing risk. *Statistical Methods in Medical Research*, 25(6):2909–2924, 2016.

59. Robust Distributed Inference .... Professor P. Loh

The past decade has brought increased interest in distributed models of computation, spurred by the need to process large volumes of data in an efficient manner, or respect criteria such as privacy of individuals/data centers. In statistics, the goal is to devise inference procedures that achieve optimal performance when i.i.d. data are distributed among local machines, and characterize the effect of such distributed procedures in relation to fully centralized methods.

However, one characteristic of distributed learning is that information must be communicated between machines and/or a central server. A finer-grained analysis of distributed systems therefore involves understanding the amount of communication (e.g., bits of information) required to be passed at any given time, and how the accuracy is affected by the amount of information allowed to be transmitted. In this sense, one can view rounds of communication as a "channel" in an information-theoretic sense; the channel may have constraints such as a budget of bits, a constraint of privacy, or robustness to noise in the transmitted data.

The goal of this essay is to survey the literature on distributed statistical inference and develop new variants of procedures which are robust to noisy communication. A good starting point might be the survey papers [1, 5] or the papers [2, 3, 4], which formulate and study some robust distributed estimation problems. The procedures proposed in the essay should then be analyzed theoretically and/or empirically.

Relevant Courses

Useful: Robust Statistics, Information Theory

References

1. Y. Gao, W. Liu, H. Wang, X. Wang, Y. Yan, and R. Zhang. A review of distributed statistical inference. *Statistical Theory and Related Fields*, pages 89–99, 2022.
2. S. Minsker. Distributed statistical estimation and rates of convergence in normal approximation. *Electronic Journal of Statistics*, 2019.
3. A. Van Elst, I. Colin, and S. Clémençon. Robust distributed estimation: Extending gossip algorithms to ranking and trimmed means. arXiv preprint arXiv:2505.17836, 2025.
4. D. Yin, Y. Chen, K. Ramchandran, and P. Bartlett. Byzantine-robust distributed learning: Towards optimal statistical rates. In *International Conference on Machine Learning*, pages 5650–5659, 2018.
5. L. Zhou, Z. Gong, and P. Xiang. Distributed computing and inference for big data. *Annual Review of Statistics and Its Application*, 2024.

60. Robust Estimation of Graph Statistics .... Professor P. Loh

As network-structured data have become increasingly prevalent in real-world applications, a plethora of interesting problems have been studied by theoretical statisticians, where the goal is to infer something about the structure of the underlying data-generating process based on observations at the nodes of the graph. Examples include estimating parameters of random graphs, recovering communities in stochastic block models, and estimation of the edge structure of a graphical model.

What has received somewhat less attention is the *robustness* of these methods to contamination in the data. Even defining a reasonable notion of contamination is a largely open research direction; for instance, it might come in the form of a small number of perturbations in the edges of an observed adjacency matrix, which might occur randomly or adversarially.

The goal of this essay is to explore the literature concerning robustness of graph estimation algorithms. Some relevant literature includes [1] for graph parameter estimation, [2, 8, 9] for graphical models, and [3, 6, 7, 10] for stochastic block models. In order to attain the appropriate depth, it is recommended that an essay focus on one or two graph estimation problems, although reading through the methods used in other problem settings may help generate new ideas. The essay should first survey related work, comparing and contrasting the assumptions and notions of robustness studied in the papers, and then propose a new estimation setting and/or notion of robustness that is analyzed theoretically or empirically. It may be interesting to consider the notions of graph robustness in contrast to classical robustness theory, and consider whether estimators coming from the robust statistics literature may be leveraged to develop new robust procedures [4, 5].

Relevant Courses

Useful: Robust Statistics

References

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10. T. Cai and X. Li. Robust and computationally feasible community detection in the presence of arbitrary outlier nodes. *Annals of Statistics*, pages 1027–1059, 2015.

61. Subgroup Selection ....

Mr M. M. Müller and Professor R. J. Samworth

Imagine that there is a global pandemic, and that you are in charge of the vaccine trial. In order to be approved by the regulators, your vaccine needs to be 80% effective, but you find to your horror that it is only 70% effective. Just before you discard the decade of research that went into developing your vaccine, your colleague says: 'If you look at women over 50, the vaccine is 95% effective. Can we just give it to them?'. The important point is that the subgroup of interest (women over 50 in this instance) was chosen *after* seeing the data, leading to concerns of data snooping, cherry picking and p-hacking.

This is an example of a *post-selection inference* problem, and in a regression setting, one would often be interested in regions of the covariate space where the regression function is at least at some pre-specified level τ (so $\tau = 0.8$ in the example above). In [1] and [2], a new framework was proposed that allows the practitioner to use the data to select a subset $\hat{A}$ of the covariate space with the guarantee that with high probability, and uniformly over a wide class of underlying distributions, the infimum of the regression function on $\hat{A}$ is at least τ, i.e., $\hat{A}$ is a subset of the τ-superlevel set of the regression function. Subject to this constraint, we would like $\hat{A}$ to have measure as large as possible. The setting of [1] works with nonparametric smoothness classes for the regression function, while [2] considers the case where the regression function is isotonic (monotone). In the context of linear models, the same high-probability guarantee has been considered by [3].

A candidate may wish to explore the literature discussing this problem in the context of clinical trials (e.g. [4], [5]) and the range of methods used to address it in the past (e.g. [6], [7]), including flexible machine learning approaches, such as the one developed very recently by [8]. They may want to compare and contrast these approaches with the framework considered in [1-3]. An ambitious candidate may investigate the role that the flexible technique due to [9] may play in the context of linear regression described by [3], for instance in the presence of heavy-tailed or multi-modal noise. They may even consider this in the framework of heterogeneous treatment effect estimation (e.g. [2, Section 4.3]).

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods

References

1. Reeve, H. W. J., Cannings, T. I. and Samworth, R. J. (2023). Optimal subgroup selection. *Annals of Statistics*, **51**, 2342–2365.

2. Müller, M. M., Reeve, H. W. J., Cannings, T. I. and Samworth, R. J. (2025). Isotonic subgroup selection. *Journal of the Royal Statistical Society Series B: Statistical Methodology*, **87**, 132–156.

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9. Feng, O. Y., Kao, Y-.C., Xu, M. and Samworth, R. J. (2025+) Optimal convex M-estimation via score matching. *Annals of Statistics (to appear)*. https://arxiv.org/pdf/2403.16688.

62. Score Matching and Diffusion Models ....
Dr W. G. Underwood and Professor R. J. Samworth

Score matching is the task of estimating the gradient of the logarithm of a probability density function [3]. As the core statistical procedure behind diffusion models [2, 6], it has attracted a great deal of interest in recent years [4, 5]. Indeed, machine learning approaches based on score matching and diffusion are currently state-of-the art for generative AI, including high-quality image and video sampling, editing, in-painting and style transfer [9]. However, many basic questions about score matching are only now starting to be answered. Recent developments include results on the fundamental limits of score estimation [8] and sampling guarantees for score-based diffusion models [1], as well as fundamental statistical applications such as model attribution and AI image detection [7].

One approach to this essay might involve surveying existing methods for score matching and diffusion models, comparing their relative strengths and weaknesses in theory or in empirical studies. An ambitious essay may address potential improvements to these methods, possibly by weakening the assumptions imposed, generalising to different data distributions or diffusion schemes, or improving the empirical or computational performance.

Relevant Courses

Essential: Topics in Statistical Theory

Useful: Modern Statistical Methods

References

1. Dou, Z., Kotekal, S., Xu, Z. and Zhou, H. H. (2024) From optimal score matching to optimal sampling. arXiv:2409.07032.

2. Ho, J., Jain, A. and Abbeel, P. (2020) Denoising diffusion probabilistic models. *Advances in Neural Information Processing Systems*, **33**, 6840–6851.

3. Hyvärinen A. and Dayan, P. (2005) Estimation of non-normalized statistical models by score matching. *Journal of Machine Learning Research*, **6**.

4. Shen, Z., Wang, H., Riabiz, M. and Oates, C. J. (2024) Operator-informed score matching for Markov diffusion models. arXiv:2406.09084.

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63. Statistical Aspects of Gradient Boosting .... Dr E. H. Young and Professor R. J. Samworth

Gradient boosting has emerged as one of the most successful machine learning methods for regression and classification, with libraries such as XGBoost [1] dominating many machine learning competitions. Boosting seeks to minimize an empirical loss function (e.g. squared error loss) by aggregating 'weak learners' that are constructed sequentially by applying a base estimator/learner to updated samples that depend on the performance of the previous iterations of the boosting procedure. In spirit they share many similarities with bagging [2, 3, 4], another popular machine learning technique. The performance of gradient boosting has been studied theoretically in terms of its bias–variance tradeoff for a range of models and base learners [5, 6, 7]. Recent work on gradient boosting includes an analysis of the limiting distributions of specific boosting procedures [7] for performing inference and an extension of boosting to complex loss functions that have desirable statistical properties [8].

One approach for this essay could involve an analysis of existing gradient boosting strategies, comparing their relative strengths and weaknesses in theory and practice, for instance comparing the use of different base learners. An ambitious essay could address one or more of the following: potential improvements to the existing boosting strategies described above, e.g. by relaxing assumptions in theory or adapting methodology; providing stronger theoretical results for pre-existing boosting methods, e.g. distributional limits of certain boosting estimators, or performing inference; or analysing boosting with new base learners yet to be studied theoretically.

Relevant Courses

Useful: Topics in Statistical Theory, Modern Statistical Methods, Statistical Learning in Practice.

References

1. Chen, T. and Guestrin, C. (2016) Xgboost: A scalable tree boosting system. In *Proceedings of the 22nd ACM SIGKDD International Conference on Knowledge Discovery and Data Mining. Association for Computing Machinery*, 785–794.

2. Bühlmann, P. and Yu, B. (2002) Analyzing bagging. *Ann. Statist.*, **30**, 927–961.

3. Hall, P. and Samworth, R. J. (2005) Properties of bagged nearest-neighbour classifiers. *J. Roy. Statist. Soc., Ser. B*, **67**, 363–379.

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64. Causal Inference Beyond Average Treatment Effects .... Professor R. D. Shah

Understanding treatment effects is a central problem in causal inference. While randomised controlled trials are considered the gold standard for estimating causal effects, they are often impractical due to ethical, logistical, or financial constraints. Consequently, researchers must often rely on observational data, which introduces the challenge of confounding. Nevertheless, when all confounders are measured, the average treatment effect can be estimated from data under relatively mild assumptions, and this problem has been the focus of a great deal of research.

The global average however can for example obscure important heterogeneity in treatment responses across individuals. For example, patients with different clinical profiles may respond differently to the same medical intervention. Ideally, we would like to tailor treatments to individuals who are most likely to benefit, motivating the need to estimate the conditional average treatment effect (CATE): the expected treatment effect given covariates. This essay could explore recent methodological and theoretical advances in CATE estimation ([1–4]). Another direction concerns inference for treatment effect heterogeneity. A key question is whether certain covariates, such as age, income, or genetic markers, interact with the effect of treatment ([5–8]).

While the mean of the treatment effect is a natural target, we may also like to understand how the entire distribution of outcomes shifts under treatment. For instance, a treatment might reduce the variance of outcomes or affect tail risks. Recent work has proposed alternative estimands such as conditional quantile treatment effects, or counterfactual densities, which can offer richer insights into the impact of interventions beyond the mean ([9, 10]). Another direction could instead pursue these developments.

Overall, the aim of this essay would be to survey recent literature in one of the three areas mentioned above, and to compare and contrast the methods, assumptions, and inferential goals. There may also be scope for original contributions, such as modifying an existing method, proposing a hybrid approach, though this is not required.

Relevant Courses

Essential: Modern Statistical Methods

Useful: Topics in Statistical Theory

References

1. Wager, Stefan, and Susan Athey. "Estimation and inference of heterogeneous treatment effects using random forests." Journal of the American Statistical Association 113.523 (2018): 1228-1242.

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3. Kennedy, Edward H. "Towards optimal doubly robust estimation of heterogeneous causal effects." Electronic Journal of Statistics 17.2 (2023): 3008-3049.

4. Kennedy, Edward H., et al. "Minimax rates for heterogeneous causal effect estimation." Annals of statistics 52.2 (2024): 793.

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10. Kennedy, Edward H., Sivaraman Balakrishnan, and L. A. Wasserman. "Semiparametric counterfactual density estimation." Biometrika 110.4 (2023): 875-896.

65. Proximal Causal Inference .... Dr P. Zhao and Professor Q. Zhao

When randomized experiments are infeasible or unethical, empirical researchers typically rely on observational data to draw causal inference. A common concern with observational studies is that there may be some unmeasured confounders. To address this, a recent promising approach proposes a clever way to use "proxy variables" of the unmeasured confounders [1], which is intimately related to the concept of "negative control variables" [2].

This essay should, at the minimum, provide a brief overview of proximal causal inference. The essay could further explore one of the following aspects of proximal causal inference:

1. connections between nonparametric causal identification [3,4,5] and generic identification in linear structural equation models [6,7], possibly starting from the short note [8].

2. semiparametric inference on the average treatment effect and other functionals [5,9,10].

3. applications of proximal causal inference or negative control variables in epidemiology [2,11,12,13].

Relevant Courses

Useful: Modern Statistical Methods; Statistics in Medical Practice; Workshop on "Foundations of Causal Inference" (19–23 January, 2026, https://www.newton.ac.uk/event/cifw01/).

References

1. Tchetgen Tchetgen, E. J., Ying, A., Cui, Y., Shi, X., & Miao, W. (2024). An Introduction to Proximal Causal Inference. *Statistical Science*, 39(3), 375–390. https://doi.org/10.1214/23-STS911

2. Lipsitch, M., Tchetgen, E. T., & Cohen, T. (2010). Negative controls: a tool for detecting confounding and bias in observational studies. *Epidemiology*, 21(3), 383-388.

3. Miao, W., Geng, Z., & Tchetgen Tchetgen, E. J. (2018). Identifying causal effects with proxy variables of an unmeasured confounder. *Biometrika*, 105(4), 987–993. https://doi.org/10.1093/biomet/asy038

4. Kallus, N., Mao, X., & Uehara, M. (2021). Causal inference under unmeasured confounding with negative controls: A minimax learning approach. https://doi.org/10.48550/arXiv.2103.14029

5. Bennett, A., Kallus, N., Mao, X., Newey, W., Syrgkanis, V., & Uehara, M. (2022). Inference on strongly identified functionals of weakly identified functions. https://doi.org/10.48550/arXiv.2208.08291

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12. Swanson, S. A., Miller, M., (2024) Toward a clearer understanding of what works to reduce gun violence: the role of falsification strategies. *American Journal of Epidemiology*, 193(8), Pages 1061–1065, https://doi.org/10.1093/aje/kwae036.

13. Ashby, E., et al. (2025) Validating and leveraging non-SARS-CoV-2 respiratory infection as a negative control outcome in a phase 3 COVID-19 vaccine trial with extended observational follow-up. *American Journal of Epidemiology*, kwaf176, https://doi.org/10.1093/aje/kwaf176

66. Restart Schemes in Riemannian Optimization .... Dr M. Colbrook and Professor C.-B. Schönlieb

Optimization on manifolds has become increasingly important across scientific and engineering disciplines where data naturally lie on non-Euclidean spaces. Examples include the analysis of diffusion tensors, protein conformations, color spaces, and rotations in 3D imaging. In such problems, Euclidean optimization techniques fail to capture intrinsic geometric structure, and their naïve application can lead to slow convergence or even divergence. Riemannian optimization provides a principled framework to generalize classical first-order methods—such as gradient descent and proximal algorithms—to curved spaces while preserving geometric constraints.

Recently, the concept of approximate sharpness has been introduced in Euclidean optimization as a weak regularity condition that ensures acceleration through restarts without requiring knowledge of problem-specific constants. Extending this framework to the Riemannian setting offers a path towards parameter-free acceleration on manifolds, unifying ideas from differential geometry and optimization theory. This would allow faster convergence of manifold-based algorithms in applications where curvature and noise complicate analysis.

The goal of this essay is to study how restart schemes subject to approximate sharpness can be formulated and analyzed for optimization problems on Riemannian manifolds. We will investigate how geometric quantities such as sectional curvature and retraction mappings influence convergence, and how approximate sharpness can be generalized using intrinsic distance functions. Theoretical developments will be complemented by experiments on classical problems such as the Riemannian barycentre, median, and manifold-valued denoising.

The essay will proceed in the following stages:

1. **Literature Review:** A review of restart schemes and approximate sharpness in Euclidean spaces [1] and an overview of Riemannian optimization algorithms, including gradient descent, subgradient descent, and proximal point methods on manifolds.

2. **Theoretical Development:** Formulation of an approximate sharpness condition on Riemannian manifolds, incorporating curvature and distance metrics, and derivation of convergence guarantees for restarted Riemannian first-order methods.

3. **Numerical Investigation:** Numerical tests on the Riemannian barycentre and ROF problems, illustrating acceleration effects on manifolds such as S^2, $SO(3)$, and the manifold of positive-definite matrices.

Optimisation on manifolds is essential for modern data analysis, physics, and imaging—where the underlying data are inherently geometric. Developing restart schemes in this context combines deep geometry with algorithmic efficiency, offering new perspectives on how curvature interacts with first-order convergence.

Relevant Courses

Essential: Optimization and Convexity, Differential Geometry

Useful: Numerical Analysis, Machine Learning, Manifolds and Topology

References

1. Adcock, Ben, Matthew J. Colbrook, and Maksym Neyra-Nesterenko. "Restarts subject to approximate sharpness: a parameter-free and optimal scheme for first-order methods." Foundations of Computational Mathematics (2025): 1-56.
2. Smith, Steven T. "Optimization Techniques on Riemannian Manifolds." Fields Institute Communications 3 (1994).
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4. Ferreira, O. P., and P. R. Oliveira. "Proximal point algorithm on Riemannian manifolds." Optimization 51.2 (2002): 257-270.
5. Bacák, Miroslav. "Computing medians and means in Hadamard spaces." SIAM journal on optimization 24.3 (2014): 1542-1566.
6. Bergmann, Ronny, Johannes Persch, and Gabriele Steidl. "A parallel Douglas–Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifolds." SIAM Journal on Imaging Sciences 9.3 (2016): 901-937.
7. Bergmann, Ronny, Johannes Persch, and Gabriele Steidl. "A parallel Douglas–Rachford algorithm for minimizing ROF-like functionals on images with values in symmetric Hadamard manifolds." SIAM Journal on Imaging Sciences 9.3 (2016): 901-937.
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67. Stochastic Restart Schemes and Approximate Sharpness .... Dr M. Colbrook

First-order optimization algorithms are central to modern data science and machine learning, where high-dimensional stochastic problems are ubiquitous. Classical gradient-based methods are simple and scalable but often converge slowly, especially when data are noisy or the objective function lacks strong convexity. Restart schemes—where an optimization algorithm is periodically reinitialized with modified parameters—have recently emerged as powerful tools for accelerating convergence.

Recent theoretical advances have shown that acceleration can be achieved under approximate sharpness, a general condition that bounds the distance to the solution set by a fractional power of the objective suboptimality plus a perturbation constant. This framework allows for optimal or near-optimal convergence rates without requiring knowledge of problem-specific parameters. However, existing restart schemes have largely been developed in deterministic settings, whereas most large-scale applications (e.g., in deep learning or inverse problems) rely on stochastic first-order methods such as stochastic gradient descent (SGD).

The aim of this essay is to develop and study restart schemes for stochastic optimization algorithms under approximate sharpness conditions. This involves extending the deterministic theory to settings where gradients and objective evaluations are noisy or randomly sampled. Both convergence in expectation and with high probability will be analyzed. Applications include stochastic versions of LASSO, logistic regression, and sparse recovery problems, where approximate sharpness naturally arises.

The essay will proceed through the following stages:

1. **Literature Review:** Survey the theory of restarts and sharpness in deterministic optimization [1] and review stochastic optimization algorithms such as SGD, mirror descent, and stochastic primal–dual methods.

2. **Theoretical Development:** Formulate a stochastic analogue of approximate sharpness and design restart strategies that achieve provable convergence in expectation and with high probability.

3. **Numerical Investigation:** Implement restarted stochastic algorithms and test them on benchmark problems (e.g., LASSO and logistic regression) to assess empirical acceleration and robustness to noise.

This project connects modern ideas in convex optimization, random processes, and algorithmic acceleration. It will involve both mathematical analysis and computational experiments, providing insight into how deterministic regularity conditions can be leveraged in stochastic settings. There will also be an opportunity to work with Ben Adcock (Simon Fraser University), who will jointly supervise the essay.

Relevant Courses

Essential: Optimization and Convexity, Basic Probability

Useful: Numerical Analysis, Statistics, or Stochastic Differential Equations

References

1. Adcock, Ben, Matthew J. Colbrook, and Maksym Neyra-Nesterenko. "Restarts subject to approximate sharpness: a parameter-free and optimal scheme for first-order methods." Foundations of Computational Mathematics (2025): 1-56.

2. Zhao, Renbo. "Accelerated stochastic algorithms for convex-concave saddle-point problems." Mathematics of Operations Research 47.2 (2022): 1443-1473.

3. Li, Chris Junchi, et al. "On the convergence of stochastic extragradient for bilinear games using restarted iteration averaging." International Conference on Artificial Intelligence and Statistics. PMLR, 2022.

4. Lu, Haihao, and Jinwen Yang. "Nearly optimal linear convergence of stochastic primal-dual methods for linear programming." arXiv preprint arXiv:2111.05530 (2021).

5. Wang, Bao, et al. "Scheduled restart momentum for accelerated stochastic gradient descent." SIAM Journal on Imaging Sciences 15.2 (2022): 738-761.

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68. What Can Artificial Neural Networks Tell Us About Real Neural Networks? . . . . Professor S. J. Eglen

Artificial neural networks are now among the most popular and effective tools used in machine learning today. They were created in the 1940s [1] and 1950s [2] as a simple approximation to how brains were thought to *learn* tasks. In 2025, given the success of artificial neural networks, what can these networks tell us about how brains might learn and function?

The essay would review recent approaches to applying artificial neural networks to problems in neuroscience and to critically evaluate what we might learn from artificial networks. Two example papers that you might wish to study are provided [3,4] but you can also review other recent papers if you wish.

Relevant Courses

Essential: None.

Useful: Some background in artificial neural networks.

References

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2. Rosenblatt F (1958) The perceptron: a probabilistic model for information storage and organization in the brain. Psychol Rev 65:386–408 https://www.ling.upenn.edu/courses/cogs501/Rosenblatt1958.pdf

3. Schaeffer R, Khona M, Fiete I (2022) No Free Lunch from deep learning in Neuroscience: A case study through models of the entorhinal-hippocampal circuit. bioRxiv http://dx.doi.org/10.1101/2022.08.07.503109

4. Song Y, Millidge B, Salvatori T, Lukasiewicz T, Xu Z, Bogacz R (2024) Inferring neural activity before plasticity as a foundation for learning beyond backpropagation. Nat Neurosci 27:348–358 http://dx.doi.org/10.1038/s41593-023-01514-1

69. Formalizing Lieb's Concavity Theorem .... Professor H. Fawzi

Lieb's concavity theorem states that the function $(A, B) \mapsto \text{tr}[A^\alpha B^{1-\alpha}]$ of a pair of positive definite matrices (A, B) is concave when $\alpha \in [0, 1]$. This theorem has important applications in quantum information. The goal of this essay is to write a formal proof of this theorem in Lean. The first step in this essay is to read the paper [1] which gives a proof of Lieb's concavity theorem. The second step is to get familiar with Lean and how it can be used to formalize mathematics [2]. The third step is to write the proof in Lean. The reference [3] gives a recent example of formalizing a theorem in quantum information in Lean, and may be helpful to read (at least the introduction).

Relevant Courses

Real analysis. Interest in computation and the formalization of mathematics is also necessary.

References

1. I. Nikoufar, A. Ebadian, M. Eshaghi Gordji, The simplest proof of Lieb concavity theorem, Advances in Mathematics (2013)
2. Mathematics in Lean, https://leanprover-community.github.io/mathematics_in_lean/index.html
3. A. Meiburg, L. Lessa, R. Soldati, A Formalization of the Generalized Quantum Stein's Lemma in Lean, arXiv:2510.08672

70. Lipschitz-Constrained Neural Networks: Theory and Applications ..... Dr D. Murari and Professor C.-B. Schönlieb

Description

Lipschitz constraints on neural networks play a central role in a range of settings where controlled sensitivity is essential: certified adversarial robustness [3,5,11], stability of deep residual architectures, efficient estimation of Wasserstein distances [8], and modelling of provably convergent fixed-point maps as in the Plug-and-Play algorithm [3,12]. This project will survey why the Lipschitz property matters, and how it is obtained in practice—either by architectural design (e.g. spectral/orthogonal layers, convex-potential residual flows [3,4,9], GroupSort/1-Lipschitz activations [1,2], tailored self-attention [6,7]) or by training objectives where the constraint is enforced or promoted through regularisation [8,10] (e.g. gradient penalties, spectral penalties).

Scope and aims

Start from a concise review of applications that fundamentally require Lipschitz control, highlighting trade-offs between expressivity, certified robustness, speed, and memory. Then provide an overview of principal mechanisms used to enforce the constraint (exact-by-design layers, operator norm control, orthogonality, residual flows) and to promote it via regularisation. After this overview, pursue one of the following paths:

1. **Theory track (universal approximation).** Present a universal approximation theorem for Lipschitz-constrained networks. You may focus, for instance, on GroupSort/spline activations with norm-bounded weights [1,2] or on residual (1-Lipschitz) architectures [9]. Clearly state assumptions, approximation class (e.g. K-Lipschitz functions on compact sets), and proof ideas (e.g. Restricted Stone–Weierstrass; piecewise-affine density; lifting/projection tricks for fixed width).

2. **Practice track (implementation & comparison).** Implement representative 1-Lipschitz models (e.g. spectral-normalised Conv/Dense, convex-potential layers, GroupSort/spline activations, Lipschitz-normalised self-attention). Train them all to classify the Fashion MNIST dataset, or a sufficiently complicated dataset of your choice. For the training, different loss functions could be considered, for example, contrasting the multiclass cross-entropy loss with the multiclass hinge loss. Compare the obtained results in terms of clean accuracy, and how the performance drops under the action of adversarial attacks, such as the ℓ^2-projected gradient descent attack. A good reference paper to follow for designing such an analysis is [5], where a similar comparison has already been performed. If other applications are of interest, please let us know, and we can adapt the project to accommodate those requests individually.

Relevant Courses

Essential: Linear algebra, basics of numerical optimisation and approximation theory.

Useful: Numerical analysis and dynamical systems (for residual-flow views), convex analysis and monotone operator theory.

References

1. Cem Anil, James Lucas, and Roger Grosse. "Sorting Out Lipschitz Function Approximation." In Proceedings of the 36th International Conference on Machine Learning (PMLR 97), pp. 291–301, 2019.

2. Sebastian Neumayer, Alexis Goujon, Pakshal Bohra, and Michael Unser. "Approximation of Lipschitz Functions Using Deep Spline Neural Networks." SIAM Journal on Mathematics of Data Science, 5(2):306–322, 2023.

3. Ferdia Sherry, Elena Celledoni, Matthias J. Ehrhardt, Davide Murari, Brynjulf Owren, and Carola-Bibiane Schönlieb. "Designing Stable Neural Networks using Convex Analysis and ODEs." Physica D: Nonlinear Phenomena, 463:134159, 2024.

4. Laurent Meunier, Blaise J. Delattre, Alexandre Araujo, and Alexandre Allauzen. "A Dynamical System Perspective for Lipschitz Neural Networks." In Proceedings of the 39th International Conference on Machine Learning (PMLR 162), pp. 15484–15500, 2022.

5. Bernd Prach, Fabio Brau, Giorgio Buttazzo, and Christoph H. Lampert. "1-Lipschitz Layers Compared: Memory, Speed, and Certifiable Robustness." In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition (CVPR), pp. 24574–24583, 2024.

6. George Dasoulas, Kevin Scaman, and Aladin Virmaux. "Lipschitz Normalization for Self-Attention Layers with Application to Graph Neural Networks." In Proceedings of the 38th International Conference on Machine Learning (PMLR 139), pp. 2456–2466, 2021.

7. Kim Hyunjik, George Papamakarios, and Andriy Mnih. "The Lipschitz Constant of Self-Attention." In International Conference on Machine Learning, pp. 5562-5571. PMLR, 2021.

8. Ishaan Gulrajani, Faruk Ahmed, Martin Arjovsky, Vincent Dumoulin, and Aaron C. Courville. "Improved Training of Wasserstein GANs." In Advances in Neural Information Processing Systems 30 (NeurIPS 2017), pp. 5767–5777, 2017.

9. Davide Murari et al. "Approximation Theory for 1-Lipschitz ResNets." arXiv:2505.12003, 2025. (NeurIPS 2025)

10. Sebastian Lunz, Ozan Öktem, and Carola-Bibiane Schönlieb. "Adversarial Regularizers in Inverse Problems." In Advances in Neural Information Processing Systems 31 (NeurIPS 2018), 2018.

11. Yusuke Tsuzuku, Issei Sato, and Masashi Sugiyama. "Lipschitz-Margin Training: Scalable Certification of Perturbation Invariance for Deep Neural Networks." In Advances in Neural Information Processing Systems 31 (NeurIPS 2018), 2018.

12. Johannes Hertrich, Sebastian Neumayer, and Gabriele Steidl. "Convolutional Proximal Neural Networks and Plug-and-Play Algorithms." Linear Algebra and its Applications, 631:203–234, 2021. Elsevier.

71. Convective Instabilities in Galaxy Clusters .... Professor H. N. Latter

Galaxy clusters are gravitationally bound astrophysical structures comprising hundreds, even thousands, of galaxies. Well known examples 'near' us include the Virgo, Coma, and Hercules clusters. A characteristic feature of these structures is the extremely hot ($\sim 10^7$ K) ionised gas that permeates the space between the galaxies. This gas, referred to as the intracluster medium (ICM), is weakly collisional and, as a consequence, the conduction of heat and momentum is anisotropic, aligning itself with the local magnetic field. The anisotropy of the heat transport, in particular, is associated with two unusual 'convective' instabilities: the magnetothermal instability and the heat-flux buoyancy instability (MTI and HBI) (see references [1], [2], and [3]). Currently, researchers are testing how these (and the disordered flows they initiate) influence the global structure and properties of galaxy clusters.

In this essay you should discuss the basic physics of weakly collisional and magnetised plasma, and then review the linear theory of the MTI and HBI, paying attention to how the familiar convective stability results are altered by the anisotropic heat flux. You may then survey the nonlinear simulations (e.g., [4],[5],[6],[7]), their potential role in conundrums such as the 'cooling flow problem' ([8]), or you could have a look at 'micro-instabilities' caused by the plasma's pressure anisotropy ([9]).

Relevant Courses

Essential: Astrophysical fluid dynamics

Useful: Stellar structure and evolution

References

1. Balbus, 2000, Astrophysical Journal, 534, 420.
2. Balbus 2001, Astrophysical Journal, 562, 909.
3. Quataert, 2008, Astrophysical Journal, 673, 758.
4. Parrish and Stone, 2007, Astrophysical Journal, 664, 135.
5. Parrish et al., 2009, Astrophyical Journal, 703, 96.
6. Perrone and Latter, 2022, MNRAS, 513, 4605.
7. Kempf and Rincon, 2025, Astronomy & Astrophysics, 694, 25.
8. Peterson and Fabian, 2006, Physics Reports, 427, 1.
9. Schekochihin et al., 2005, Astrophysical Journal, 629, 139.

72. The Vertical Shear Instability in Protoplanetary Discs .... Professor H. N. Latter

Protoplanetary discs are subject to several hydrodynamical instabilities that may influence their evolution, observability, and the process of planet formation ([1]). Perhaps the one of most interest to researchers currently is the 'vertical shear instability' (VSI), which feeds off the slight deviations from Keplerian orbital motion induced by the central protostar's irradiation ([2]). The linear theory of the instability is remarkably rich, connecting various ideas and processes such as centrifugal instability, double diffusion, discoseismology, and ray theory. The nonlinear saturation of the instability, on the other hand, notably supports large-scale features, such as global wavetrains that corrugate the disc, amidst a morass of small-scale turbulent motions.

In this essay, students are asked to survey the linear dynamics of the VSI. First, the essay should discuss global equilibrium disc models, how vertical shear is ubiquitous in them, and provide estimates of their cooling timescales ([1],[2],[3]). Next, it should present the linear VSI in its various guises, in particular, in the local Boussinesq approximation ([3]) and then a vertically stratified model ([2],[4])). Finally, the global theory of the VSI, now understood as a growing travelling wave, might be explored ([5]). Additional topics could include its nonlinear saturation ([2],6]), nonlinear corrugation waves and wave zones ([7]), secondary instabilities acting upon the waves ([3],[8]), and small-scale critically balanced turbulence ([6]).

Relevant Courses

Essential: Astrophysical Fluid Dynamics, Dynamics of Astrophysical Discs

Useful: Fluid Dynamics of the Environment

References

1. Lesur et al., 2023. In: Inutsuka et al. (Eds), *Protostars and Planets VII*, p. 465.
2. Nelson, Gressel, Umurhan, 2013. MNRAS, 435, 2610.

3. Latter and Papaloizou, 2018. MNRAS, 474, 3110.

4. Barker and Latter, 2015. MNRAS, 450, 21.

5. Ogilvie, Latter, Lesur, 2025. MNRAS, 537, 3349.

6. Lesur, Latter, Ogilvie, 2025. Astronomy & Astrophysics, accepted: eprint arXiv:2508.20839.

7. Svanberg, Cui, Latter, 2022. MNRAS, 514, 4581.

8. Cui and Latter, 2022. MNRAS, 512, 1639.

73. Eccentric Astrophysical Discs .... Professor G. I. Ogilvie

Closed Keplerian orbits around a massive body are generally non-circular. A thin Keplerian disc may be composed of nested elliptical orbits whose eccentricity e and longitude of pericentre ϖ vary continuously with semi-major axis a and time t. The complex eccentricity is $e\,\exp(\mathrm{i}\varpi) = E(a,t)$.

When $|E|$ and $|\partial E/\partial \ln a|$ are sufficiently small, a linear evolutionary equation can be derived for the complex eccentricity, which determines how the shape of the disc propagates by means of pressure, viscosity, self-gravity and other collective effects that are weak compared to gravity. More generally, $E(a,t)$ satisfies a nonlinear evolutionary equation and the presence of eccentricity affects the transport of mass and angular momentum in the disc.

Eccentric discs are thought to exist in many astrophysical situations, including narrow planetary rings around Saturn and Uranus, protoplanetary discs around young stars, circumstellar discs around rapidly rotating Be stars, accretion discs around compact objects in close binary systems, and circumbinary discs around binary stars or black holes.

This essay should discuss apsects of the dynamics and significance of eccentric discs in at least two of these areas of application. Apart from the derivation and interpretation of the evolutionary equation(s) for eccentric discs, theoretical topics that might be discussed include the stability of fluid flows with elliptical streamlines, the gravitational interaction of orbiting companions with a disc, and the numerical simulation of eccentric discs.

A selection of starting references is provided below. Use of the Astrophysics Data System `ui.adsabs.harvard.edu` is recommended.

Relevant Courses

Useful: Astrophysical Fluid Dynamics, Dynamics of Astrophysical Discs, Planetary System Dynamics

References

1. Borderies, N., Goldreich, P. and Tremaine, S. (1983). *Astron. J.* **88**, 1560–1568

2. Goodchild, S. and Ogilvie, G. (2006). *Mon. Not. R. Astron. Soc.* **368**, 1123–1131

3. Kley, W. and Dirksen, G. (2006). *Astron. Astrophys.* **447**, 369–377

4. Lubow, S. H. (1991). *Astrophys. J.* **381**, 259–267

5. Ogilvie, G. I. (2001). *Mon. Not. R. Astron. Soc.* **325**, 231–248
6. Ogilvie, G. I. and Barker, A. J. (2014). *Mon. Not. R. Astron. Soc.* **445**, 2621–2636
7. Ogilvie, G. I. and Lynch, E. M. (2019). *Mon. Not. R. Astron. Soc.* **483**, 4453–4469
8. Papaloizou, J. C. B. (2005). *Astron. Astrophys.* **432**, 743–755
9. Thun, D., Kley, W. and Picogna, G. (2017) *Astron. Astrophys.* **604**, A102

74. Effects of a Magnetic Field on Stellar Oscillations .... Professor G. I. Ogilvie

Many stars are believed to have a strong magnetic field in their deep interiors, but until recently this has been very difficult to constrain observationally. In the last few years, however, detailed measurements of oscillation modes, together with developments in theory, have provided evidence for strong magnetic fields in some stellar interiors. To explain the suppression of dipole modes in some red giants, a picture has emerged according to which inwardly propagating gravity waves (g modes) are converted into outwardly propagating magnetic waves and dissipated, if they enter a region where the magnetic field exceeds a critical strength [1,2,3]. This concept has also been used to estimate the field strength in a massive main-sequence star [4] and to propose a mechanism of dissipating tidal disturbances [5]. Complementary ideas are that stellar oscillation modes can be dissipated through Alfvén resonances [6] or become trapped or chaotic [7]. Magnetic fields in stellar interiors have also been detected through the asymmetric splitting of oscillation modes and the application of perturbation theory [8].

This essay should give a review of several of these recent developments.

Use of the Astrophysics Data System `ui.adsabs.harvard.edu` is recommended. The theory of MHD waves in a stratified atmosphere is treated at length (but with different applications in mind) in [9].

Relevant Courses

Essential: Astrophysical Fluid Dynamics
Useful: Structure and Evolution of Stars

References

1. Fuller, J. et al. (2015). *Science* **350**, 423
2. Lecoanet, D. et al. (2017). *MNRAS* **466**, 2181
3. Rui, N. Z. and Fuller, J. (2023). *MNRAS* **523**, 582
4. Lecoanet, D. et al. (2022). *MNRAS* **512**, L16
5. Duguid, C. et al. (2024). *Astrophys. J.* **966**, L14
6. Loi, S. T. and Papaloizou, J. C. B. (2017). *MNRAS* **467**, 3212
7. Loi, S. T. and Papaloizou, J. C. B. (2018). *MNRAS* **477**, 5338
8. Li, G. et al. (2022). *Nature* **610**, 43
9. Cally, P. S. and Bogdan, T. J. (2024). arXiv:2408.01591

75. Ocean Circulation on Ice-Covered Moons ....
Dr N. Shibley

Several ice-covered satellites in the Solar System are thought to maintain liquid-water oceans [1]. However, the dynamics and circulation of these ocean systems are not well-known. Various ideas pertaining to ocean stratification [2], planetary rotation [3,4], and bottom heating [5], among others, may all influence the expected circulation. The essay will synthesize the current literature on the Solar System's ice-covered moons, as it pertains to the dynamics of subsurface oceans. It will describe and reflect on different existing theories for expected circulations, characterizing the physical differences between such theories. The essay will identify specifically why such theories lead to different expected circulation results and comment on these different approaches. Finally, the essay will compare and contrast how properties of different moons (i.e., Enceladus and Europa) may lead to different expected circulation patterns.

Relevant Courses

Useful: Fluid Dynamics of Climate, Extrasolar Planets: Atmospheres and Interiors, Planetary System Dynamics

References

1. Nimmo, F., and R. T. Pappalardo (2016), Ocean worlds in the outer solar system, J. Geophys. Res. Planets, 121, 1378–1399, doi:10.1002/2016JE005081.

2. Zhu, P., et al. (2017), The influence of meridional ice transport on Europa's ocean stratification and heat content, Geophys. Res. Lett., 44, 5969–5977, doi:10.1002/2017GL072996.

3. Ashkenazy, Y., Tziperman, E. (2021), Dynamic Europa ocean shows transient Taylor columns and convection driven by ice melting and salinity. Nat. Commun. 12, 6376. doi:10.1038/s41467-021-26710-0.

4. Soderlund, K. M. (2019), Ocean dynamics of outer solar system satellites. Geophys. Res. Lett., 46, 8700–8710. doi:10.1029/2018GL081880

5. Bire, S., et al. (2022), Exploring ocean circulation on icy moons heated from below, J. Geophys. Res.: Planets, 127, e2021JE007025, doi.org/10.1029/2021JE007025

76. Spontaneous Generation of Gravity Waves ....
Professor P. Haynes

Large-scale atmospheric and oceanic flows are typically close to geostrophic balance (Coriolis force balancing pressure gradient) and the 'slow' evolution of such flows is, at least qualitatively, described by the quasi-geostrophic equations which exclude 'fast' waves, i.e. internal inertia-gravity waves. The Rossby adjustment problem provides a canonical description of how a fluid evolves from an arbitrary initial condition, through the emission of inertia-gravity waves (or Poincaré waves in the shallow-water equation version), to a state of geostrophic balance. (It is convenient to use the general term 'gravity waves' to describe these waves.)

The separation between 'slow' and 'fast' motion is not perfect. A flow that is initially geostrophically balanced and contains no gravity-wave motion may evolve continuously in time to reach a state where there is a significant amount of gravity wave activity. This phenomenon is observed in the real atmosphere and ocean and in mathematical models and is typically described as 'spontaneous generation' of gravity waves.

Understanding of spontaneous generation has improved markedly over the last 20 years or so. Early numerical simulations appeared to show spontaneous generation but there were doubts over whether this apparent generation was a numerical artifact and would not appear in a real fluids and also over whether gravity waves had been completely excluded from the initial conditions. These doubts have now largely been resolved. Alongside this there has been progress in constructing relevant mathematical models and analysing their behaviour. Furthermore it is recognised that these spontaneously emitted waves may have important effects, e.g. triggering convection in the atmosphere or mixing events in the ocean.

It is suggested that reading for an essay on this topic starts with the reviews by Vanneste [1] and Plougonven and Zhang [2], the former putting more emphasis on a mathematical description of spontaneous generation, the latter more concerned with the phenomenon of spontaneous gravity-wave generation in the atmosphere. The essay itself should begin with a general review of spontaneous generation, defining clearly what it is, explaining why it is potentially important and theoretically challenging and summarising some of the key results on which current understanding is based. The essay could then move on to survey a particular sub-topic, for example simple mathematical models and accompanying theory (e.g. Vanneste and Yavneh [3]), or numerical simulation (e.g. Plougonven and Snyder [4]), or spontaneous generation in a particular flow configuration such as frontogenesis (e.g. Shakespeare and Taylor [5]).

Relevant Courses

Essential: Undergraduate course in fluid dynamics.

Useful: Fluid Dynamics of Climate. (This essay topic does relate closely to parts of the Fluid Dynamics of Climate course and anyone considering this essay who is not taking that course is advised – and welcome - to discuss with the setter.)

References

1. Vanneste, J., 2013: Balance and Spontaneous Wave Generation in Geophysical Flows. Annu. Rev. Fluid Mech. 2013. 45:147–72.

2. Plougonven, R., Zhang, F., 2014: Internal gravity waves from atmospheric jets and fronts, Rev. Geophys., 52, 33–76.

3. Vanneste, J, Yavneh, I., 2004: Exponentially small inertia-gravity waves and the breakdown of quasi-geostrophic balance. J. Atmos. Sci. 61, 211–23.

4. Plougonven, R, Snyder, C., 2007: Inertia-gravity waves spontaneously excited by jets and fronts. Part I: Different baroclinic life cycles. J. Atmos. Sci. 64:2502–20.

5. Shakespeare, C.J., Taylor, J.R., 2015: The spontaneous generation of inertia-gravity waves during frontogenesis forced by large strain: Numerical solutions. J. Fluid Mech., 772, 508–34.

77. Rain Evaporation and Downdraft Formation ....

Dr Q. Kriaa

Understanding the transport of water in the atmosphere—whether as vapour, cloud droplets, or rain—is essential to accurately model cloud formation, precipitation, and cloud organization. In particular, during rainfall, not all raindrops reach the ground—part of the rain evaporates as it falls through dry layers of the atmosphere. As a consequence of evaporation, the air may cool down sufficiently to sink in the atmosphere as a *downdraft*, spreading along the ground and forming what is known as *a cold pool*. Being able to accurately model the rain evaporation is crucial as cold pools play a key role in organizing convection and influencing the life cycle of clouds (e.g. [1]).

This essay will explore different modelling approaches to describe rain evaporation, including on downdraft formation. The essay will start with a literature review. After a brief discussion on the evaporation of a single raindrop, the essay should focus on simple models developed at the scale of an atmospheric column. The student will then implement a numerical solution of this problem using one or several approaches, depending on their interests. The objective will be to explore the ability of models of varying complexity to reproduce a given timeseries from atmospheric observations. Further references and guidance are available on request.

Relevant Courses

Essential: Undergraduate fluid mechanics.

Useful: Fluid Dynamics of the Environment, Fluid Dynamics of the Climate.

References

1. Meyer, B., & Haerter, J. O. (2020). Mechanical forcing of convection by cold pools: Collisions and energy scaling. Journal of Advances in Modeling Earth Systems, 12(11).

2. Srivastava, R. C. (1985). A simple model of evaporatively driven dowadraft: Application to microburst downdraft. Journal of Atmospheric Sciences, 42(10), 1004-1023.

3. Kruger, E. (2020). Dynamics of downdraughts and cold pools: an experimental and numerical study (Doctoral dissertation). https://doi.org/10.17863/CAM.61806

4. Takemi, T. (1999). Evaporation of rain falling below a cloud base through a deep atmospheric boundary layer over an arid region. Journal of the Meteorological Society of Japan. Ser. II, 77(2), 387-397.

5. Vicroy, D. D. (1992). Assessment of microburst models for downdraft estimation. Journal of Aircraft, 29(6), 1043-1048.

78. Learning Rheology of Antarctic Iceshelves ....

Dr K. Shah

Sea level rise is a significant challenge faced by society. A major contributor to rising sea levels is mass loss from the Antarctic icesheet. Iceshelves are extensions of the grounded icesheet that form when ice flows out over the surrounding ocean. They buttress the icesheet, slowing its mass loss. The deformation of the ice in response to stress, known as rheology, is key to understanding how ice flows and to estimating discharge from the grounded ice into the ocean and subsequent mass loss. Characterising iceshelf rheology is a grand challenge in glaciology. Machine learning techniques have recently been leveraged to predict iceshelf rheology directly from observations, revealing flow physics that goes beyond conventional understanding.

The essay will begin by explaining the different types of iceshelf rheology and the physics of ice deformation to which they correspond [1]. It will then layout key open questions about the ice flow law and motivate why they are important to address. The essay will give an exposition of the use of remote sensing measurements to observationally constrain iceshelf rheology [2]. In light of these findings, areas where machine learning approaches can improve ice flow models will then be discussed. The essay will introduce physics-informed neural networks, focusing on their application to data-driven solutions of inverse problems [3]. It will then give an exposition of a recent application of deep learning to predict iceshelf rheology, highlighting the novelty of their approach and what the key findings are [4]. A good essay should include an outlook on progress towards the open questions that the essay writer posed at the start of the essay, other applications for these deep learning techniques, and promising areas for future research.

Relevant Courses

Essential: Undergraduate courses on machine learning, calculus, partial differential equations.

Useful: Fluid Dynamics of Solid Earth and/or Non-Newtonian Flows, knowledge of neural networks.

References

1. Cuffey, Kurt M., and William Stanley Bryce Paterson. "The Physics of Glaciers" (Chapter 3). Academic Press, 2010.

2. Millstein, Joanna D., Brent M. Minchew, and Samuel S. Pegler. "Ice viscosity is more sensitive to stress than commonly assumed." *Communications Earth & Environment* 3, no. 1 (2022): 57.

3. Raissi, Maziar, Paris Perdikaris, and George E. Karniadakis. "Physics-informed neural networks: A deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations." *Journal of Computational Physics* 378 (2019): 686-707.

4. Wang, Yongji, Ching-Yao Lai, David J. Prior, and Charlie Cowen-Breen. "Deep learning the flow law of Antarctic ice shelves." *Science* 387, no. 6739 (2025): 1219-1224.

79. The Polar Vortex .... Dr A. Ming

The stratospheric polar vortex forms each winter about 15–50 km above Earth's surface as the poles are plunged into darkness. As the polar stratosphere cools, a large meridional temperature gradient develops. Strong eastward winds spin up, producing a Coriolis force that balances the resulting pressure gradient. These circumpolar winds are known as the polar-night jet, which acts as a mixing barrier, trapping cold air inside. In the Southern Hemisphere, this process eventually leads to the springtime ozone hole.

The strength of the vortex during winter is modulated by large-scale waves propagating from the troposphere that disturb the winds. Weak wave activity results in a strong and stable vortex, whereas strong wave activity can weaken or even destroy the vortex during events known as sudden stratospheric warmings [1]. These disruptions can have downward influences, altering the Arctic Oscillation and North Atlantic Oscillation, and are often associated with extreme winter weather across Europe and North America. Thus, the polar vortex serves as a critical bridge between high-altitude stratospheric dynamics and surface climate variability.

This essay will begin by describing the theoretical frameworks used to understand the polar vortex, including the Transformed Eulerian Mean framework [2] and the concept of wave–mean flow interaction, which encompasses the Charney–Drazin criterion [3]. It will then review the current understanding of the polar vortex and its downward influence. Possible extensions to this discussion include the use of idealized frameworks to study the vortex, ozone feedbacks on its evolution, and the coupling between different atmospheric layers and the polar vortex.

Relevant Courses

Essential: An undergraduate course in fluid dynamics

Useful: Fluid Dynamics of Climate

References

1. Baldwin, M. P., Ayarzagüena, B., Birner, T., Butchart, N., Butler, A. H., Charlton-Perez, A. J., et al. (2021). Sudden stratospheric warmings. Reviews of Geophysics, 59, e2020RG000708. https://doi.org/10.1029/2020RG000708
2. Andrews, D. G., Holton, J. R., & Leovy, C. B. (1987). Middle Atmosphere Dynamics. Academic Press.
3. Charney, J. G., and P. G. Drazin (1961), Propagation of planetary-scale disturbances from the lower into the upper atmosphere, J. Geophys. Res., 66(1), 83–109, doi:10.1029/JZ066i001p00083.

80. Granular Column Collapse and Critical State Theory $\mu(I)$ Rheological Models ....

Professor J. A. Neufeld

The collapse of a column of granular material is an archetypal experimental test of granular material, and of the rheological models which characterise their deformation and flow. Amongst the key observable parameters are the distance to which the granular pile might deform and slump, the angle which the pile ultimately achieves, and the time over which the pile reaches the end, static state. Models of the slumping of the granular column have focused on either using numerical models to simulate the frictional contact between a collection of grains, or on continuum models which aim to encapsulate these frictional granular interactions using an effective rheology, $\mu(I)$, which depends on the inertial number I which can be interpreted as the micro-scale rearrangement time to the large-scale shear rate.

Recently, the original granular slumping experiments have been revisited using a cohesive granular material. Cohesive forces between the grains may support finite stress, and as a result granular columns which start with small enough aspect ratio may remain undeformed even when released. The addition of cohesion between grains presents a non-trivial extension to classical $\mu(I)$ rheologies. This essay will survey the $\mu(I)$ granular rheology, focusing on the granular column collapse as the archetypal setting, and consider how cohesion may be incorporated into the general modelling framework. Students may wish to comment on how this extension is related to the *Cam clay* model of cohesive soils.

Relevant Courses

Useful: Slow Viscous Flow, Solidification of the Solid Earth

References

1. Lajeunesse, E. and Monnier, J. B. and Homsy, G. M. 2005 Granular slumping on a horizontal surface. *Phys. Fluids*, **17**:10, 103302.

2. Gans, A. *et al* 2023 Collapse of a cohesive granular column. *J. Fluid Mech.* **959**, A41.

3. Blatny, L., Gray, J.M.N.T., Gaume, J. 2024 A critical state $\mu(I)$-rheology model for cohesive granular flows. *J. Fluid Mech.* **997**, A67.

81. Visco-Diffusive Instability at Ocean Fronts .... Professor J. R. Taylor

Recent observations have revealed a series of thin layers with a width of ~ 1 m in the northern Gulf of Mexico where freshwater from the Mississippi/Atchafalaya meets salty waters in the Gulf of Mexico. The mechanisms that generate these layers are unclear, but the dynamics could influence mixing of oxygen, nutrients, heat, and other important tracers. One candidate mechanism for the formation of these layers is the McIntrye instability [1]. This instability can develop for flows in thermal wind balance when the fluid viscosity is different from the diffusivity of the fluid density. In the case of the Gulf of Mexico, the density is dominated by the salinity, which has a diffusivity that is almost 1000 times smaller than the fluid viscosity.

The McIntyre instability has been suggested as an explanation for observed layers around mid-depth eddies [2]. However, the length scale of the observed layers does not match the most unstable mode of the McIntyre instability. A similar conundrum occurs for inviscid symmetric instability which exhibits an ultraviolet catastrophe whereby the most unstable mode has an infinite wavenumber. In that case, it has been suggested that a secondary shear instability can arrest the primary symmetric instability and set the scale of the dominant remaining modes [3].

Here, the student will explore secondary instabilities that could arise in response to the McIntyre instability in a stratified, rotating fluid. Following a similar analysis to [3], the student should explore the time and lengthscales associated with the development of secondary instability developing from the primary McIntyre instability. The student could then comment on the possible relevance of the McIntyre instability to ocean fronts using a scaling analysis and/or numerical simulations.

Relevant Courses

Essential: Fluid Dynamics of Climate

References

1. McIntyre, M.E., 1970. Diffusive destabilisation of the baroclinic circular vortex. Geophysical and Astrophysical Fluid Dynamics, 1(1-2), pp.19-57.

2. Le Bars, M., 2021. Numerical study of the McIntyre instability around Gaussian floating vortices in thermal wind balance. Physical Review Fluids, 6(9), p.093801.

3. Taylor, J.R. and Ferrari, R., 2009. On the equilibration of a symmetrically unstable front via a secondary shear instability. Journal of Fluid Mechanics, 622, pp.103-113.

82. Precise Predictions of the Lightest CP-even Higgs Boson Mass in the Minimal Supersymmetric Standard Model .... Professor B. C. Allanach

The Minimal Supersymmetric Standard Model (MSSM) is still regarded by some to be an attractive TeV-scale extension to the Standard Model (however, in this essay, you should not review supersymmetry or the MSSM at all). The prediction of a Higgs boson whose properties match those of the experimentally discovered particle is an obvious priority. In the Standard Model, the Higgs boson mass can be fixed as a free parameter. This is not the case in the MSSM however: it is predicted in terms of other parameters of the model.

In many schemes, the MSSM Higgs boson mass has large radiative corrections and is calculated to a relatively high order in perturbation theory in various different schemes and approximations. It has been a question of research as to which scheme or approximation best links a precise Higgs boson mass prediction to the rest of the model.

The purpose of this essay is to find out and present the issues of the most precise predictions of the lightest CP-even Higgs boson mass prediction in the MSSM.

The essay should set the scene in terms of presenting motivation for precise predictions of the lightest CP-even Higgs boson mass. Then, it should review the tree-level plus leading order quantum correction predictions in terms of the other parameters of the model. It should go on to describe the important issues that arise in different calculations of the most precise predictions, along with a comparison of attempts to address them.

Relevant Courses

Essential: Supersymmetry, Advanced Quantum Field Theory

Useful: Standard Model

References

1. P. Kant, R. V. Harlander, L. Mihaila and M. Steinhauser, JHEP **08** (2010), 104.

2. R. L. Workman *et al.* [Particle Data Group], PTEP **2022** (2022), 083C01.

3. R. V. Harlander, J. Klappert and A. Voigt, Eur. Phys. J. C **77** (2017) no.12, 814.

4. P. Slavich *et al.*, Eur. Phys. J. C **81** (2021) no.5, 450.

5. D. de Florian *et al.* [LHC Higgs Cross Section Working Group], CERN Yellow Rep. Monogr. **2** (2017), 1-869.

83. Quantum Near-Extremal Black Holes ....

Dr A. Castro

The gravitational path integral (GPI) has long been understood to be a key tool for discovering new quantum gravity effects and for gaining conceptual insights. In recent years, it has provided detailed statistical information about black holes: the GPI has demonstrated that low-temperature black holes are unexpectedly dominated by quantum gravity corrections. The aim of this essay is to reproduce key aspects of this result. Some of the topics the student is expected to cover are:

1. An understanding of the geometry and fields that describe extremal and near-extremal black holes, with particular emphasis on the near-horizon background. Two possible examples are the four-dimensional Reissner-Nordström black hole or the three-dimensional BTZ black hole. Students should focus on one solution, and the following steps apply to the chosen theory/black hole.

2. Evaluate the classical Euclidean action of the black hole. Using the principles advocated by Gibbons-Hawking, show how to reproduce the Bekenstein-Hawking entropy from it.

3. Discuss properties of the Euclidean GPI, including one-loop corrections. In particular, address gauge fixing, Fadeev-Popov ghosts, and the conformal mode.

4. Show and discuss why the path integral of the near-horizon solution for the extremal case is plagued by zero modes. A key part of this analysis includes constructing the so-called tensor modes and describing their properties. (Note: there are additional zero modes, such as vector and photon modes; the student is only asked to quantify in detail the tensor modes.)

5. Discuss how the tensor modes are modified in the near-extremal background. Evaluate the GPI to leading order in temperature and show that quantum corrections dominate it.

The above list is indicative, and the specific contents can be modified upon discussion with the essay setter.

Relevant Courses

Essential: Advanced QFT, General Relativity.

Useful: Black Holes, String Theory, Gauge/Gravity Duality.

References

1. L. V. Iliesiu, S. Murthy and G. J. Turiaci, "Revisiting the logarithmic corrections to the black hole entropy," arxiv.org/abs/2209.13608.

2. G. J. Turiaci, "New insights on near-extremal black holes," arxiv.org/abs/2307.10423.

3. N. Banerjee and M. Saha, "Revisiting leading quantum corrections to near extremal black hole thermodynamics," arxiv.org/abs/2303.12415.

4. D. Kapec, Y. T. A. Law and C. Toldo, "Quasinormal corrections to near-extremal black hole thermodynamics," arxiv.org/abs/2409.14928.

5. M. J. Blacker, A. Castro, W. Sybesma and C. Toldo, "Quantum corrections to the path integral of near extremal de Sitter black holes," arxiv.org/abs/2503.14623.

84. Green-Schwarz Superstrings .... Dr R. A. Reid-Edwards

Supersymmetry is an important ingredient in quantum theories of extended objects, including strings. Starting with a bosonic string theory, describing the embedding of a worldsheet Σ into some target space $\mathcal{M}$

$$X : \Sigma \to \mathcal{M},$$

there are several ways of consistently introducing supersymmetry. The best understood and most widely used technique is the RNS construction, in which the woldsheet theory has local supersymmetry and Σ becomes a supermanifold. The subject of this essay will the Green-Schwarz construction, wherein the target space $\mathcal{M}$ has manifest supersymmetry.

Crucial to the consistency of the theory is a local fermionic symmetry on Σ called κ-symmetry. In this essay you will explore Green-Schwarz constructions in examples such as when $\mathcal{M}$ is Minkowski spacetime and $AdS_5 \times S^5$. You will investigate the geometric origins of κ-symmetry and understand why it works.

Time permitting, the essay could then study the connection between classical integrability of the models when $\mathcal{M} = AdS_5 \times S^5$ and the implications for the AdS/CFT correspondence.

Relevant Courses

Essential: Quantum Field Theory

Useful: String Theory, Supersymmetry

References

1. M. B. Green, J. H. Schwarz and E. Witten, "SUPERSTRING THEORY. VOL. 1: INTRODUCTION," 1988, ISBN 978-0-521-35752-4

2. G. Arutyunov and S. Frolov, "Foundations of the $\mathrm{AdS}_5 \times S^5$ Superstring. Part I," J. Phys. A **42** (2009), 254003 doi:10.1088/1751-8113/42/25/254003 [arXiv:0901.4937 [hep-th]].

3. R. R. Metsaev and A. A. Tseytlin, "Type IIB superstring action in AdS(5) x S**5 background," Nucl. Phys. B **533** (1998), 109-126 doi:10.1016/S0550-3213(98)00570-7 [arXiv:hep-th/9805028 [hep-th]].

4. I. Bena, J. Polchinski and R. Roiban, "Hidden Symmetries of the $\mathrm{AdS}_5 \times S^5$ Superstring," Phys. Rev. D **69** (2004), 046002 doi:10.1103/PhysRevD.69.046002 [arXiv:hep-th/0305116 [hep-th]].

85. The ADHM Equations .... Professor D. Skinner

The Yang-Mills equations on S^4 admit solitonic solutions known as instantons. Using little more than linear algebra (and a good deal of insight), Atiyah, Hitchin and (independently) Drinfeld & Manin were able to give an explicit construction of all instantons. The ADHM equations can be variously understood as describing the moduli space of holomorphic bundles on $\mathbb{CP}^3$, bound states of certain D-branes in string theory, and a moment map for the construction of hyperkähler manifolds.

Your essay will present Yang-Mills instantons and explain the role of the ADHM equations in their solution. It will then go on to explore one or other of the additional suggested topics.

Relevant Courses

Essential: Solitons, Instantons & Geometry

Useful: The Advanced Quantum Field Theory and String Theory courses may be useful.

References

1. S. Donaldson, *The ADHM construction of Yang-Mills instantons*, https://arxiv.org/abs/2205.08639

2. D. Tong, *TASI Lectures on Solitons*, http://www.damtp.cam.ac.uk/user/tong/tasi.html

3. K. Costello *Lecture Course on Mathematical Physics 777, April 2025*, https://pirsa.org/speaker/kevin-costello?page=0

86. Chiral Symmetry on the Lattice ....

Professor D. Tong

Massless fermions have an extra chiral symmetry that rotates the left- and right-handed parts independently. However, if space is made discrete – so it is a lattice – it can be hard to realise this symmetry, a fact closely connected to anomalies in quantum field theory. Recently it was understood how chiral symmetry emerges in a simple $d = 1 + 1$ lattice model and in more complicated models in $d = 3+1$ known as Weyl semi-metals. Surprisingly, the chiral symmetry does not commute with particle number and, combined, these give a structure known as the Onsager algebra. The purpose of this essay is to review these developments finding chiral symmetries on the lattice.

Relevant Courses

Essential: QFT, Advanced QFT, Standard Model.

Useful: Symmetries, Particles, and Fields.

References

1. An introduction to putting fermions on the lattice can be found in chapter 4 of my lecture notes on Gauge Theory: `https://www.damtp.cam.ac.uk/user/tong/gaugetheory.html`

2. Chiral symmetry for fermions in $d = 1+1$ dimensions was exhibited in Arkya Chatterjee, Sal Pace, and Shu-Heng. Shao, *Quantized Axial Charge of Staggered Fermions and the Chiral Anomaly*, Phys.Rev.Lett. 134 (2025) 2, 021601 arXiv:2409..12220

3. This was extended to 3d Weyl semi-metals by Lei Gioia and Ryan Thorngren, *Exact Chiral Symmetries of 3+1D Hamiltonian Lattice Fermions*, arXiv:2503.07708

87. Axion-like Particles Signatures at the Large Hadron Collider .... Professor M. Ubiali

Axion-like particles (ALPs) represent a significant class of hypothetical pseudo-scalar bosons that arise naturally in many extensions of the Standard Model (SM). Originally motivated by the strong CP problem through the QCD axion, ALPs more generally emerge as pseudo-Nambu-Goldstone bosons from the spontaneous breaking of approximate global symmetries at high energy scales.

Their weak couplings to SM fields and potentially light masses make them attractive candidates for physics beyond the SM, with important implications for both particle physics and cosmology. This essay explores how the Large Hadron Collider (LHC) can probe ALP physics through precision measurements and dedicated searches.

The first part of the essay, after discussing the original solution by Peccei and Quinn [4], will establish the theory framework for describing ALP interactions with SM particles, via an Effective Field Theory approach [5], systematically constructing the leading operators that couple ALPs to photons, gluons, and electroweak bosons. Following the formalism of Ref. [1], the linear and non-linear realisations of electroweak symmetry breaking will be explored, clarifying how different theoretical assumptions translate into distinct phenomenological predictions.

The second part will focus on the collider phenomenology of ALPs at the LHC, following Ref. [2]. The dominant production mechanisms for ALPs at hadron colliders will be explored, including resonant production, production through decays of heavy SM resonances such as the Higgs and Z bosons, and associated production channels. Explicit calculations of ALP decay rates to various final states using one-loop computations will be reproduced in detail. The essay will describe how these decay patterns depend on the ALP mass and coupling structure, and discuss the experimental signatures that result from different parameter space regions.

Optionally, the essay will survey the current experimental constraints from LHC searches and discuss the discovery potential of future runs and proposed colliders, following Ref. [3].

Relevant Courses

Essential: Quantum Field Theory
Useful: Particles and Symmetries, Advanced Quantum Field Theory

References

1. I. Brivio, M. B. Gavela, L. Merlo, K. Mimasu, J. M. No, R. del Rey and V. Sanz, *"ALPs Effective Field Theory and Collider Signatures,"* Eur. Phys. J. C **77** (2017) no.8, 572 doi:10.1140/epjc/s10052-017-5111-3, arXiv:1701.05379 [hep-ph].

2. M. Bauer, M. Neubert and A. Thamm, *"Collider Probes of Axion-Like Particles,"* JHEP **12** (2017), 044 doi:10.1007/JHEP12(2017)044, arXiv:1708.00443 [hep-ph]

3. M. Bauer, M.ZHeiles, M.ZNeubert and A.ZThamm, *"Axion-Like Particles at Future Colliders,"* Eur. Phys. J. C **79** (2019) no.1, 74 doi:10.1140/epjc/s10052-019-6587-9, arXiv:1808.10323 [hep-ph]

4. R. D. Peccei,
   "The Strong CP problem and axions,"
   Lect. Notes Phys. **741** (2008), 3-17 doi:10.1007/978-3-540-73518-2_1, arXiv:hep-ph/0607268

5. H. Georgi, D. B. Kaplan and L. Randall,
   "Manifesting the Invisible Axion at Low-energies,"
   Phys. Lett. B **169** (1986), 73-78, doi:10.1016/0370-2693(86)90688-X

88. Lattice Gauge Theory . . . .

Professor M. B. Wingate

Many physical phenomena are described by strongly coupled nonabelian gauge theories, where perturbation theory does not apply. Discretizing spacetime on a hypercubic lattice is an elegant way to regulate these quantum field theories. This essay concerns the lattice formulation of nonabelian gauge theory, primarily where the gauge group is $SU(N)$.

The essay should introduce gauge fields on the lattice as link variables which are elements of the gauge group. A connection should be made between Wilson's plaquette action and the Yang–Mills Lagrangian of the continuum [1, 2, 3].

Furthermore, a successful essay should discuss the following in some detail:

- How to couple gauge-field degrees-of-freedom to static charges in different irreducible representations of $SU(N)$ and the definition of a potential between these static charges. This topic will require a discussion or summary of the tensor method [4], perhaps not in general but at least for some examples.
- The strong coupling expansion and the question of confinement: area law vs. perimeter law with examples [1, 2, 3].
- If space permits, how to take the continuum limit (in the context of the renormalization group) [1, 2, 3].

Relevant Courses

Essential: Quantum Field Theory; Advanced Quantum Field Theory; Symmetries, Particles, and Fields

Useful: Statistical Field Theory (renormalization group)

References

1. M. Creutz, *Quarks, Gluons, and Lattices* (Cambridge Univ. Press, 1983).
2. I. Montvay and G. Münster, *Quantum Fields on a Lattice* (Cambridge Univ. Press, 1994).
3. C. Gattringer and C. B. Lang, *Quantum Chromodynamics on the Lattice* (Springer–Verlag, 2010).
4. T. P. Cheng and L. F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford Univ. Press, 1984), §4.3.

89. Spontaneous Symmetry Breaking and Topological Quantum Memory .. Professor B. Béri

Spontaneous symmetry breaking (SSB) is a central concept for understanding ordered phases in quantum many-body systems. For pure quantum states, SSB describes how a symmetric Hamiltonian can possess ground states that display long-range order, as revealed by order parameters. In topologically ordered systems, such as the surface code, a similar picture holds if we suitably generalise the concept of symmetries. A key feature of the surface code is its ability to serve as a quantum memory; this requires robustness against noise, a feature that can again be characterised through SSB, now formulated for a yet broader notion of symmetries encompassing mixed quantum states. The purpose of this essay is to provide a coherent account of these interlinked notions of SSB. The essay should include, but not necessarily be limited to, discussions on: SSB in pure states and the SSB viewpoint on topological order, strong-to-weak SSB in mixed states, including its illustration on the surface code subject to suitable dephasing, and the relation of these ideas to the recoverability of encoded quantum information and the associated random-bond Ising model from surface-code quantum error correction.

Relevant Courses

Useful: Part II Quantum Information and Computation, Part III Quantum Entanglement in Many-body Physics, Part III Topological Quantum Matter.

References

1. S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, 2011).
2. J. McGreevy, Whence QFT?, lecture notes (2022), mcgreevy.physics.ucsd.edu/s14/index.html.
3. D. Gaiotto, A. Kapustin, N. Seiberg, and B. Willett, Generalized Global Symmetries, JHEP **2015**, 172 (2015).
4. J. McGreevy, Generalized Symmetries in Condensed Matter, Annu. Rev. Cond. Mat. Phys. **14**, 57 (2023).
5. L. A. Lessa, R. Ma, J.-H. Zhang, Z. Bi, M. Cheng, and C. Wang, Strong-to-Weak Spontaneous Symmetry Breaking in Mixed Quantum States, PRX Quantum **6**, 010344 (2025).
6. E. Dennis, A. Kitaev, A. Landahl, and J. Preskill, "Topological Quantum Memory," J. Math. Phys. **43**, 4452 (2002).

90. Communicating Classical Information Through a Quantum Channel ... Professor N. Datta, Dr B. Bergh

An important task in quantum information theory is the communication of information through quantum channels which are inherently noisy. In this essay we focus on the task of sending classical information through quantum channels. A simpler variant of this task is when the encoding of classical information into input states for the channel is already fixed, which can then be thought of as communication through classical-quantum (c-q) channels. Inputs to such channels are letters from a classical alphabet, whereas the outputs are quantum states. The question of interest is to determine the *one-shot, ε-error capacity* of a c-q channel, i.e., the maximum number of bits that can be transmitted through the channel when the sender (Alice) is allowed to use the channel only once, and when the error incurred in the communication task is required to be below a fixed threshold value (say $\varepsilon \in (0,1)$). The task of sending classical information through a generic quantum channel can then be reduced to the problem for c-q channels by optimizing over all possible classical encodings.

If in addition, Alice and Bob are allowed to use pre-shared entanglement as an auxiliary resource, then the corresponding capacity has a higher value and is called the *one-shot, ε-error entanglement assisted capacity* of the channel.

Traditionally, these capacities were studied in the so-called *asymptotic memoryless setting*, in which multiple (n) identical uses of the channel were considered, with the requirement that the error vanishes in the asymptotic limit ($n \to \infty$), and the rate at which classical information can be sent through the channels is known as its classical capacity.

In this essay, the student is asked to review papers studying the problem of (entanglement-assisted) classical communication both in the asymptotic and one-shot settings. This will include the study of error exponents, that characterize the rate at which the errors incurred in the communication tasks decay as $n \to \infty$. For this, the student will need to familiarize themselves with recent technical developments in quantum information theory such as novel expressions for quantum divergences and integral representations of operators.

Relevant Courses

Essential: Part III Quantum Information Theory

References

1. L. Wang and R. Renner, 'One-Shot Classical-Quantum Capacity and Hypothesis Testing,' Phys. Rev. Lett. 108, 200501 (2012); https://arXiv:1007.5456v3.
2. H-C. Cheng, 'Simple and Tighter Derivation of Achievability for Classical Communication over Quantum Channels,' PRX Quantum 4, 040330, 2023; https://arXiv:2208.02132v2.
3. H-C. Cheng and P-C. Liu, 'Error Exponents for Quantum Packing Problems via An Operator Layer Cake Theorem,' https://arxiv.org/abs/2507.06232v3
4. S. Khatri, L. Lami, and M. M. Wilde, Principles of Quantum Communication Theory: A Modern Approach; https://markwilde.com/PQCT-khatri-lami-wilde.pdf

91. Quantum State Summoning and Relativistic Quantum Information .... Professor A. Kent

Summoning is an operational task that probes how quantum information can be distributed in spacetime subject to the twin constraints of relativistic causality and the no-cloning theorem. A quantum state is prepared by Bob at a start point and given, without a classical description, to Alice: both parties here refer to networks of collaborating agents in space-time. Later, Bob "summons" the state from Alice, who must return it at a point that depends on where it was summoned and on the pre-agreed description of the summoning task. In one interesting type of summoning task, a "call" is made at one of several specified call points, and the state must be *revealed* at the corresponding return point.

The essay will introduce the various versions of summoning tasks and the *no-summoning theorem*, then develop the necessary and sufficient conditions for single-call summoning and explain protocols that achieve summoning using teleportation and quantum error correction. It will evaluate efficiency issues and modern improvements, and survey extensions that broaden the conceptual picture—in particular, multi-call summoning (and the "paradox of choice"), unconstrained-input formulations, and spacetime analogues of secret sharing via localization/exclusion and state-assembly tasks.

Possible topics to be discussed include:

- Worked examples in 1+1 and 3+1 dimensions illustrating when summoning is possible, and explicit constructions that combine teleportation with codeword-stabilized (or related) error-correcting codes.
- Resource considerations: entanglement and classical communication requirements; why naive recursive schemes are inefficient and how later constructions improve scaling.
- Links to relativistic cryptography (e.g. bit commitment and location-dependent tasks) and to broader frameworks for quantum tasks in Minkowski space.
- The related task of entanglement sharing.
- Interesting unsolved problems related to summoning and worked examples.

The essay is intended to be largely expository, with rigorous statements of the main results and at least one complete construction worked through in detail. A reader who has taken Part III courses in quantum information and is familiar with special relativity should be able to follow the arguments and see how spacetime geometry constrains quantum protocols.

Relevant Courses

Essential: Part II *Quantum Information and Computation*; Part IA *Dynamics and Relativity* (or equivalent background in causal structure and light cones).

Useful: Part III *Quantum Information, Foundations and Gravity*; Part III *Quantum Information Theory*; . (Background in quantum error correction and teleportation will be helpful.)

References

1. A. Kent, "A no-summoning theorem in relativistic quantum theory," *Quantum Information Processing* **12**, 1023–1032 (2013). arXiv:1101.4612.

2. P. Hayden and A. May, "Summoning information in spacetime, or where and when can a qubit be?," *J. Phys. A: Mathematical and Theoretical* **49**, 175304 (2016). arXiv:1210.0913.

3. E. Adlam and A. Kent, "A quantum paradox of choice: More freedom makes summoning a quantum state harder," *Phys. Rev. A* **93**, 062327 (2016). arXiv:1509.04226.

4. A. Kent, "Unconstrained summoning for relativistic quantum information processing," *Phys. Rev. A* **98**, 062332 (2018). arXiv:1806.01736.

5. P. Hayden and A. May, "Localizing and excluding quantum information; or, how to share a quantum secret in spacetime," *Quantum* **3**, 196 (2019). arXiv:1806.04154.

6. A. Kent, “Quantum tasks in Minkowski space,” *Classical and Quantum Gravity* **29**, 224013 (2012). arXiv:1204.4022.

7. Y.-D. Wu, A. Khalid, and B. C. Sanders, “Efficient code for relativistic quantum summoning,” *New Journal of Physics* **20**, 063052 (2018). arXiv:1711.10594.

8. A. Cross, G. Smith, J. A. Smolin, and B. Zeng, “Codeword Stabilized Quantum Codes,” *(preprint)* (2007). arXiv:0708.1021.

9. C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, “Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels,” *Phys. Rev. Lett.* **70**, 1895–1899 (1993).

10. R. Cleve, D. Gottesman, and H.-K. Lo, “How to share a quantum secret,” *Phys. Rev. Lett.* **83**, 648–651 (1999).

11. Zahra Khanian, Dongjin Lee, Debbie Leung, Zhi Li, Alex May, Takato Mori, Stanley Miao, Farzin Salek, Jinmin Yi and Beni Yoshida, “ Entanglement sharing schemes”, https://arxiv.org/abs/2509.21462, (2025)

92. Topological State Sums and String Nets .... Dr A. Turzillo

The study of topological quantum phases of matter has benefited from its connections to topological field theory (TFT). A partial dictionary between the two allows us to translate concepts from one picture to the other, where new insights can be gleaned. The correspondence is more of an intuition than a rigorous statement (and recent studies have identified exceptions to it), but there are settings where it can be made precise. This essay aims to understand the connection between topological phases and TFT through concrete models.

On the phases side, we will study two dimensional exactly solvable lattice models called string nets [1,2]. These local Hamiltonians generalize quantum double models (such as the toric code) [3] and have ground states that are superpositions of closed strings. The ends of open strings are emergent excitations called anyons, whose fusion and braiding capture topological invariants.

On the field theory side, we will study state sum constructions of TFTs. The idea of a state sum is to write the partition function as products of local amplitudes on simplices that triangulate the spacetime manifold; then, by imposing local relations on these amplitudes, the partition function can be made independent of the triangulation; that is, made to be topological.

It may be useful to review the Fukuma-Hosono-Kawai state sum construction of TFTs in two spacetime dimensions [4] and its relation to one dimensional lattice models [5]. Then moving up to three spacetime dimensions, we will begin with Dijkgraaf-Witten theory [6] and quantum double models [7] before turning to Turaev-Viro-Barrett-Westbury theories [6,7] and string net models. Some relevant work in this direction includes Refs. [8,9].

This is a broad (and deep) topic area, so an understanding of some subset of these topics – string net models, state sum models, or their relation in a specific setting (e.g. Dijkgraaf-Witten and quantum doubles) – will be adequate for the completion of this essay.

Relevant Courses

Useful: Topological Quantum Matter, Quantum Entanglement in Many-Body Physics, and algebraic topology at the undergraduate level

References

1. M. Levin, X.-G. Wen, *String-net condensation: A physical mechanism for topological phases*, Phys. Rev. B 71 045110 (2005).
2. C.-H. Lin, M. Levin, F. J. Burnell, *Generalized string-net models: A thorough exposition*, Phys. Rev. B 103 195155 (2021).
3. Kitaev, A. Yu. (2003-01-01). *Fault-tolerant quantum computation by anyons*, Annals of Physics. 303 (1): 2–30 (2003).
4. S. Fukuma, A. Hosono, H. Kawai, *Lattice Topological Field Theory in two dimensions*, Comm. Math. Phys. 161, no. 1, 157–175 (1994).
5. A. Kapustin, A. Turzillo, M. You, *Topological Field Theory and Matrix Product States*, Phys. Rev. B 96 075125 (2017).
6. R. Dijkgraaf, E. Witten, *Topological gauge theories and group cohomology*, Communications in Mathematical Physics 129 393–429 (1990).
7. A. Bullivant, C. Delcamp, *Tube algebras, excitations statistics and compactification in gauge models of topological phases*, J. High Energ. Phys. 2019, 216 (2019).
8. J. Barrett, B. Westbury, *Invariants of piecewise-linear 3-manifolds*, Trans. Amer. Math. Soc. 348, no. 10, 3997–4022 (1996).
9. J. Barrett, B. Westbury, *Spherical Categories*, Adv. Math. 143:357-375, (1999).
10. A. Kirillov Jr, *String-net model of Turaev-Viro invariants*, arXiv: 1106.6033 (2011).
11. C. Delcamp, N. Ishtiaque, *Symmetry topological field theory and non-abelian Kramers-Wannier dualities of generalised Ising models*, arXiv:2408.06074 (2024).

93. Symmetry Protected Topological Phases of Free and Interacting Electrons
Dr. A. Turzillo

A large class of topological quantum phases are realized by systems of non-interacting electrons, with their universal features captured by the topologies of their electron bands and classified by K-theory [1,2]. These phases include many topological insulators and, more generally, symmetry protected topological (SPT) phases, defined by the property that their topological features rely on “protection” by a symmetry, without which the phase is disordered.

In the presence of strong effective interactions, electron band theory breaks down and different approaches such as topological field theory and tensor networks are needed. Interacting SPT phases have been classified by fermionic variants of group cohomology [3,4,5] and cobordism [6].

A natural question is how SPT phases of free electrons are related to their interacting counterparts. Some free phases are known to be destabilized by interactions; for example, a stack of eight Majorana chains (1D topological superconductors) is connected to the disordered phase by deformations through the interacting parameter space [7]. On the other hand, strong interactions open the possibility of new SPT phases, whose topological features have no realization in free electron systems. These two phenomena have been studied in part [8].

This essay aims to understand SPT phases of electronic systems in both the free and interacting settings and to develop the theoretical connections between them.

Relevant Courses

Useful: Topological Quantum Matter, Quantum Entanglement in Many-Body Physics

References

1. A. Kitaev, *Periodic table for topological insulators and superconductors*, Advances in Theoretical Physics: Landau Memorial Conference, AIP Conference Proceedings 1134, pp. 22-30 (2008) [arXiv:0901.2686].
2. A. P. Schnyder, S. Ryu, A. Furusaki, A. W. W. Ludwig, *Classification of Topological Insulators and Superconductors*, ibid. pp. 10-21 (2008) [arXiv:0905.2029].
3. X. Chen, Z.-C. Gu, Z.-X. Liu, X.-G. Wen, *Symmetry protected topological orders and the group cohomology of their symmetry group*, Phys. Rev. B 87, 155114 (2013).
4. Q.-R. Wang, Z.-C. Gu, *Construction and classification of symmetry protected topological phases in interacting fermion systems*, Phys. Rev. X 10, 031055 (2020).
5. A. Turzillo, M. You, *Fermionic Matrix Product States and One Dimensional Short Range Entangled Phases with Anti-Unitary Symmetries*, Phys. Rev. B 99 035103 (2017).
6. A. Kapustin, R. Thorngren, A. Turzillo, Z. Wang, *Fermionic Symmetry Protected Topological Phases and Cobordisms*, J. High Energ. Phys. 2015, 52 (2014).
7. L. Fidkowski, A. Kitaev, *The effects of interactions on the topological classification of free fermion systems*, Phys. Rev. B 81, 134509 (2010).
8. Y.-A. Chen, A. Kapustin, A. Turzillo, M. You, *Free and interacting short-range entangled phases of fermions: Beyond the tenfold way*, Phys. Rev. B 100, 195128 (2019).

94. Quantum Groups and Generalized Symmetries .... Professor F. Verstraete

A very active research domain in theoretical physics is the study of generalized symmetries. In this essay, generalized symmetries will be studied for one-dimensional quantum lattice systems. This is relevant as those generalized symmetries are the key in identifying all possible phases of (topological) quantum matter. It is also of strong interest from the mathematical point of view, as those generalized symmetries involves the representation theory of fusion categories (generalizing the representation theory of groups). It turns out that tensor networks, and more specifically matrix product operators, provide a natural framework for developing such a representation theory.

The specific topic of this essay is to relate the representation theory of fusion categories to the one of quantum groups - as the conjecture is that quantum group symmetries can be understood as the continuum analogue of categorical synmmetries.

Relevant Courses

Essential: Quantum Entanglement in Many-Body Physics (part III)

Useful: Quantum Information Theory (part III)

References

1. Matrix product operator symmetries and intertwiners in string-nets with domain walls, L Lootens, J Fuchs, J Haegeman, C Schweigert, F Verstraete, SciPost Physics 10 (3), 053

2. Matrix quantum groups as matrix product operator representations of Lie groups, R Couvreur, L Lootens, F Verstraete, arXiv preprint arXiv:2202.06937

3. Trading Mathematical for Physical Simplicity: Bialgebraic Structures in Matrix Product Operator Symmetries, Y Liu, A Molnar, XQ Sun, F Verstraete, K Kato, L Lootens, arXiv preprint arXiv:2509.03600

95. Riemannian Black Hole Uniqueness Conjecture .... Professor M. Dunajski

A combination of results of Hawking, Carter, Israel and D. Robinson implies that, under suitable technical assumptions, any vacuum static or stationary asymptotically flat black hole exterior region must be isometric with the Schwarzschild or the Kerr solution.

A similar result was conjectured to hold in the Riemannian context, with the Schwarzschild and Kerr gravitational instantons being unique asymptotically flat (AF) and Ricci–flat Riemannian metrics. The recent examples of AF Ricci–flat metrics constructed by Chen and Teo disproved this conjecture.

The essay will explore the subject, starting off with a review of asymptotically flat and asymptotically locally flat gravitational instantons, and comparison with asymptotically locally Euclidean gravitational instantons (where the uniqueness theorem is known to hold). In the second part of the essay the Chen–Teo solutions will be explored together with a detailed discussion of either their rod structure.

Relevant Courses

Essential: General Relativity, Black Holes

Useful: Solitons, Instantons, and Geometry

References

1. Aksteiner, S. and Andersson, L. (2021) Gravitational Instantons and special geometry. `arXiv:2112.11863`

2. Chen, Y. and Teo, E. (2011) A new AF gravitational instanton. Physics Letters **B 703** 359–362.

3. Chen, Y. and Teo, E. (2015) Five-parameter class of solutions to the vacuum Einstein equations. Phys. Rev. **D 91**, 124005.

4. Dunajski, M. (2024) Solitons, Instantons, and Twistors (2nd Edition) *Oxford Graduate Texts in Mathematics*, Oxford University Press. (Chapter 10).

5. Lapedes, A. S. (1980) Black-hole uniqueness theorems in Euclidean quantum gravity Phys. Rev. **D 22**, 1837.

96. Causality and Stability of Relativistic Viscous Hydrodynamics .... Dr L. Gavassino

Relativistic viscous hydrodynamics describes the behaviour of fluids near local thermodynamic equilibrium when both dissipative processes and relativistic effects are relevant. It provides an effective macroscopic description for a wide range of physical systems, from the quark–gluon plasma produced in heavy-ion collisions to accretion flows around compact astrophysical objects.

Formulating a consistent theory of relativistic dissipative fluids, however, is far from straightforward. The earliest relativistic generalizations of the Navier–Stokes equations, proposed by Eckart [1] and and Landau & Lifshitz [2], were later shown to suffer from severe pathologies. Linear perturbation analyses demonstrated that these formulations admit exponentially growing modes, indicating dynamical instabilities [3]. Furthermore, they permit the propagation of signals at superluminal speeds, thereby violating relativistic causality. It was subsequently recognized that these two issues are deeply intertwined: any acausal dissipative theory is necessarily unstable according to some inertial observer [4].

In recent years, the Bemfica–Disconzi–Noronha–Kovtun (BDNK) framework [5,6] has provided a reformulation of first-order relativistic hydrodynamics that is both causal and stable. This is achieved through a careful redefinition of the hydrodynamic variables and by allowing a more general class of constitutive relations compatible with relativistic thermodynamics.

The essay should explain the concept of hydrodynamic frame, the link between acausality and instabilities in relativistic fluid theories, and how the BDNK approach resolves these problems. Possible directions of discussion may include: the analysis of linear perturbations and characteristic speeds, the role of frame choices in determining causality, the concept of hydrodynamic frame transformation, the consistency of BDNK with the second law of thermodynamics [7], and applications to astrophysical or high-energy systems.

Relevant Courses

Essential: General Relativity

Useful: Some familiarity with the non-relativistic Navier-Stokes equations will be very helpful.

References

1. C. Eckart, *Phys. Rev.* **58**, 919 (1940).
2. L. Landau and E. Lifshitz, *Fluid Mechanics*, v. 6 (Elsevier Science, 2013).
3. W. Hiscock and L. Lindblom, *Physical Review D: Particles and Fields* **31**, 725 (1985).
4. L. Gavassino, *Phys. Rev. X* **12**, 041001 (2022).
5. P. Kovtun, *Journal of High Energy Physics* **2019**, 34 (2019).
6. F. S. Bemfica, M. M. Disconzi, and J. Noronha, *Phys. Rev. X* **12**, 021044 (2022).
7. L. Gavassino, M. Antonelli, and B. Haskell, *Physical Review D* **102** (2020).

97. Cosmological Billiards .... Professor S. A. Hartnoll

It was argued many decades ago that close to a spacelike singularity Einstein's equations would at once simplify and become choatic [1]. This behaviour was seen explicitly in a 'minisuperspace' model developed by Misner [2]. More recent systematic descriptions of this dynamics are known as the cosmological billiard, because the Einstein equations are mapped onto a chaotic hyperbolic billiard problem [3, 4].

This essay will have two parts. The first part will describe the near-singularity limit of Einstein's equations and the map onto a billiard problem. Attention should be paid to difference between minisuperspace and full models. The second part should treat at least one more advanced topic. This is at the discretion of the student, examples of appropriate topics are the chaotic nature of the dynamic map between Kasner eras [5, 6], canonical quantisation of billiard dynamics and arithmetic quantum chaos [7,8], connections to extended symmetry algebras and higher dimensions [9] and numerical evidence for BKL dynamics [10]. These references are intended as possible entry points to the relevant literature.

Relevant Courses

Essential: General Relativity.

Useful: Canonical Gravity. Symmetries, Particles and Fields. Quantum Field Theory.

References

1. V. A. Belinsky, I. M. Khalatnikov and E. M. Lifshitz, "Oscillatory approach to a singular point in the relativistic cosmology," Adv. Phys. **19** (1970), 525-573.
2. C. W. Misner, "Mixmaster universe," Phys. Rev. Lett. **22** (1969), 1071-1074.
3. T. Damour, M. Henneaux and H. Nicolai, "Cosmological billiards," Class. Quant. Grav. **20** (2003), R145-R200.
4. V. Belinski and M. Henneaux, "The Cosmological Singularity," Cambridge Univ. Pr., 2017.
5. J. D. Barrow, Chaotic behaviour in general relativity, Phys. Rep. 85, 1–49, 1982.
6. I. M. Khalatnikov, E. M. Lifshitz, K. M. Khanin, L. N. Shchur and Y. G. Sinai, On the stochasticity in relativistic cosmology, J. Stat. Phys. 38, 97–114, 1985.
7. R. Graham, Chaos and quantum chaos in cosmological models, Chaos, Solitons & Fractals 5, 1103–1122, 1995.
8. P. Sarnak, Arithmetic quantum chaos, Blythe Lectures, https://publications.ias.edu/node/527, 1993.
9. A. J. Feingold, A. Kleinschmidt and H. Nicolai, Hyperbolic Weyl groups and the four normed division algebras, J. Algebra 322, 1295–1339, 2009, [arXiv:0805.3018 [math.RT]].
10. D. Garfinkle and F. Pretorius, Spike behavior in the approach to spacetime singularities, Phys. Rev. D 102, 124067, 2020, [arXiv:2010.01399 [gr-qc]].

98. Dark Energy .... Professor E. Pajer

Dark energy is the name we give to whatever is causing the expansion of the Universe to speed up at late times. A convenient way to describe it is through a pressure–to–energy–density ratio w, which for a perfectly constant "cosmological constant" would be $w = -1$, but in more general scenarios may differ slightly from -1 or vary with time. The stakes are high: if dark energy is truly a constant vacuum energy, then its observed tiny value raises deep questions in high-energy physics about why quantum fields contribute so little to the energy of empty space; if instead it arises from new, very light fields or from a change to gravity on cosmic distances, then our understanding of fundamental interactions and general relativity would need to be extended. Observationally, dark energy is probed by distance and growth measurements—using, for example, type-Ia supernovae, the cosmic microwave background, and the clustering and weak lensing of galaxies. Recently, the DESI DR2 publications have delivered sharper distance measurements from millions of galaxies and quasars, providing a timely dataset to test whether the data prefer a strictly constant dark energy or allow room for gentle time evolution in w. This essay should give a clear and critical account of the current status of dark energy, balancing *theoretical modelling* and *observational constraints*. Either component may dominate the discussion, but neither should account for less than 20% of the overall essay.

The essay may include:

- A concise *review of dark energy*, with emphasis on *dynamical dark energy* and *modified gravity*, contrasting these with the cosmological constant of ΛCDM.
- A brief but sharp discussion of the *cosmological constant problem* (magnitude, radiative stability/naturalness, "why now?").
- An *observational overview* (choose 2–3 probes to treat in depth), explaining what current data imply for Λ vs. dynamical models. Suitable probes include BAO (with DESI DR2 highlights), SNe Ia, CMB (late-time geometry/ISW), and LSS/weak lensing (growth, μ/Σ or α_i forecasts).
- A focused section on one of the following topics::
  1. *Effective Field Theory of Dark Energy (EFT-DE):* the main theoretical framework to describe time evolution in general relativity and how it can be used to model very general gravitational dynamics in field theory.
  2. *Dynamical Dark Energy in light of recent surveys:* use w_0–w_a or model-informed parameterizations to interpret current constraints (e.g. DESI DR2 BAO combined with CMB and SNe) and assess whether data prefer time-variation of $w(z)$.
  3. Specific *models of dynamical dark energy* (e.g. quintessence, k-essence, Horndeski, DHOST) and their observational status.
  4. *Modifications of general relativity* on cosmological scales and the tight constraints they are subject to from theoretical consistency and observations.

The essay should be written at a level accessible to a Part III student who has attended all relevant courses in the program.

The selection of references below is indicative. Students may choose to focus on one review or more and they may include additional primary sources, provided they are properly integrated into the essay's narrative.

Relevant Courses

Essential: General Relativity.

Useful: Cosmology, QFT, AQFT, Field theory in Cosmology.

References

Reviews:

1. Polchinski, Joseph. "The cosmological constant and the string landscape." arXiv preprint hep-th/0603249 (2006).

2. Weinberg, Steven. "The cosmological constant problem." Reviews of modern physics 61.1 (1989): 1.

3. Joyce, Austin, et al. "Beyond the cosmological standard model." Physics Reports 568 (2015): 1-98.

4. Clifton, Timothy, et al. "Modified gravity and cosmology." Physics reports 513.1-3 (2012): 1-189.

5. Huterer, Dragan, and Daniel L. Shafer. "Dark energy two decades after: observables, probes, consistency tests." Reports on Progress in Physics 81.1 (2017): 016901.

6. Tsujikawa, Shinji. "Modified gravity models of dark energy." Lectures on cosmology: Accelerated Expansion of the Universe. Berlin, Heidelberg: Springer Berlin Heidelberg, 2010. 99-145.

EFT of Dark Energy:

1. Gubitosi, Giulia, Federico Piazza, and Filippo Vernizzi. "The effective field theory of dark energy." Journal of Cosmology and Astroparticle Physics 2013.02 (2013): 032.

2. Bellini, Emilio, and Ignacy Sawicki. "Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity." Journal of Cosmology and Astroparticle Physics 2014.07 (2014): 050.

3. Frusciante, Noemi, and Louis Perenon. "Effective field theory of dark energy: A review." Physics Reports 857 (2020): 1-63.

Observations:

1. Abdul-Karim, M., et al. "DESI DR2 Results II: Measurements of Baryon Acoustic Oscillations and Cosmological Constraints." arXiv preprint arXiv:2503.14738 (2025).

2. Gu, Gan, et al. "Dynamical Dark Energy in light of the DESI DR2 Baryonic Acoustic Oscillations Measurements." arXiv preprint arXiv:2504.06118 (2025).

99. Quasinormal Modes of Black Holes ....
Professor H. S. Reall

Quasinormal modes are linear perturbations of a black hole that describe damped oscillations of the hole. A quasinormal mode has a definite, complex, frequency. Any black hole has a characteristic spectrum of quasinormal frequencies. If a hole is perturbed then its quasinormal modes describe how it relaxes back to equilibrium. This is observable in the gravitational waves produced by a black hole merger.

The essay should explain what quasinormal modes are and why they are relevant for the late time behaviour of a perturbed black hole. It should then discuss some applications. These might include: quasinormal modes of the Kerr black hole (via the Teukolsky equation); methods for calculating quasinormal modes e.g. WKB or continued fractions; the role of quasinormal modes in the AdS/CFT correspondence [3]; a rigorous mathematical approach to quasinormal modes [4]; the connection between quasinormal modes and strong cosmic censorship [5] or testing General Relativity through observations of quasinormal modes in gravitataional waves [6].

Relevant Courses

Essential: General Relativity, Black Holes.

References

1. K. Kokkotas and B. Schmidt "Quasinormal modes of stars and black holes", Living Reviews in Relativity 2 (1999), 2.
2. E. Berti, V. Cardoso and A. Starinets, "Quasinormal modes of black holes and black branes", Class. Quant. Grav. 26, 163001 (2009)
3. G. Horowitz and V. Hubeny, "quasinormal modes of AdS black holes and the approach to thermal equilibrium", Phys. Rev. D62, 024027 (2000)
4. C. Warnick, "on quasinormal modes of asymptotically anti-de Sitter black holes", Commun. Math. Phys. 333, 959 (2015)
5. O. Dias, H. Reall and J. Santos, "strong cosmic censorship: taking the rough with the smooth", JHEP 1810,001 (2018)
6. LIGO collaboration, "GW250114: Testing Hawking's Area Law and the Kerr Nature of Black Holes", Phys. Rev. Lett. 135, 111403 (2025)

100. Perturbations of Higher-Dimensional Black Holes ....
Professor J. E. Santos

Four spacetime dimensions yield an elegant and relatively constrained picture of black hole physics. When the dimensionality is increased, however, that simplicity breaks down: higher-dimensional black holes are not uniquely fixed by their conserved charges, they can possess multiple distinct horizon topologies, and many solutions show dynamical instabilities across wide swathes of parameter space. These differences are not merely formal - they underpin concrete examples where the weak cosmic censorship conjecture is called into question.

This essay should give a thorough, up-to-date review of work on the stability of asymptotically flat black holes in more than four spacetime dimensions. It should explain the key ideas and techniques used to probe stability (linear perturbation analysis, numerical evolutions, thermodynamic and mode analyses), summarise recent discoveries, and discuss their implications for weak cosmic censorship and higher-dimensional gravity. The presentation will avoid unnecessary technicalities so that students taking Part III courses in relativity or field theory can follow the arguments and appreciate both the physical intuition and the main open questions.

Relevant Courses

Essential: General Relativity and Black Holes

References

1. Roberto Emparan and Harvey S. Reall, "Black Holes in Higher Dimensions", Living Rev. Rel. 11 (2008), 6.
2. G. T. Horowitz, *et. al*, "Black Holes in Higher Dimensions", Cambridge University Press, 2012.
3. Ó. J. C. Dias, G. S. Hartnett and J. E. Santos, "Quasinormal modes of asymptotically flat rotating black holes," Class. Quant. Grav. **31** (2014) no.24, 245011.

101. Vector Charge Theory for Granular Materials – Is Skepticism Warranted? ....

Professor M. E. Cates

Vector charge theory (VCT) is a generalization of classical electromagnetism whose excitations include 'fractons' [1]. The latter are objects that cannot move freely but only in a restricted fashion, such as having to move in the direction of their (vector) charge. Recently, a connection was proposed between the electrostatics sector of VCT and the equations of static equilibrium for certain types of granular packings [2,3]. However, some of the arguments appear questionable: overall, the value of the VCT/granular analogy is not yet clear. The primary goal of this essay is to assess this work with a critical eye, and reach a reasoned view of how seriously it should be taken. This will involve surveying the older granular literature to explore earlier models of isostatic packings, such as [4,5]. Following on from (or perhaps *en route* to) the assessment of the VCT/granular analogy, several avenues are possible. In particular, the essay could address other recent innovative papers on granular elasticity, which range from extremely elegant [6], to distinctly questionable [7]. Along the way, there may be the opportunity for original research... but this will depend on finding some solid ground on which to build!

Relevant Courses

Essential: There are no essential courses for this essay.

Useful: Statistical Field Theory and Noisy Mechanics could be helpful, although neither addresses granular systems directly.

References

1. M. Pretko, Generalized electromagnetism of subdimensional particles: A spin liquid story. Phys. Rev. B 96, 035119 (2017)

2. J. N. Nampoothiri, Y. Wang, K. Ramola, J. Zhang, S. Bhattacharjee, and B. Chakraborty, Emergent elasticity in amorphous solids, Phys. Rev. Lett. 125, 118002 (2020)

3. J. N. Nampoothiri, M. D'Eon, K. Ramola, and B. Chakraborty, Tensor electromagnetism and emergent elasticity in jammed solids. Phys. Rev. E 106, 065004 (2022)

4. M. E. Cates, J. P. Wittmer, J.-P. Bouchaud, and P. Claudin, Jamming, force chains, and fragile matter, Phys. Rev. Lett. 81,1841 (1998)

5. R. C. Ball, and R. Blumenfeld, Stress field in granular systems: Loop forces and potential formulation, Phys. Rev. Lett. 115505 (2002)

6. E. Di Giuli, Field theory for amorphous solids, Phys. Rev. Lett. 121, 118001 (2018)

7. H. Charan, M. Moshe, and I. Procaccia, Anomalous elasticity and emergent dipole screening in three-dimensional amorphous solids, Phys. Rev. E 107, 055005 (2023)

102. Pattern Formation in Mixtures of Active and Passive Particles .... Professor R. L. Jack

Brownian motion of particles or microorganisms is *passive*, in that particles move only as a result of random collisions with their surrounding fluid. Recently, there has been a lot of work on *active* particles, which exploit sources of energy to self-propel themselves. When active and passive particles are mixed together, new phenomena can arise, including the formation of dynamical patterns, such as travelling clusters of particles.

Mathematical models of this effect can either describe the stochastic motion of interacting particles, or their collective behaviour at hydrodynamic level, via partial differential equations. The essay will discuss model systems in which this behaviour can be analysed at both levels of description. Subject material could include the nature of the resulting patterns, and the relationships between particle-based and hydrodynamic descriptions.

Relevant Courses

Useful: Noisy Mechanics

References

1. J Mason, RL Jack and M Bruna, *Dynamical patterns and nonreciprocal effective interactions in an active-passive mixture through exact hydrodynamic analysis*, Nature Comms. **16**, 6017 (2025).

2. Z You, A Baskaran and MC Marchetti, *Nonreciprocity as a generic route to traveling states*, Proc. Nat. Acad. Sci. **117**, 19767 (2020).

3. S Saha, J Agudo-Canalejo, and R Golestanian, *Scalar Active Mixtures: The Nonreciprocal Cahn-Hilliard Model*, Phys. Rev. X **10**, 041009 (2020).

4. J Stenhammar, R Wittkowski, D Marenduzzo and ME Cates, *Activity-Induced Phase Separation and Self-Assembly in Mixtures of Active and Passive Particles*, Phys Rev. Lett. **114**, 018301 (2015).

103. The Scallop Theorem .... Professor E. Lauga

In his seminal 1977 paper, Edward Purcell introduced the foundation of what is now known, in the context of microorganism locomotion, as the Scallop Theorem. In essence, the theorem states that a swimmer attempting to move through a Newtonian fluid at low Reynolds number, using a reciprocal motion (one whose sequence of shapes is identical under time-reversal symmetry) cannot achieve net propulsion. A formal proof of the theorem was provided some 35 years later. In this essay, students will outline the biophysical context that motivated the original discussion of the theorem, develop the physical intuition underlying it, present its formal proof in detail, discuss experimental evidence (either confirmations or violations), and explore mechanisms by which the constraints imposed by the theorem may be circumvented in real or practical systems. They will also discuss its applications to robotic designs (i.e. artificial swimmers). In addition, eager students are encouraged to investigate modern extensions or alternative formulations of the theorem in complex (non-Newtonian) fluids, where weaker or modified versions may arise. The ideal candidate will have a strong interest in problems at the intersection of applied mathematics, biology and fluid mechanics.

Relevant Courses

Essential: Vector Calculus (1A), Fluid dynamics (1B and II), Methods (1B), or equivalent for students who did not study in Cambridge

Useful: Perturbation Methods (III), Biological Flows (III), Slow Viscous Flows (III).

References

1. E. M. Purcell "Life at low Reynolds number" American Journal of Physics 45, 3 (1977).
2. E. Lauga "Life around the scallop theorem" Soft Matter 7, 3060 (2011).
3. K. Ishimoto and Y. Michio "A coordinate-based proof of the scallop theorem" SIAM Journal on Applied Mathematics 72, 1686 (2012).
4. T. Qiu et al. "Swimming by reciprocal motion at low Reynolds number" Nature Communications 5, 5119 (2014).
5. M. Hubert et al. "Scallop theorem and swimming at the mesoscale" Physical Review Letters 126, 224501 (2021).

104. General Relativity as a Dynamical PDE .... Dr R. Teixeira da Costa

Einstein's equations in General Relativity can be seen a second order PDE whose solution is a Lorentzian manifold. This PDE has no concrete character until one breaks the diffeomorphism invariance of the equations. In doing so with a clever choice of (wave) gauge, Choquet-Bruhat [3] showed that Einstein's equations have a wave-like character and are locally well-posed as dynamical equations. This classical result has been revisited and improved [2,4]. As in the case of wave equations, there is a natural notion of maximal development predicted by the initial data (i.e. globally hyperbolic) [1,6]. See also the book [5].

The goal of this essay is to discuss the classical proof of existence and uniqueness of the maximal global development of initial data for the Einstein equations.

Relevant Courses

Essential: Part III Analysis of PDE ; *either* Part III Differential Geometry *or* Part III General Relativity

References

1. Choquet-Bruhat, Y., & Geroch, R. (1969). Global aspects of the Cauchy problem in General Relativity. Communications in Mathematical Physics, 14(4), 329–335. https://doi.org/10.1007/BF01645389

2. Fischer, A. E., & Marsden, J. E. (1972). The Einstein evolution equations as a first-order quasi-linear symmetric hyperbolic system, I. Communications in Mathematical Physics, 28(1), 1–38. https://doi.org/10.1007/BF02099369

3. Fourès-Bruhat, Y. (1952). Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. Acta Mathematica, 88, 141–225. https://doi.org/10.1007/BF02392131

4. Hughes, T. J. R., Kato, T., & Marsden, J. E. (1977). Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. Archive for Rational Mechanics and Analysis, 63(3), 273–294. https://doi.org/10.1007/BF00251584

5. Ringström, H. (2009). The Cauchy Problem in General Relativity. European Mathematical Society Publishing House. https://doi.org/10.4171/053

6. Sbierski, J. (2016). On the Existence of a Maximal Cauchy Development for the Einstein Equations: a Dezornification. Annales Henri Poincaré, 17(2), 301–329. https://doi.org/10.1007/s00023-015-0401-5

105. Regularity Theory of Energy Minimising Maps between Riemannian Manifolds ....

Professor N. Wickramasekera

Given smooth Riemannian manifolds (M^n, g) and (N^m, h), consider a C^1 map $u : M \to N$. Associated with u there is a natural notion of *Dirichlet energy* of u, defined to be the integral over M of the trace (with respect to the metric g) of the pull back $u^\star h$ of the metric h. Critical points of Dirichlet energy are called harmonic maps, among which are local minimisers of Dirichlet energy. The existence theory of Dirichlet energy minimising maps requires relaxing the regularity assumptions on u and working in an appropriately defined (non-linear) Sobolev space of maps. In the simplest case $N = \mathbb{R}^m$ with the Euclidean metric, it is well-known, and easy to show, that such maps satisfy Laplace's equation weakly, and so their regularity is easy to establish. For general N however the Euler-Lagrange equations are nonlinear and more care is required. In fact interestingly, in contrast to the linear case (namely, the case $N = \mathbb{R}^m$), in the general case a minimiser u can have a small, non-empty singular set (defined as the complement of the set of points in M where u is regular, i.e. smooth).

This essay concerns the regularity theory of locally energy minimising maps into compact Riemannian manifolds N, and in particular bounding the size of the singular set of such a map from above in some appropriate way. Schoen and Uhlenbeck proved a regularity result in 1982 (extending the work of Giaquinta and Guisti around the same time) which in the first instance implies that a locally energy minimising map is regular in the interior away from a closed set of Hausdorff dimension at most $n - 2$. With some more work, they improved this dimension bound on the singular set to $n-3$, which is the optimal bound independent of N. Their method relied on characterising regular points as the points where a certain energy density falls below a fixed threshold.

The essay should start by explaining the ideas behind the Schoen-Uhlenbeck regularity theorem, and how the Hausdorff dimension bound $n - 2$ is deduced, by following [1]. Then it should proceed by exploring the technical lemmas and proofs in depth, discussing also examples. If desired, the proof of the sharp dimension bound $n - 3$ can be explored, again following [1]. Other standard relevant background can be found in [2].

Relevant Courses

Essential: Elliptic PDEs

Useful: Differential Geometry, Analysis of PDEs

References

1. L. Simon, *Theorems on Regularity and Singularity of Energy Minimising Maps*, Birkhäuser (1996).

2. D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag (1983).

106. Cannon–Thurston Maps ....
Dr M. Kudlinska

Let M be a closed hyperbolic 3-manifold which admits the structure of a fiber bundle over the circle with fiber Σ. Such manifolds arise as mapping tori of pseudo-Anosov diffeomorphisms of closed surfaces [3].

The inclusion $\Sigma \hookrightarrow M$ lifts to an equivariant inclusion of the universal covers,

$$\widetilde{\Sigma} \simeq \mathbb{H}^2 \to \mathbb{H}^3 \simeq \widetilde{M}.$$

Remarkably, Cannon–Thurston showed that the inclusion map extends to a continuous map of the compactifications $\mathbb{B}^2 = \mathbb{H}^2 \cup \mathbb{S}^1_\infty$ and $\mathbb{B}^3 = \mathbb{H}^3 \cup \mathbb{S}^2_\infty$ [1]. Since the fundamental group $\pi_1(\Sigma)$ embeds as a normal subgroup of $\pi_1(M)$, a classical argument shows that $\pi_1(\Sigma)$ and $\pi_1(M)$ have the same limit sets in $\mathbb{H}^3 \cup \mathbb{S}^2_\infty$. It follows that the induced map of the boundaries

$$\partial\widetilde{\Sigma} \simeq \mathbb{S}^1 \to \mathbb{S}^2 \simeq \partial\widetilde{M}$$

is a continuous surjection from the circle onto the 2-sphere, i.e. a *Peano curve.*

The aim of this essay is to give an account of the construction of Cannon–Thurston in [1]. This involves a description of pseudo-Anosov diffeomorphisms of closed surfaces and the associated invariant measured laminations, and how such measured laminations give rise to totally geodesic subspaces in $\widetilde{M}$.

An excellent essay could go on to discuss generalisations of Cannon–Thurston maps to the setting of word hyperbolic groups and their associated Gromov boundaries, giving an overview of the result of Mj that proves if G is word hyperbolic and $H \trianglelefteq G$ is a normal hyperbolic subgroup then the embedding $H \hookrightarrow G$ extends to a continuous embedding

$$H \cup \partial H \to G \cup \partial G$$

where ∂H and ∂G are the Gromov boundaries of H and G, respectively.

Relevant Courses

Essential: Part II Algebraic Topology, Part IB Geometry

Useful: Part II Differential Geometry, Part III Geometric Group Theory (essential for the extension task)

References

1. J. W. Cannon and W. P. Thurston, Group invariant Peano curves, Geom. Topol. **11** (2007), 1315–1355; MR2326947
2. M. Mj, Cannon-Thurston maps for hyperbolic group extensions, Topology **37** (1998), no. 3, 527–538; MR1604882
3. W. P. Thurston, Hyperbolic structures of 3-manifolds, II: Surface groups and 3-manifolds which fiber over the circle, unpublished

107. Poincaré Duality Groups .... Dr M. Kudlinska

The homology and cohomology groups of a compact manifold are related via *Poincaré–Lefschetz duality.* Analogously, a group G is said to be a *Poincaré Duality group of degree n* (or PD^n, for short) if for every G-module M and every integer k, there is a natural isomorphism

$$H^k(G; M) \cong H_{n-k}(G; \mathbb{Z} \otimes M).$$

Fundamental groups of closed aspherical n-manifolds are the porototypical examples of PD^n-groups; C.T.C. Wall famously asked whether there exist PD^n-group which do not arise in this way.

Whilst it was shown that every PD^2-group is the fundamental group of a compact surface [4, 5], Davis constructed PD^n-groups, for every $n > 2$, which are not finitely presented and thus cannot be fundamental groups of compact manifolds [2]. It is still open whether there exists a finitely presented PD^n-group which is not the fundamental group of a closed aspherical manifold.

This essay would start by developing the necessary background knowledge in group (co)homology. The standard text for this is [1]. It would go on to define PD^n-groups and the related notion of PD^n-pairs, and discuss equivalent characterisations.

One goal of the essay could be to give an account of Davis' construction of non-finitely presented PD^n-groups. This involves developing the theory of (infinite) Coxeter groups and Davis complexes, which are their geometric realisations, and the computation of their homology groups with coefficients in the group ring.

Another goal could be to give an account of the work in [4, 5] (summarised in [3]), showing that PD^2-groups are fundamental groups of surfaces. This involves studying amalgamated free product splittings of PD^2-groups and how such splittings are detected in the homology of the group.

Relevant Courses

Essential: Part IB Groups, Rings and Modules, Part II Algebraic Topology

Useful: Part III Algebraic Topology, Part III Geometric Group Theory (essential for the PD^2 task)

References

1. K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, 87, Springer, New York-Berlin, 1982; MR0672956

2. M. W. Davis, The cohomology of a Coxeter group with group ring coefficients, Duke Math. J. **91** (1998), no. 2, 297–314; MR1600586

3. B. Eckmann. Poincaré duality groups of dimension two are surface groups, Combinatorial Group Theory and Topology (eds. S.M. Gersten and J. Stallings), Annals of Math. Studies 111 (1987), 35-51, Princeton Univ. Press, Princeton

4. B. Eckmann and P. A. Linnell, Poincaré duality groups of dimension two. II, Comment. Math. Helv. **58** (1983), no. 1, 111–114; MR0699010

5. B. Eckmann and H. Müller, Poincaré duality groups of dimension two, Comment. Math. Helv. **55** (1980), no. 4, 510–520; MR0604709

108. The KPZ Fixed Point and Brownian Absolute Continuity .... Dr S. Sarkar

Since Kardar, Parisi and Zhang introduced the KPZ equation in their seminal paper in 1986 as a model of interface growth, the equation has made appearances everywhere from the edge of a bacterial colony in a petri dish to the advancing edge of a coffee stain on a surface; and this field has become a subject of intense research interest in both Physics and Mathematics for the last couple of decades. The random growth processes that are expected to have the same scaling and asymptotic fluctuations as the KPZ equation and converge to the universal limiting object called the KPZ fixed point are said to lie in the KPZ universality class. A central object to describe the random growth models in the KPZ universality class is the so-called Airy line ensemble. The directed landscape, the conjectured full scaling limit of models in the KPZ class, was constructed in terms of the Airy line ensemble in [1], which also gives a description of the KPZ fixed point as a variational formula. The Airy line ensemble, in turn, has a very simple resampling property with respect to independent Brownian motions, called the Brownian Gibbs property. Using the variational formula and the Brownian Gibbs property, one can prove that the KPZ fixed point is absolutely continuous with respect to Brownian motion on compacts, a long-standing central question in this area. Recently, [3] shows that the KPZ fixed point increments started from some special initial data exhibit strong quantitative comparison with respect to Brownian motion on compacts. The problem remains open to extend these to all initial data and to $L^{\infty-}$ comparison.

A successful essay should give an account of these developments. It should also include proofs of the important results from [2] and [3] only.

Relevant Courses

Advanced probability

References

1. D. Dauvergne, J. Ortmann and B. Virág. The directed landscape. *Acta Mathematica*, 2022.

2. S. Sarkar and B. Virág. Brownian absolute continuity of the KPZ fixed point with arbitrary initial condition. *Ann. Probab.,* 2021.

3. P. Tassopoulos and S. Sarkar. Quantitative Brownian regularity of the KPZ fixed point with meagre initial data. *arXiv:2509.19415*, 2025

109. The Consistent Reasoning Paradox and Turing's Infallibility Dilemma in AI . . . .

Professor A. C. Hansen

In his celebrated 1947 lecture [4] to the London Mathematical Society, Turing discussed his perspective on artificial intelligence (AI) – that may be viewed as a prelude to his seminal 1950 paper [5] – where he asks: 'can a machine think?'. A crucial part of the 1947 lecture is his famous infallibility dilemma:

> "*If a machine is expected to be infallible, it cannot also be intelligent. There are several mathematical theorems which say almost exactly that. But these theorems say nothing about* ***how much intelligence*** *may be displayed if a machine makes no pretence at infallibility.*"
>
> From 'Lecture on the automatic computing engine' *Oxford University Press* (1947).

Recent results on the Consistent Reasoning Paradox (CRP) [1] allow one to shed light on this long-standing issue of Turing. Consistent reasoning – at the core of logical reasoning – is the ability to consistently answer equivalent questions phrased differently (e.g. (i) 'Is $1 > 0$?' and (ii) 'Is one greater than 0?'); if you are correct on (i), you should also be correct on (ii). The CRP shows that there are problems, e.g. in basic arithmetic, where any AI that always answers and strives to reason consistently will *hallucinate* (wrong, yet plausible answers) infinitely often. The paradox is that there exists a non-consistently reasoning AI – which is not on the level of human intelligence – that will be correct on the same set of problems.

Hence, the CRP is strongly related to Turing's infallibility dilemma and helps identify how much intelligence may be displayed if an AI makes no pretence at infallibility. By contrast, the classical results of Turing – the Halting problem [3] – and Gödel [2] (this is what Turing is referring to in the quote) do not tell us 'how much intelligence may be displayed'. The CRP can quantify this: consistent reasoning. This is surprising, as one can show that making a specialised AI reason consistently is strictly easier than the Halting problem, yet fallibility is necessary for consistent reasoning.

This project is about logical reasoning in AI – going back to the roots of Turing – investigating the consequences of the CRP for developing AI that is capable of logical reasoning. An immediate corollary of the CRP is that trustworthy AI must compute a so-called 'indeterminacy function' – or slightly less formal – an 'I don't know function' in order to say 'I don't know'. How do design such functions and how to compute them is very much an open problem.

Relevant Courses

Useful: Part II: Linear Analysis (and some knowledge of optimisation), Automata and Formal Languages, Logic and Set Theory.

References

1. A. Bastounis, P. Campodonico, M. van der Schaar, B. Adcock and A. C. Hansen. On the consistent reasoning paradox of intelligence and optimal trust in AI: The power of 'I don't know'. *arXiv:2408.02357*, (2024).

2. T. Franzén. *Gödel's theorem: an incomplete guide to its use and abuse.* AK Peters/CRC Press, 2005.

3. A. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, S2-42(1):230, 1936.

4. A. Turing. Lecture on the automatic computing engine (1947). In *The Essential Turing.* Oxford University Press, 2004.

5. A. Turing. I.-Computing machinery and intelligence. *Mind*, LIX(236):433–460, 1950.

110. Undecidability and Indeterminacy in Analysis .... Professor A. C. Hansen

In 2016 [2], T. Tao initiated a program on blow-up and undecidability in PDEs. In particular, he suggested the following:

> "*Somewhat analogously to how a quantum computer can be constructed from the laws of quantum mechanics, or a Turing machine can be constructed from cellular automata such as Conway's 'Game of Life', one could hope to design logic gates entirely out of ideal fluid (perhaps by using suitably shaped vortex sheets to simulate the various types of physical materials one would use in a mechanical computer). If these gates were sufficiently 'Turing complete', and also 'noise-tolerant', one could then hope to combine enough of these gates together to 'program' a von Neumann machine consisting of ideal fluid that, when it runs, behaves qualitatively like the blowup solution...*"
>
> From: 'Finite time blowup for an averaged three-dimensional Navier-Stokes equation' *J. Amer. Math. Soc.* (2016), §1.3.

A fascinating phenomenon is that such an embedding of a Turing machine or von Neumann machine into the flow of a PDE that yields Turing completeness (that is, the flow of the PDE simulates a universal Turing machine) typically implies undecidability of statements about the location of the solution to the PDE. That is, it becomes impossible to prove statements about the precise location of the solution in the standard axioms of mathematics (ZFC). Tao proved exactly this for a particular ODE in [3], moreover, as he points out:

> "*Remarkably, the above results also hold if one works instead with the nonlinear wave equation*
> $$\partial_t^2 u - \Delta u = -(\nabla V)(u)$$
> *on a torus instead of a particle in a potential well, or if one replaces the target domain* $\mathbb{R}^m$ *by a more general Riemannian manifold.*"

In particular, Tao establishes that there is an indeterminacy phenomenon in standard PDEs, where statements about the location of the solution become independent of the standard axioms of mathematics.

The topic of this essay is to investigate this phenomenon further, potentially extending to other PDEs and other problems in analysis. Interestingly, similar results have recently been established regarding indeterminacy of solutions to standard optimisation problems that also become undecidable [1]. Hence, indeterminacy of solutions to basic problems in analysis may be much more common than previously known.

See also the following blogposts:

https://terrytao.wordpress.com/2014/02/04/finite-time-blowup-for-an-averaged-three-dimensional-navier-stokes-equation/

https://terrytao.wordpress.com/2017/07/11/on-the-universality-of-potential-well-dynamics/

Relevant Courses

Essential: Part III analysis either Functional Analysis or PDEs.

Useful: Part II: Automata and Formal Languages, Logic and Set Theory.

References

1. A. Bastounis, P. Campodonico, M. van der Schaar, B. Adcock and A. C. Hansen. On the consistent reasoning paradox of intelligence and optimal trust in AI: The power of 'I don't know'. *arXiv:2408.02357*, (2024).

2. T. Tao. Finite time blowup for an averaged three-dimensional Navier–Stokes equation. *J. Amer. Math. Soc.*, 29(3):601–674, 2016.

3. T. Tao. On the universality of potential well dynamics. *Dynamics of Partial Differential Equations*, 14(3):219–238, 2017.

111. Principled Approaches to Score Matching in Diffusion Models .... Dr G. Maierhofer

Modern generative models aim to estimate complex probability distributions underlying datasets such as images, videos, or text. In state-of-the-art diffusion models (e.g. DALL·E and Stable

Diffusion), this task is achieved through score matching: rather than modelling the data distribution directly, one learns the score function - the gradient of the log-probability density. This score function determines the reverse dynamics of stochastic diffusion processes and is central to the success of diffusion-based generative modeling.

Score matching thus lies at the intersection of probability theory, stochastic analysis, and machine learning. The goal of this essay is to study of how these disciplines contribute to the theoretical foundations and practical computation of the score function. In particular, the essay will:

1. Provide a coherent account of score-based diffusion models, summarising the conceptual and mathematical foundations that link diffusion processes with generative modeling objectives;
2. Describe and compare two distinct approaches to obtaining the score function:
   (a) the standard method based on time reversal of stochastic differential equations and neural network approximations of the score function (cf. [3]);
   (b) a novel approach based on conditional expectation proposed in [4], which derives the reverse equation through statistical averaging;
3. Explore the numerical implementation of both formulations on a simple, illustrative example, discussing their respective strengths, limitations, and computational implications.

Relevant Courses

Essential: Part III Advanced Probability; basic proficiency in Python; an introductory course to ML such as Part II Mathematics of Machine Learning;

Useful: Part III Stochastic Calculus with Applications to Finance, prior experience with PyTorch

References

1. Chen, M., Mei, S., Fan, J. & Wang, M., 2024. An overview of diffusion models: Applications, guided generation, statistical rates and optimization. arXiv preprint arXiv:2404.07771
2. Song, Y. & Ermon, S., 2020. Improved techniques for training score-based generative models. Advances in Neural Information Processing Systems, 33, pp.12438–12448. Available at: https://arxiv.org/pdf/2006.09011
3. Song, Y., Sohl-Dickstein, J.N., Kingma, D.P., Kumar, A., Ermon, S. and Poole, B., 2021. Score-based generative modeling through stochastic differential equations. In: Proceedings of the International Conference on Learning Representations (ICLR 2021). Available at: https://arxiv.org/abs/2011.13456
4. Wong, E., 2025. Conditional expectation and generative AI. In preparation. University of California, Berkeley.

112. Regularity Criterion for 3D Navier-Stokes Equations based on Coarse Spatial Measurements ....

Professor E. S. Titi

The conventional theory of turbulence posits that turbulent flows are governed by a finite number of degrees of freedom. Building upon this foundational principle, rigorous mathematical results have been established that identify these degrees of freedom and provide quantitative estimates for their number in relation to the relevant physical parameters of the two-dimensional (2D) Navier–Stokes equations (cf. [4], [6–8]). Moreover, a novel data assimilation algorithm has been developed to reconstruct exact solutions of the 2D Navier–Stokes equations from coarse spatial observations (cf. [1], [3], [5]). Extending this framework to the three-dimensional (3D) Navier–Stokes equations, a new global regularity criterion was introduced in [2].

This essay focuses primarily on the newly proposed regularity criterion for the 3D Navier–Stokes equations [2] and examines its mathematical validity and implications.

Relevant Courses

Essential: Analysis of Partial Differential Equations. Introduction to Nonlinear Analysis.

Useful: Mathematical Analysis of the Incompressible Navier-Stokes Equations. Some knowledge of Fluid Mechanics.

References

1. A. Azouani, E. Olson and E.S. Titi, Continuous data assimilation using general interpolant observables, J Nonlinear Sci **24** (2014), 277–304.

2. A. Biswas and R. Price, Continuous data assimilation for the three-dimensional Navier-Stokes equations, SIAM J. Math. Anal. **53** (2021), 6697–6723.

3. A. Biswas, C. Foias, C. Mondaini and E.S. Titi, Downscaling data assimilation algorithm with applications to statistical solutions of the Navier–Stokes equations, Ann. I. H. Poincaré – AN **36** (2019), 295–326.

4. B. Cockburn, D.A. Jone and E.S. Titi, Estimating the number of asymptotic degrees of freedom for nonlinear dissipative systems, Math. Comput. **66** (1997), 1073–1087.

5. C. Foias, M.S. Jolly, D. Lithio and E.S. Titi, One-Dimensional Parametric Determining form for the Two-Dimensional Navier–Stokes Equations, J Nonlinear Sci **27** (2017), 1513–1529.

6. C. Foias, G. Prodi, Sur le comportement global des solutions non-stationnaires des équations de Navier- Stokes en dimension 2, Rend. Semin. Mat. Univ. Padova **39** (1967), 1–34.

7. D.A. Jones and E.S. Titi, Determining finite volume elements for the 2D Navier–Stokes equations, Physica D **60** (1992), 165–174.

8. D.A. Jones and E.S. Titi, Upper bounds on the number of determining modes, nodes and volume elements for the Navier–Stokes equations, Indiana Univ. Math. J., **42** (1993), 875–887.

113. Neural Networks for Solving Differential Equations on Calabi–Yau Manifolds ....

Dr S. Krippendorf

Calabi-Yau (CY) manifolds play a central role as compactification spaces in string theory. Understanding low-energy physics requires access to geometric data such as the (approximate) Ricci-flat metric and spectra of differential operators. This essay should survey neural-network-based methods for solving the relevant differential equations on compact CY manifolds. Following a discussion on approximating Ricci-flat metrics and you can choose to focus on computing eigenvalues and eigenfunctions of the associated scalar Laplacian or how to obtain the metric for multi-moduli complete intersection Calabi-Yau manifolds. Where appropriate, you may validate your approach against existing numerical packages (cyjax, cymetric).

Relevant Courses

This essay has few pre-requisites beyond undergraduate level and should certainly be accessible to people following either QFT and related subjects, e.g. string theory and/or supersymmetry, *or* differential geometry and related courses.

References

1. M. Gerdes and S. Krippendorf, *CYJAX: A package for Calabi–Yau metrics with JAX*, *Mach. Learn. Sci. Tech.* **4** (2023) 025031, doi:10.1088/2632-2153/acdc84, arXiv:2211.12520 [hep-th].
2. L.B. Anderson, M. Gerdes, J. Gray, S. Krippendorf, N. Raghuram and F. Ruehle, *Moduli-dependent Calabi–Yau and SU(3)-structure metrics from Machine Learning*, *JHEP* **05** (2021) 013, doi:10.1007/JHEP05(2021)013, arXiv:2012.04656 [hep-th].
3. M. Larfors, A. Lukas, F. Ruehle and R. Schneider, *Numerical metrics for complete intersection and Kreuzer–Skarke Calabi–Yau manifolds*, *Mach. Learn. Sci. Tech.* **3** (2022) 035014, doi:10.1088/2632-2153/ac8e4e, arXiv:2205.13408 [hep-th].
4. V. Braun, T. Brelidze, M.R. Douglas and B.A. Ovrut, *Eigenvalues and Eigenfunctions of the Scalar Laplace Operator on Calabi–Yau Manifolds*, *JHEP* **07** (2008) 120, doi:10.1088/1126-6708/2008/07/120, arXiv:0805.3689 [hep-th].

114. Quantum Gibbs Sampling ....

Professor F. Verstraete

This essay aims to explore efficient quantum algorithms to prepare samples of the Gibbs state of statistical physics Hamiltonians. The importance of this task lies in exhibiting quantum advantage in the form of efficient simulation of equilibrium properties of quantum many-body systems, providing a potential advantage of quantum computing over classical computing.

After the initial attempt [1] at preparing a direct quantum equivalent of the ubiquitous Metropolis-Hastings algorithm, recently algorithms that simulate the Lindbladian dynamics [2] have been proposed, with a mixing time that is polynomial in the system size, and even logarithmic at high enough temperatures [3]. Algorithms that focus on specific k-local Hamiltonians have also been discussed, with recent advances in poly-depth quantum algorithms that sample from the exact distribution of error correcting codes which are also sums of commuting local terms [4], and approximating a Gibbs state by reducing the problem to a classical decoding problem [5]. Ideas that sample from Pauli Hamiltonians have been further improved [6], by improving the polynomial bounds proposed in [4] to linear in the linear system size for 2 error correcting codes, saturating the Lieb-Robinson bound, and introducing a logarithmic depth circuit that samples from the Gibbs state of the 1D Ising model.

This essay should cover recent advances in the field, going through the importance of performing quantum Gibbs sampling, and explaining relevant algorithms that sample from the approximate or exact Gibbs state of local Hamiltonians. If time permits, the author may want to explore unsolved problems such as sampling from higher-dimensional Ising models and the transverse field Ising model using a quantum algorithm.

Relevant Courses

- Part II Quantum Information and Computation,
- Part III Quantum Entanglement in Many-Body Physics (Lieb-Robinson bounds, topological quantum order),
- Part III Quantum Information Theory (Lindbladian evolution)

References

1. K. Temme, T. J. Osborne, K. G. Vollbrecht, D. Poulin, and F. Verstraete, Quantum metropolis sampling
2. C. Rouze, D. S. Franca, A. M. Alhambra, Efficient thermalization and universal quantum computing with quantum Gibbs samplers
3. C. Rouze, D. S. Franca, A. M. Alhambra, Optimal algorithm for Gibbs state preparation
4. Pablo Páez Velasco, Niclas Schilling, Samuel O. Scalet, Frank Verstraete, Angela Capel, Efficient and simple Gibbs state preparation of the 2D toric code via duality to classical Ising chains
5. Alexander Schmidhuber, Jonathan Z. Lu, Noah Shutty, Stephen Jordan, Alexander Poremba and Yihui Quek, Hamiltonian decoded quantum interferometry
6. I. H. C. Shum, A. Capel, Efficient quantum Gibbs sampling of stabilizer codes using hybrid computation

115. Solving the Technical Challenge of Constraining Non-Gaussianity from the Cosmic Microwave Background .... Professor J. Fergusson

Non-Gaussianity is a powerful probe of the physics of the early universe. Specifically, the bispectrum encodes information on any dynamics beyond the free field model of inflation opening a unique window into the physics of the early universe. The equations for creating an estimator for the bispectrum are easy to derive, but naive implementations are computationally intractable, particularly for non-separable shapes. This essay will explore the multitude of technical advances required to make this task possible in practice.

The essay should cover the following topics:

- Motivation for studying non-Gaussianity
- Derivation of the bispectrum estimator
- Discussion of methods for constraining seperable bispectra (KSW,Binned,Wavelets)
- Discussion of methods for constraining non-separable bispectra (Modal/CMBBest) with a discussion of the following challenges:
  - Ensuring the stability and orthogonality of the basis functions and the projections from primordial to observed bispectrum.
  - Accurate tetrapyd integration approaches.
  - Diffusive inpainting of masked regions to reduce mode coupling.

Relevant Courses

Essential: Cosmology

Useful: Field Theory in Cosmology

References

1. Planck 2018 results. IX. Constraints on primordial non-Gaussianity
   Planck Collaboration
   (General review of area)
   https://arxiv.org/abs/1905.05697

2. High-resolution CMB bispectrum estimator with flexible modal basis
   Sohn, W. and Fergusson, J. R. and Shellard, E. P. S.
   (Description of CMBBest and tetrapyd integration techniques)
   https://arxiv.org/abs/2305.14646

3. Using inpainting to construct accurate cut-sky CMB estimators
   Gruetjen, H. F. and Fergusson, J. R. and Liguori, M. and Shellard, E. P. S.
   (Description of inpainting technique)
   https://arxiv.org/abs/1510.03103

4. Efficient optimal non-Gaussian CMB estimators with polarisation
   Fergusson, J. R.
   (Description of Modal estimator)
   https://arxiv.org/abs/1403.7949

116. Efficacy of the Bispectrum as a Cosmological Observable .... Professor P. Shellard

Despite some apparent tensions in cosmological parameters, there is compelling evidence for the inflationary Λ-CDM model, which has become *the standard cosmology.* Nevertheless, there remain a multitude of models for early universe inflation that remain consistent with cosmological observations, even after comparison with CMB data from the Planck satellite. This degeneracy arises because the ability to distinguish between different models using only the power spectrum is largely limited to two parameters, the tensor-to-scalar ratio, r, and the spectral index, n_s.

However, there is a good prospect that new insights will come from improving measurements of the bispectrum (or three-point correlation function) and other correlators of primordial fluctuations, especially using next-generation CMB (such as the Simons Observatory) and galaxy surveys (DESI, Euclid satellite, LSST etc). In principle, non-Gaussian correlations encode an enormous amount of information about fundamental physics in the early universe, and provide a powerful tool to discriminate between specific microscopic models (see [1] for an overall review).

The purpose of this essay is to explore the prospects for primordial non-Gaussianity to probe models of inflation. This essay should cover the following three topics, but these need not be equal, with students choosing the which parts to emphasise–either on bispectrrum calculations for specific models of primordial non-Gaussianity or on the confrontation between these predictions and observations of the CMB and LSS.

1. A review of key concepts and theoretical underpinnings, why the bispectrum offers the first insight into non-linear interactions and evolutionary effects. Predictions for standard single-field slow-roll inflation models [5, 3]. Brief survey of alternative mechanisms for larger non-Gaussianity (refer to reviews such as [6], [9] and references therein, or see the model section of the 2018 Planck NG paper [10] and/or [1]).

2. A choice of a specific class of non-Gaussian models for for an in-depth discussion, describing how NG is generated and its 'shape'. E.g. a contemporary example could be the so-called "cosmological collider" signal which can reveal the presence of additional massive fields during inflation (see, for example, [4], [7], and [8]).

3. A review of observational techniques for detecting non-Gaussianity. (See e.g. [11] and the Planck NG constraints [10]) The outlook for measuring non-Gaussianity from the chosen models using upcoming survey data of large-scale structure or the CMB. (See [1] and references within)

Relevant Courses

Essential: Cosmology.

Useful: Field Theory in Cosmology, Quantum Field Theory, General Relativity.

References

1. P. Daniel Meerburg and others., "Astro2020 Science White Paper - Primordial Non-Gaussianity", [arXiv:1903.04409]

2. Akrami, Y and others, "Planck 2018 results. X. Constraints on inflation", A & A, 641 (2020) A10 [1807.06211].

3. J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," JHEP **0305**, 013 (2003) [astro-ph/0210603].

4. N. Arkani-Hamed, and J.M. Maldacena, "Cosmological Collider Physics", arXiv:1503.08043 (2015)

5. P Creminelli and M Zaldarriaga. "Single-Field Consistency Relation for the 3-Point Function." JCAP, 0410:006, 2004 [arXiv:astro-ph/0407059].

6. X. Chen, "Primordial Non-Gaussianities from Inflation Models," Adv. Astron. **2010**, 638979 (2010) [arXiv:1002.1416 [astro-ph.CO]].

7. W. Sohn, D-G. Wang, Fergusson, J.R. and Shellard, E. P. S., "Searching for cosmological collider in the Planck CMB data", (arxiv:2404.07203) JCAP, 09, 016, (1924).
8. G. Cabass, O. Philcox, M. Ivanov, K. Akitsu, S-F. Chen, Shi-Fan, M. Simonović, and M. Zaldarriaga, "BOSS Constraints on Massive Particles during Inflation: The Cosmological Collider in Action", arXiv:2404.01894 (2024).
9. S. Renaux-Petel, "Primordial non-Gaussianities after Planck 2015: an introductory review", [arXiv:1508.06740]
10. Akrami, Y. and others, "Planck 2018 results. IX. Constraints on primordial non-Gaussianity", A & A , 641, A9 (2020) [arXiv:1905.05697] (astro-ph)]; Ade, P. A. R. and others, "Planck 2015 results. XVII. Constraints on primordial non-Gaussianity", [arXiv:1502.01592]; See also the review section of Planck 2013 results.
11. M. Liguori, E. Sefusatti, J. R. Fergusson and E. P. S. Shellard, "Primordial non-Gaussianity and Bispectrum Measurements in the Cosmic Microwave Background and Large-Scale Structure," Adv. Astron. **2010**, 980523 (2010) [arXiv:1001.4707 [astro-ph.CO]].

117. Neutrinos and Light Relic Particles in Cosmology ....
Professor B. D. Sherwin

Relativistic, weakly interacting particles such as neutrinos are an important component in the early universe, with the cosmic neutrino background background making up $\approx 40\%$ of the energy density at high redshifts.

There are several open questions in neutrino physics that cosmology is best positioned to answer. For example, oscillation experiments imply that neutrinos have different masses, but the value of these masses is still unknown. A determination of the unknown neutrino mass would reveal a new scale in physics and could determine the neutrino mass ordering; it could even contribute to understanding the mechanism by which neutrinos obtain their (surprisingly small) mass.

At the same time, cosmological probes have the ability to constrain the effective number of neutrinos in the early universe. Beyond just constraining the well-known number of standard model neutrino species, this implies that cosmology can search for new, weakly interacting relativistic particle species. With clear targets, upcoming experiments will be able to either discover or rule out broad classes of such beyond-the-standard-model particles.

This essay should discuss the insights cosmology can give us about i) the neutrino mass and other properties of neutrinos and ii) the presence of new, as yet undiscovered light relic particles in our Universe. While both topics should be covered, students may choose to emphasize one.

Within topic i), after some background review of the physics of cosmic neutrinos and the neutrino mass, the essay should explain how cosmology can probe the mass of neutrinos. Students should provide some discussion of the observational status of and the prospects for the mass measurement, and may wish to comment on claims of a "negative neutrino mass" tension with results from the DESI experiment. Students could consider also briefly discussing the complementarity of cosmological measurements with particle physics experiments.

Within topic ii), the essay should explain how constraints on N_{eff}, the number of effective neutrino species, enable searches for new light particles in the early Universe. The essay should discuss targets for such searches and which kinds of new physics could be discovered or constrained by such measurements. The essay should also explain carefully how light relics affect cosmological observations and comment on observational bounds and prospects for future experiments.

Relevant Courses

Essential: Cosmology

Useful: Field Theory in Cosmology, Quantum Field Theory, General Relativity

References

1. CMB-S4 Science Book 2016, arXiv:1610.02743 (relevant chapters on neutrino mass and light relics)
2. Lesgourges, J. and Pastor, S., 2012, arxiv:1212.6154 (further details can be found in arXiv:astro-ph/0603494 by the same authors)
3. Green, D. and Meyers, J. arXiv:2407.07878
4. Lynch, G. and Knox, L. arXiv:2503.14470
5. Elbers et al. 2025 arXiv:2503.14744
6. Hou, Z. et al. 2013, Physical Review D, vol. 87, Issue 8, id. 083008
7. Baumann, D., Green, D., and Wallisch, B., 2016, Journal of Cosmology and Astroparticle Physics, Volume 2016, Issue 01, pp.007-007

118. The Averaged Null Energy Condition .... Professor A. C. Wall

Many theorems in classical general relativity rely on assuming some sort of energy condition. However, all local energy conditions (defined at a single point) can be violated at a single point. It is therefore important to know what energy conditions might hold even for quantum fields in curved spacetime.

The Averaged Null Energy Condition (ANEC) states that the integral of the null-null component of the stress-energy tensor along a lightray is non-negative. This energy condition has now been proven for a broad range of interacting QFTs, at least in Minkowski spacetime. Your essay should explain at least one of these general proofs of the ANEC. It should also discuss the status of the ANEC in curved spacetime, and the use of the ANEC to rule out various types of unusual spacetimes.

Optionally, your essay may also consider other proposed quantum versions of energy conditions, including "quantum energy inequalities", or the "quantum null energy condition". But this is not necessary to receive a good mark. An alternative would be to consider the ANEC in greater depth.

Relevant Courses

Essential: General Relativity, Quantum Field Theory

Useful: Black Holes, Advanced QFT, Quantum Information

References

1. S. Gao and R.M. Wald "Theorems on Gravitational Time Delay", Class. Quant. Grav. 17 (2000) 4999, arXiv:gr-qc/0007021.

2. N. Graham and K.D. Olum, "Achronal averaged null energy condition", Phys.Rev.D 76, 064001 (2007), arXiv:0705.3193.

3. T. Faulkner, R.G. Leigh, O. Parrikar, H. Wang, "Modular Hamiltonians for Deformed Half-Spaces and the Averaged Null Energy Condition", JHEP 09 (2016) 038, arXiv:1605.08072.

4. T. Hartman, S. Kundu, A. Tajdini, "Averaged Null Energy Condition from Causality", JHEP 07 (2017) 066, arXiv:1610.05308.

119. Black Hole Bombs ....
Dr Z. Wyatt

A black hole is a "region of no escape". Thus, it came as great surprise when Penrose argued that energy could be extracted from a rotating black hole through a process called superradiant scattering. Not long after, Press and Teukolsky speculated that by placing a mirror around such a black hole to reflect the superradiant frequencies, one could create a positive feedback loop thus resulting in a "black-hole bomb".

This essay will focus on these ideas in the most-developed context of the Einstein Klein-Gordon system. The essay should outline the Penrose process of energy extraction from a black hole [6]; superradiant mode solutions of the Klein-Gordon equation via heuristic (e.g. [3]) and analytic methods [5]. The essay could then focus on closely related behaviour that occurs the Einstein Klein-Gordon equations, such as (1) the existence of non-decaying soliton-like solutions called boson stars [1], or the existence of time-periodic black hole solutions via numerical [4] and analytic methods [2].

Relevant Courses

Essential: General Relativity

Useful: Black Holes

References

1. Bizon and Wasserman, "On existence of mini-boson stars," Comm. Math. Phys. **215** (2000)

2. Chodosh and Shlapentokh-Rothman, "Time-Periodic Einstein–Klein–Gordon Bifurcations of Kerr," Comm. Math. Phys. **356**, (2017)

3. Detweiler, "Klein-Gordon Equation and Rotating Black Holes," Phys. Rev. D **22** (1980)

4. Herdeiro and Radu, "Kerr black holes with scalar hair," Phys. Rev. Lett. **112** (2014)

5. Shlapentokh-Rothman, "Exponentially growing finite energy solutions for the Klein-Gordon equation on sub-extremal Kerr spacetimes," Commun. Math. Phys. **329** (2014)

6. Wald, "General Relativity," University of Chicago Press. 1984.

120. Functional and Harmonic Analysis in Verification of Neural Networks in PDEs ....

Professor A. C. Hansen

In the 1980s and 1990s, the analysis community in Princeton based around E. Stein worked on a program on well-posedness of the classical wave equation [3,5,6,7]:

$$\begin{cases} (\partial_t^2 - \Delta)u(x,t) = F(x,t), & (x,t) \in \mathbb{R}^d \times [0,T] \\ u(x,0) = g_1(x), \quad \partial_t u(x,0) = g_2(x) \end{cases} . \tag{2}$$

This lead to well-posedness results of the form [2,4,5,6]:

$$\|u\|_{L_t^\infty L_x^p} \le C_{p,d,T}(\|g_1\|_{W^{k,p}} + \|g_2\|_{W^{k-1,p}} + \|F\|_{L^1([0,T];W^{k-1,p})}), \quad \text{if} \quad k \ge (d-1)\left|\frac{1}{2} - \frac{1}{p}\right|. \tag{3}$$

Such important bounds are now experiencing new relevance, due to a new wave of learning-based methods for solving PDEs [1]. In particular, some training procedure will produce a neural network φ that approximates the solution u. However, one typically does not have any a priori error bound on the error $e = \varphi - u$, see [1].

Thus, to verify if φ is sufficiently close to u, one can plug e into (2) and use (3) to compute an error bound for e, providing a posteriori bounds (see [1] for details). The catch is that in order to compute an error bound $\|e\|_{L_t^\infty L_x^p}$ using (3), one needs to compute an upper bound for $C_{p,d,T}$ from (3). Without knowing a numerical upper bound $C_{p,d,T}$, this technique of error estimation becomes hard to verify. This begs the following question:

How to provide upper bounds for $C_{p,d,T}$ (preferably sharp)? (*)

This project revisits classical tools from functional and harmonic analysis used to prove (3), with the aim of advancing question ($*$) stated above, but with a twist. The classical proof techniques providing statements a la 'there exists a constant' must be modified to a constructive approach yielding statements of the form 'this constant is bounded by the integer N'.

Relevant Courses

Essential: Part II: Linear Analysis, Part III: Functional Analysis

References

1. E. Haugen, A. Stepanenko, and A. C. Hansen. Trustworthy ai in numerics: On verification algorithms for neural network-based PDE solvers, 2025. arXiv:2509.26122.

2. A. Miyachi. On some estimates for the wave equation in L^p and H^p. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 27(2):331–354, 1980.

3. J. C. Peral. L^p estimates for the wave equation. *J. Functional Analysis*, 36(1):114–145, 1980.

4. S. Sjöstrand. On the Riesz means of the solutions of the Schrödinger equation. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)*, 24:331–348, 1970.

5. A. Seeger, C. D. Sogge, and E. M. Stein. Regularity properties of fourier integral operators. *Annals of Mathematics*, 134(2):231–251, 1991.

6. C. D. Sogge. L^p estimates for the wave equation and applications. In *Journées "Équations aux Dérivées Partielles" (Saint-Jean-de-Monts, 1993)*, pages Exp. No. XV, 12. École Polytech., Palaiseau, 1993.

7. E. M. Stein. *Singular Integrals and Differentiability Properties of Functions.* Princeton University Press, Princeton, 1971.

121. The Muon Anomalous Magnetic Moment and Lattice QCD .... Professor C. E. Thomas

Quantum loop effects cause the magnetic moment of the muon to deviate from the value predicted by the Dirac equation and this deviation is parameterized by a quantity called the muon anomalous magnetic moment, a_μ. Comparing high-precision experimental measurements of a_μ with theoretical calculations provides a stringent test of the Standard Model of particle physics and constraints on new physics. In recent years there have been tantalizing signs of a discrepancy between experiment and theory, but also suggestions that uncertainties in the theoretical calculations are not well enough understood.

In the Standard Model, a_μ can be expressed in terms of a QED component, an electroweak component and a hadronic component, a_μ^{Had}. The first two components can be computed in perturbation theory. However, a_μ^{Had}, involving the strongly-interacting regime of Quantum Chromodynamics (QCD), cannot and instead a number of methods have been used to relate it to other measurable quantities. There has recently been progress in attempting to reduce the uncertainties in a_μ^{Had} by performing first-principles non-perturbative calculations in a numerical approach called lattice QCD.

The essay should give an introduction to the muon anomalous magnetic moment and describe the current theoretical and experimental situation. It should then discuss some relevant lattice QCD calculations, focusing mainly on what can be computed and how this can be used as an input to a_μ^{Had} rather than general aspects or technical details of lattice QCD.

Relevant Courses

Essential: Quantum Field Theory.

Useful: Advanced Quantum Field Theory; Symmetries, Particles and Fields; Standard Model.

References

1. Chapter V of J. F. Donoghue, E. Golowich and B. R. Holstein, *Dynamics of the Standard Model*, Cambridge University Press, 2nd edn. 2014.

2. Review on "*Muon Anomalous Magnetic Moment*" in S. Navas *et al.* (Particle Data Group), *2024 Review of Particle Physics*, Phys. Rev. D 110, 030001 (2024) [http://pdg.lbl.gov/].

3. Christine Davies, *Muon $g-2$*, PoS LATTICE2024 019 (2025) [arXiv:2503.03364].

4. R. Aliberti *et al.*, *The anomalous magnetic moment of the muon in the Standard Model: an update*, Phys. Rept. 1143 (2025) [arXiv:2505.21476].

122. The Local Langlands Correspondence for GL_2 .... Dr D. Whitmore

The Langlands programme is a central topic within modern pure mathematics, relating areas such as number theory, harmonic analysis, representation theory, and algebraic geometry. For F a non-archimedean local field, the local Langlands correspondence describes the representation theory of $\mathrm{GL}_n(F)$ in terms of the representation theory of the Galois group of F. An understanding of local Langlands would be very useful for anyone interested in further studying the Langlands programme or the representation theory of p-adic groups. The goal of this essay is to state the local Langlands correspondence for GL_n, defining the objects involved and computing examples in the $n = 2$ case to illustrate aspects of the theory.

On the Galois side, the essay should cover the notion of Weil–Deligne representations and prove Grothendieck's ℓ-adic monodromy theorem, relating them to Galois representations. Regarding the representation theory of p-adic groups, you should describe how general irreducible representations are obtained from supercuspidal representations via parabolic induction, and discuss the role of Hecke algebras associated to compact open subgroups. For explicit examples, firstly cover the $n = 1$ case and its relation to local class field theory. You could then include the spherical case (stating and proving the Satake isomorphism), before describing the correspondence for GL_2 explicitly in the case of principal series representations and possibly also for the (tame) supercuspidal representations.

Relevant Courses

Essential:

Part II Representation Theory (or equivalent), Part III Local Fields

References

1. Colin J Bushnell and Guy Henniart. The local Langlands conjecture for GL(2). Springer, 2006.
2. Guy Henniart. Une preuve simple des conjectures de Langlands pour GL(n) sur un corps p-adique. Inventiones mathematicae, 139(2):439–455, 2000.
3. Michael Harris and Richard Lawrence Taylor. The Geometry and Cohomology of Some Simple Shimura Varieties. Princeton University Press, 1999.
4. François Rodier. Représentations de GL(n, k) où k est un corps p-adique. Séminaire Bourbaki, 1981:92–93, 1982.
5. John Tate. Number theoretic background. In Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part, volume 2, pages 3–26, 1979

123. Primordial Gravitational Waves .... Professor P. Shellard

The discovery of a stochastic background of primordial gravitational waves (GWs) would open a new window on the very early universe and could offer profound insights about fundamental physics (see recent references in the Snowmass review [2] or older overviews [3]). These sources range from inflationary tensor modes (see e.g. [4]), GWs generated by non-linear processes during preheating or violent phase transitions (e.g. [5, 6]), gravitational waves created by large scalar fluctuations during inflation (primordial black holes) [1], and cosmic strings or other massive compact objects created in the early universe (see e.g. [7]). The purpose of this essay is to survey possible sources of primordial gravitational waves which may be accessible to future experiments and then to provide an exposition about one of these generation mechanisms and the potential impact of detection. The essay should introduce the underlying basic theory for describing gravitational waves, focussing on the generation mechanisms for a stochastic GWs background, and the distinguishing features of the resulting spectra such as their scale-dependence, amplitudes, and correlation functions. After this general overview of the subject area, it is suggested that candidates should focus more detailed attention on a specific primordial GWs scenario, noting current experimental constraints and future prospects for detection of a stochastic background. Two prominent examples might be tensor modes from inflation or gravitational waves from cosmic strings, but the literature for other scenarios can also be investigated.

Quantum fluctuations during inflation can naturally create both scalar and tensor fluctuations [8, 4]. These can leave an imprint in the polarization of the cosmic microwave background (a B-mode signal), although only upper bounds exist at present [9, 10]; in 2014, long wavelength B-modes generated by dust were mistakenly attributed to inflationary tensors, but more sensitive measurements continue to be made or are planned (e.g. Simons Observatory and LiteBIRD [2, 11]). You may wish to address further questions such as: What would we learn about inflation from the discovery of such B-modes? Are there inflationary scenarios in which primordial tensor modes might be detected on much smaller scales, such as by LIGO or LISA, e.g. those associated with the creation of primordial black holes [1]?

An alternative scenario for creating a stochastic GWs background that has attracted enduring interest is cosmic strings [7]. Cosmic string networks evolve in such a manner that they create a scale-invariant GWs background across all the frequency windows observed by current and planned experiments, whether LIGO, LISA or Pulsar Timing Arrays (PTA). There is considerable theoretical uncertainty in making GW predictions for these nonlinear objects, as can be seen from the differing constraints interpreted from the LIGO data [12, 13]. There is currently a purported detection of a stochastic GWs background claimed by the PTA NANOGrav collaboration which could be speculatively attributed to cosmic strings, but only improved spectral information will distinguish this from possible astrophysical sources [14].

Relevant Courses

Essential: General Relativity, Cosmology

Useful: Field Theory in Cosmology, Quantum Field Theory, Advanced Quantum Field Theory

References

1. Sasaki, Misao, "Primordial black holes and gravitational waves from inflation", Gen. Rel. Grav. 57, 5, (2025) 82 (or similar reviews on arxiv).

2. R. Caldwell et al, "Detection of Early-Universe Gravitational Wave Signatures and Fundamental Physics" arXiv:2203.07972 (gr-qc).

3. Other older reviews: R.A. Battye & E.P.S. Shellard, "Primordial gravitational waves : a probe of the early universe", arXiv:astro-ph/9604059 (simple overview); M. Maggiore,

"Gravitational wave experiments and early universe cosmology," Phys. Rept. 331, 283 (2000) [arXiv:gr-qc/9909001] (review article).

4. See, for example, Liddle and Lyth, "Cosmological Inflation and Large-Scale Structure", CUP textbook (2009). Or online Baumann lecture notes, chapter 6 in : http://cosmology.amsterdam/education/cosmology .

5. M. Kamionkowski, A. Kosowsky and M. S. Turner, "Gravitational radiation from first order phase transitions," Phys. Rev. D 49, 2837 (1994) [arXiv:astro-ph/9310044].

6. R. Easther and E. A. Lim, "Stochastic gravitational wave production after inflation," JCAP 0604, 010 (2006) [arXiv:astro-ph/0601617].

7. A. Vilenkin and E.P.S. Shellard, "Cosmic strings and other topological defects", CUP (2000).

8. A.A. Starobinsky, "Spectrum of relict gravitational radiation and the early state of the universe," JETP Lett. 30 (1979) 682.

9. Akrami et al, "Planck 2018 results. X. Constraints on inflation", arXiv:1807.06211,

10. Ade, et al, "The Latest Constraints on Inflationary B-modes from the BICEP/Keck Telescopes", A & A 641, A10 (2020) [arXiv:2203.16556].

11. Allys, et al, "Probing Cosmic Inflation with the LiteBIRD Cosmic Microwave Background Polarization Survey", arXiv:2202.02773.

12. J. Blanco-Pillado, K. Olum, X. Siemens, "New limits on cosmic strings from gravitational wave observation", Phys. Lett. B 778 (2018) 392 [arXiv:1709.02434].

13. R. Abbott, et al, "Constraints on cosmic strings using data from the third Advanced LIGO-Virgo observing run," Phys. Rev. Lett.126.241102 [arXiv:2101.12248]

14. J. Blanco-Pillado, K. Olum, J. Wachter, "Comparison of cosmic string and superstring models to NANOGrav 12.5-year results", Phys. Rev D.103.103512 [arXiv:2102.08194]

124. Inverse Poincare Constants and Bayesian Data Assimilation .... Professor R. Nickl

This project will examine Bayesian data assimilation for dissipative nonlinear parabolic PDEs where the initial condition is modelled by a Gaussian process prior. Recent results in the Navier-Stokes setting show that given sufficiently many discrete noisy measurements of the velocity field, the posterior distribution will concentrate around the ground-truth and that the rate of this convergence can in general not be faster than inverse logarithmic in sample size. However, it is also shown that when an inverse Poincare inequality (the Laplacian being Lipschitz in L^2 norm) holds on the set of initial conditions a faster rate of convergence is attained. The essay should aim to give a summary of the Bayesian approach to Inverse Problems for PDEs and then to investigate the rate of convergence for the Reaction-Diffusion setting where the existence of an inertial manifold (a subset of the dynamical system that attracts solutions exponentially in time) with this inverse Poincare property can be explored. Finally, the statistical consequences of the PDE results given in the essay should be discussed.

Relevant Courses

Essential: Gaussian Processes, Analysis of Functions

Useful: Topics in Statistical Theory, Advanced Probability, Mathematical Analysis of the Incompressible Navier-Stokes Equations, Introduction to Nonlinear Analysis, Analysis of PDEs

References

1. R. Nickl, Bayesian non linear statistical inverse problems. lecture notes, 2022.
2. James C. Robinson. Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Inertial Manifolds.
3. Nickl, R. (2025). Bernstein–von Mises theorems for time evolution equations. arXiv preprint arXiv:2407.14781.

125. Neural Operators: Theory, Algorithms, and Applications to PDEs .... Dr M. Colbrook

Neural operators are a recent class of machine-learning architectures designed to approximate mappings between function spaces, with prominent examples including the Fourier Neural Operator (FNO), Deep Operator Network (DeepONet), and various kernel- and graph-based formulations. These models aim to learn solution operators of partial differential equations (PDEs) directly from data, offering an alternative to classical numerical methods. Their rapid development has led to a growing theoretical literature on approximation, generalisation, and stability, together with a wide range of applications in fluid dynamics, materials science, and parametric PDEs.

This essay will provide a structured introduction to neural operators, beginning with a survey of the foundational ideas: operator learning, universal approximation results, spectral architectures such as the FNO (Li et al., 2020), and the DeepONet framework (Lu et al., 2021). The student will review key theoretical results on expressivity and error bounds (e.g., Kovachki et al., 2023), comparing different formulations and identifying common mathematical structures. For example, the FNO can be viewed as learning in a truncated spectral basis; convergence properties of operator networks often depend on regularity and spectral decay of the underlying PDE solution operators. Particular emphasis will be placed on understanding how these models differ from standard neural networks and how their design reflects analytical properties of the underlying PDEs.

The second part of the essay will explore applications. Possible directions include: learning solution maps for elliptic or parabolic PDEs; comparing neural operators with classical numerical methods; or examining the behaviour of trained models under perturbations or out-of-distribution parameters. The student may implement simple examples, although substantial coding is not required and the focus should remain on mathematical understanding and critical evaluation.

Depending on progress and interest, the essay may also highlight open questions, potential connections between different architectures, or small theoretical steps motivated by the literature review. The overall aim is to develop a clear, mathematically grounded perspective on neural operators, their capabilities, and their limitations, while keeping the project well within the scope of a Part III essay.

Relevant Courses

Useful: Functional analysis, some background in PDE theory

References

There will be other references the student will uncover along the way!

1. Li, Zongyi, et al. "Fourier neural operator for parametric partial differential equations." arXiv preprint arXiv:2010.08895 (2020).

2. Lu, Lu, et al. "Learning nonlinear operators via DeepONet based on the universal approximation theorem of operators." Nature machine intelligence 3.3 (2021): 218-229.

3. Kovachki, Nikola, et al. "Neural operator: Learning maps between function spaces with applications to PDEs." Journal of Machine Learning Research 24.89 (2023): 1-97.

126. Interactive Theorem Proving and Computer-Assisted Proofs in Applied Mathematics ....

Dr M. Cranmer and Dr M. Colbrook

Interactive Theorem Provers such as Lean provide a framework in which mathematical statements and proofs can be formalised with complete logical rigour. While substantial progress has been made in pure mathematics, the formalisation of applied and computational mathematics remains at an early stage. This essay offers an introduction to the foundations of interactive theorem proving and examines how these tools can support rigorous reasoning in analysis, differential equations, and numerical computation.

The essay will begin with an overview of dependent type theory, the structure of Lean, and the organisation of the mathlib library. It will then survey recent developments relevant to applied mathematics, such as formalised one-dimensional ODE theory, components of multivariate calculus, and emerging projects aimed at connecting Lean with scientific computation. Particular attention will be given to understanding the current capabilities and limitations of these tools, and to the challenges involved in formalising even basic analytical results.

A central component of the essay will be a focused case study. Possible topics include: verifying simple properties of differential equations; exploring existing formal proofs of numerical algorithms; or analysing how an applied mathematical argument could be encoded in Lean. One possible case study for the essay is certified numerical solution of polynomial systems (for example via homotopy continuation), where the Smale's 17th problem literature analyses the computational complexity of finding approximate roots under natural condition assumptions. The goal is not to carry out extensive new formalisation, but to develop a clear understanding of how formal proofs are structured, what mathematical ingredients they require, and how they interact with executable code.

The essay is suitable for students interested in computing, analysis, numerical methods, or the mathematical aspects of scientific computing. No prior experience with Lean is required, though basic familiarity with functional programming will be helpful.

Relevant Courses

Useful: Basic calculus and programming

References

1. J. Avigad and P. Massot, Mathematics in Lean (online textbook).
2. L. de Moura et al., Theorem Proving in Lean 4 (online manual).
3. mathlib4 API documentation (Lean community).
4. J. Harrison, Floating point verification in HOL Light: the exponential function, University of Cambridge Computer Laboratory, Technical Report 428 (1997).
5. S. Smale, Mathematical problems for the next century, The Mathematical Intelligencer 20(2): 7-15 (1998). DOI: 10.1007/BF03025291
6. C. Beltran and L. M. Pardo, Smale's 17th problem: average polynomial time to compute affine and projective solutions, Journal of the American Mathematical Society 22(2): 363-385 (2009). DOI: 10.1090/S0894-0347-08-00630-9