Faculty of Mathematics
Part III Essays: 2020-21

*Titles 1 – 70*

Department of Pure Mathematics
& Mathematical Statistics

*Titles 71 – 121*

Department of Applied Mathematics
& Theoretical Physics

*Titles 122 – 134*

Additional Essays
Introductory Notes

Overview. As explained in the Part III Handbook, in place of a three-hour end-of-year examination paper you may submit an essay written during the year. The Part III Essay Booklet contains details of the approved essay titles, together with general guidelines and instructions for writing an essay. A timetable of relevant events and deadlines is included on page (iii).

In the past the great majority of Part III students have chosen to write an essay: the work is an enjoyable change and is valuable training for research.

Credit. The essay is equivalent to one three-hour examination paper and marks are credited accordingly. As noted in Appendix III of the Part III Handbook, the Faculty Board does not necessarily expect the mark distribution for essays to be the same as that for written examinations. Indeed, in recent years for many students their essay mark has been amongst their highest marks across all examination papers, both because of the typical amount of effort devoted to the essay and the different skill set tested (compared to a time-limited written examination). The Faculty Board wishes that hard work and talent thus exhibited should be properly rewarded.

Essay Titles. The titles of essays in this booklet have been approved by the Part III Examiners. If you wish to write an essay on a topic not covered in this booklet you should approach your Part III Subject Adviser/Departmental Contact or another member of the academic staff to discuss a new title. You should then ask your Director of Studies to write to the Secretary of the Faculty Board (email: undergrad-office@maths.cam.ac.uk) not later than 1 February requesting that an essay on that topic be approved. The new essay title will require the approval of the Part III Examiners. It is important that the essay should not substantially overlap with any course being given in Part III. Additional essays approved by the Part III Examiners will be announced and added to this booklet not later than 1 March. All essay titles are open to all candidates. If you request an essay title you are under no obligation to write the corresponding essay. Essay titles cannot be approved informally: the only allowed essay titles are those which appear in the final version of this document (available on the Faculty web site).

Interaction with the Essay Setter. Before attempting any particular essay, candidates are advised to meet the setter in person. Normally candidates may consult the setter up to three times before the essay is submitted. The first meeting may take the form of a group meeting at which the setter describes the essay topic and answers general questions. There is a range of practices across the Faculty for the other two meetings depending on the nature of the essay and whether, say, there is a need for further references and/or advice about technical questions. The setter may comment on an outline of the essay (for example in the second meeting), and may offer general feedback (for example, on mathematical style in general terms, or on whether clearer references to other sources are required) on a draft of the essay in the final meeting. The setter is not allowed to give students an expected grade for their essay.

Content of Essay and Originality. The object of a typical essay is to give an exposition of a piece of mathematics which is scattered over several books or papers. Originality is not usually required, but often candidates will find novel approaches. All sources and references used should be carefully listed in a bibliography. Candidates are reminded that mathematical content is more important than style.

Presentation of Essay. Your essay should be legible and may be either handwritten or produced on a word processor. There is no prescribed length for the essay in the University Ordinances, but the

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1 The titles are also published in the University’s journal of record, i.e. the Cambridge University Reporter.
2 Regulation 17 of the Regulations for the Mathematical Tripos.
3 All additional titles will also be published in the Cambridge University Reporter.
Faculty Board Advice to the Part III Examiners suggests that 5,000-8,000 words is a normal length, and exceptionally long essays (i.e. more than twice this maximum) are discouraged. If you are in any doubt as to the length of your essay, please consult either the essay setter or your Part III Subject Adviser/Departmental Contact.

**Academic Misconduct and Plagiarism.** Before starting your essay you must read

- both the University's statement on the *Definition of Academic Misconduct* available at the URL https://www.plagiarism.admin.cam.ac.uk/definition,
- and the *Faculty Guidelines on Plagiarism and Academic Misconduct* available at the URL https://www.maths.cam.ac.uk/internal/faculty/plagiarism; the latter is reproduced starting on page (v) of this document.

The University takes a very serious view of academic misconduct in University examinations. The powers of the University Disciplinary Panels extend to the amendment of academic results or the temporary or permanent removal of academic awards, and the temporary or permanent exclusion from membership of the University. Fortunately, incidents of this kind are very rare.

**Signed Declaration.** The essay submission process includes signing the following declaration. It is important that you read and understand this before starting your essay.

> I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University’s statement on the *Definition of Academic Misconduct* and the *Faculty Guidelines on Plagiarism and Academic Misconduct* and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

If you are in any doubt as to whether you will be able to sign the above declaration you should consult the member of staff who set the essay. If the setter is unsure about your situation you should consult the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) as soon as possible.

**Viva Voce Examination.** The Part III Examiners have power, at their discretion, to examine a candidate *viva voce* (i.e. to give an oral examination) on the subject of her or his essay, although this procedure is not often used.

**Time Management.** It is important to control carefully the amount of time spent writing your essay since it should not interfere with your work on other courses. You might find it helpful to construct an essay-writing timetable with plenty of allowance for slippage and then try your hardest to keep to it.

**Final Decision on Whether to Submit an Essay.** You are not asked to state which essay (if any) and which written papers you have chosen for examination until the beginning of the Easter term. At that point, you will be sent the appropriate form to complete. Your Director of Studies must counter-sign this form, and you should then send it to the Chair of Part III Examiners (c/o the Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second Thursday in Easter Full Term, which this year is **Thursday 6 May 2021**. This deadline will be strictly adhered to.

**Essay Submission.** You should submit your essay to the Chair of Part III Examiners (c/o Undergraduate Office, Centre for Mathematical Sciences) so as to arrive **not later than 12 noon** of the second
Thursday in Easter Full Term, which this year is Thursday 6 May 2021. This deadline will be strictly adhered to.

- Together with your essay you should include a completed and signed Essay Submission Form as found on page (iv) of this document.
- The title page of your essay should bear only the essay title. Please do not include your name or any other personal details on the title page or anywhere else on your essay.
- At the time of writing it has not been decided if essay submission will be electronic or in hard copy. If the latter, then it is important that you ensure that the pages of your essay are fastened together in an appropriate way, e.g. by stapling or binding them. However, please do not bind or staple the Essay Submission Form to your essay, but instead attach it loosely, e.g. with a paperclip.

Extension of Submission Deadline. If an extension is likely to be needed due to exceptional and unexpected developments, a letter of application and explanation demonstrating the nature of such developments is required from the candidate's Director of Studies. This application should be sent to the Director of Taught Postgraduate Education (email: director-tpe@maths.cam.ac.uk) by the submission date as detailed above. It is expected that such an extension would be (at most) to the following Monday at 12 noon. A student who is dissatisfied with the decision of the Director of Taught Postgraduate Education can request within seven days of the decision, or by the submission date (extended or otherwise), whichever is earlier, that the Chair of the Faculty review the decision. The provision of any such extension will be reported to the Part III Examiners.

Return of Essays. It is not possible to return essays to candidates. You are therefore advised to make your own copy before handing in your essay.

Further Guidance. Advice on writing an essay is provided in two Wednesday afternoon talks listed below. Slides from these talks will subsequently be made available on the Part III Academic Support Moodle (see https://www.vle.cam.ac.uk/course/view.php?id=203401).

Feedback. If you have suggestions as to how these notes might be improved, please write to the Chair of Part III Examiners (c/o Undergraduate Office, Centre for Mathematical Sciences).

Timetable of Relevant Events and Deadlines

**Wednesday 11 November 4:15pm**
Talk (MR2)  
*Planning your essay: reading, understanding, structuring.*

**Wednesday 27 January 4:15pm**
Talk (MR2)  
*Writing your essay: from outline to final product.*

**Saturday 1 February**  
Deadline for Candidates to request additional essays.

**Thursday 6 May, Noon**  
Deadline for Candidates to return form stating choice of papers and essays.

**Thursday 6 May, Noon**  
Deadline for Candidates to submit essays.

**Thursday 3 June**  
Part III Examinations expected to begin.

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4 Alternatively, the University’s procedure can be invoked via the Examination Access and Mitigation Committee; see the Guidance notes for dissertation and coursework extensions linked from https://www.student-registry.admin.cam.ac.uk/about-us/EAMC.
MATHEMATICAL TRIPOS, PART III 2021
Essay Submission Form

To the Chair of Examiners for Part III of the Mathematical Tripos

I declare that this essay is work done as part of the Part III Examination. I have read and understood both the University’s statement on the Definition of Academic Misconduct and the Faculty Guidelines on Plagiarism and have abided by them. This essay is the result of my own work, and except where explicitly stated otherwise, only includes material undertaken since the publication of the list of essay titles, and includes nothing which was performed in collaboration. No part of this essay has been submitted, or is concurrently being submitted, for any degree, diploma or similar qualification at any university or similar institution.

Signed: .......................................................... Date: ..........................................................

Title of Essay: .................................................................................................................................

........................................................................................................................................................

Essay Number: ..........................

Your Name: ........................................ College: .................................................................

Assessor comments:
Any essay comments we receive will be sent to your College immediately following the publication of results. Comments are not mandatory, and your assessor may not provide them. If you would prefer to receive your comments by email, please provide your preferred email address below:

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Appendix 1: Faculty of Mathematics: Guidelines on Plagiarism and Academic Misconduct

For the latest version of these guidelines please see
https://www.maths.cam.ac.uk/internal/faculty/plagiarism

University Resources

The University publishes information on *Plagiarism and Academic Misconduct*, including

- The University definition of academic misconduct;
- Information for students, covering
  - Students’ responsibilities
  - Why does plagiarism matter?
  - Collusion
- information about Referencing and Study skills;
- information on Resources and support;
- the University’s statement on proofreading;
- Plagiarism FAQs.

There are references to the University statement
- in the Part IB and Part II Computational Project Manuals, and
- in the Part III Essay Booklet (linked from the Part III Essays page),
- in the Computational Biology Handbook (linked from the Computational Biology Course page).

*Please read the University statement carefully; it is your responsibility to read and abide by this statement.*

The Faculty Guidelines

The guidelines below are provided by the Faculty to help students interpret what the University Statement means for Mathematics. However neither the University Statement nor the Faculty Guidelines supersede the University’s Regulations as set out in the Statutes and Ordinances. If you are unsure as to the interpretation of the University Statement, or the Faculty Guidelines, or the Statutes and Ordinances, you should ask your Director of Studies or Course Director (as appropriate).

What is plagiarism?

Plagiarism can be defined as the unacknowledged use of the work of others as if this were your own original work. In the context of any University examination, this amounts to passing off the work of others as your own to gain unfair advantage.

Such use of unfair means will not be tolerated by the University or the Faculty. If detected, the penalty may be severe and may lead to failure to obtain your degree. This is in the interests of the vast majority of students who work hard for their degree through their own efforts, and it is essential in safeguarding the integrity of the degrees awarded by the University.

Checking for plagiarism

Faculty Examiners will routinely look out for any indication of plagiarised work. They reserve the right to make use of specialised detection software if appropriate (the University subscribes to *Turnitin Plagiarism Detection Software*).
The scope of plagiarism

Plagiarism may be due to

- **copying** (this is using another person’s language and/or ideas as if they are your own);
- **collusion** (this is collaboration either where it is forbidden, or where the extent of the collaboration exceeds that which has been expressly allowed).

How to avoid plagiarism

Your course work, essays and projects (for Parts IB, II and III, the M.Phil. etc.), are marked on the assumption that it is your own work: i.e. on the assumption that the words, diagrams, computer programs, ideas and arguments are your own. Plagiarism can occur if, without suitable acknowledgement and referencing, you take any of the above (i.e. words, diagrams, computer programs, ideas and arguments) from books or journals, obtain them from unpublished sources such as lecture notes and handouts, or download them from the web.

Plagiarism also occurs if you submit work that has been undertaken in whole or part by someone else on your behalf (such as employing a ‘ghost writing service’). Furthermore, you should not deliberately reproduce someone else’s work in a written examination. These would all be regarded as plagiarism by the Faculty and by the University.

In addition, you should not submit any work that is substantially the same as work you have submitted, or are concurrently submitting, for any degree, diploma or similar qualification at any university or similar institution.

However, it is often the case that parts of your essays, projects and course-work will be based on what you have read and learned from other sources, and it is important that in your essay or project or course-work you show exactly where, and how, your work is indebted to these other sources. The golden rule is that the Examiners must be in no doubt as to which parts of your work are your own original work and which are the rightful property of someone else.

A good guideline to avoid plagiarism is not to repeat or reproduce other people’s words, diagrams or computer programs. If you need to describe other people’s ideas or arguments try to paraphrase them in your own words (and remember to include a reference). Only when it is absolutely necessary should you include direct quotes, and then these should be kept to a minimum. You should also remember that in an essay or project or course-work, it is not sufficient merely to repeat or paraphrase someone else’s view; you are expected at least to evaluate, critique and/or synthesise their position.

In slightly more detail, the following guidelines may be helpful in avoiding plagiarism.

**Quoting**

A quotation directly from a book or journal article is acceptable in certain circumstances, provided that it is referenced properly:

- short quotations should be in inverted commas, and a reference given to the source;
- longer pieces of quoted text should be in inverted commas and indented, and a reference given to the source.

Whatever system is followed, you should additionally list all the sources in the bibliography or reference section at the end of the piece of work, giving the full details of the sources, in a format that would enable another person to look them up easily. There are many different styles for bibliographies. Use one that is widely used in the relevant area (look at papers and books to see what referencing style is used).
Paraphrasing

Paraphrasing means putting someone else’s work into your own words. Paraphrasing is acceptable, provided that it is acknowledged. A rule of thumb for acceptable paraphrasing is that an acknowledgement should be made at least once in every paragraph. There are many ways in which such acknowledgements can be made (e.g. “Smith (2001) goes on to argue that ...” or “Smith (2001) provides further proof that ...”). As with quotation, the full details of the source should be given in the bibliography or reference list.

General indebtedness

When presenting the ideas, arguments and work of others, you must give an indication of the source of the material. You should err on the side of caution, especially if drawing ideas from one source. If the ordering of evidence and argument, or the organisation of material reflects a particular source, then this should be clearly stated (and the source referenced).

Use of web sources

You should use web sources as if you were using a book or journal article. The above rules for quoting (including ‘cutting and pasting’), paraphrasing and general indebtedness apply. Web sources must be referenced and included in the bibliography.

Collaboration

Unless it is expressly allowed, collaboration is collusion and counts as plagiarism. Moreover, as well as not copying the work of others you should not allow another person to copy your work.

Links to University Information

- Information on Plagiarism and Academic Misconduct, including
  - Students’ responsibilities;
  - Information for staff.
Appendix 2: Essay Descriptors for Part III of the Mathematical Tripos

The Part III Committee believes that the essay is a key component of Part III. It also believes that it is entirely reasonable and possible that candidates may obtain higher marks for essays than in their examination, both because of the typical amount of effort devoted to the essay, and also the different skill set which is tested compared to a time-limited written examination. In light of these beliefs, as well as the comments of both the internal examiners and the external examiners, the Part III Committee believes that it is appropriate to suggest the following descriptors for the various possible broad grade ranges for an essay. The committee trusts that these guidelines prove useful in guiding the judgement of the inevitably large numbers of assessors marking essays, and that these guidelines strengthen the mechanisms by which all essays are assessed uniformly. They are not meant to be either prescriptive or comprehensive, but rather general guidance consistent with long-standing practice within the faculty.

An Essay of Distinction Standard
Typical characteristics expected of a distinction standard essay include:

- Demonstration of a clear mastery of all the underlying mathematical content of the essay.
- Demonstration of a deep understanding and synthesis of advanced mathematical concepts.
- A well-structured and well-written essay of appropriate length (5000-8000 words) with:
  - very few grammatical or presentational issues;
  - a clear introduction demonstrating an appreciation of the context of the central topic of the essay;
  - a coherent presentation of that central topic;
  - a final section which draws the entire essay to a clear and comprehensible end, summarizing well the key points while suggesting future work.

An essay of distinction standard would be consistent with the quality expected of an introductory chapter of a PhD thesis from a leading mathematics department. A more elegant presentation and synthesis than that presented in the underlying papers, perhaps in the form of a shorter or more efficient proof of some mathematical result would be one possible characteristic of an essay of distinction standard. Furthermore, it would be expected that an essay containing publishable results would be of a high distinction standard, but, for the avoidance of doubt, publishable results are not necessary for an essay to be of high distinction standard. An exceptionally high mark (α+) should be justified by a specific extra statement from the assessor highlighting precisely which section of the essay was of particularly distinguished quality.

An Essay of Merit Standard
Typical characteristics expected of a merit standard essay include:

- Demonstration of a good mastery of most of the underlying mathematical content of the essay.
- Demonstration of understanding and synthesis of mathematical concepts typical of the content of a Part III course.
- A largely well-structured essay of appropriate length (5000-8000 words) with:
  - some, but essentially minor, grammatical or presentational issues;
  - an introduction demonstrating an appreciation of a least some context of the central topic of the essay;
  - a reasonable presentation of that central topic;
  - a final section which draws the entire essay to a comprehensible end, summarizing the key points.

An essay of merit standard would be consistent with the quality expected of a first class standard final year project from a leading mathematics department. Such essays would not typically exhibit extensive reading beyond the suggested material in the essay description, or original content.
An Essay of Pass Standard

Typical characteristics expected of a pass standard essay include:

- Demonstration of understanding of some of the underlying mathematical content of the essay.
- An essay exhibiting some non-trivial flaws in presentation through, for example:
  - an inappropriate length;
  - repetition or lack of clarity;
  - lack of a coherent structure;
  - the absence of either an introduction or conclusion.

An essay of pass standard would be consistent with the quality expected of an upper second class standard final year project from a leading mathematics department. For the avoidance of doubt, an excessively long essay (i.e. of the order of twice the suggested maximum length or more) would be likely to be of (at best) pass standard. A key aspect of the essay is that the important mathematical content is presented clearly in (at least close to) the suggested length.
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1. Intrinsic Diophantine Approximation

Consider the standard $n$-sphere $S^n$ with equation $x_0^2 + \ldots + x_n^2 = 1$. Rational points on $S^n$ are dense. How well can a typical point of $S^n$ be approximated by rational points from $S^n$? How about if one restricts to rational points whose denominators are powers of a fixed prime? What if the sphere $S^n$ is replaced by another homogeneous variety? In recent years a number of techniques have been developed to address these questions that blend non-abelian harmonic analysis, ergodic theorems on Lie groups, and advances towards the generalized Ramanujan conjecture.

The essay would focus on the cases $n = 2, 3$, describe the state of the art and give elements of proofs of some of the results following the articles suggested below.

Relevant Courses

Useful: Part II: Representation theory, Part III: Local fields, Algebraic Number theory, Diophantine Approximation

References


2. Ratner’s Theorems on Unipotent Flows

Ratner’s theorems describe the behaviour of orbits of unipotent flows on homogeneous spaces $G/\Gamma$, where $G$ is a connected Lie group and $\Gamma$ a discrete subgroup. They offer a classification of invariant measures on the one hand and a topological description of orbit closures on the other hand. They are cornerstones of modern dynamics and have many applications in number theory, in particular in the geometry of numbers, because they apply in particular to the case when $\Gamma$ is an arithmetic group, e.g. when $G = \text{SL}_n(\mathbb{R})$ and $\Gamma = \text{SL}_n(\mathbb{Z})$.

There are a number of expository sources on Ratner’s theorems and its applications (in particular to the Oppenheim conjecture, proved by Margulis) including books and lecture notes. The essay would discuss the theorems as well as some of its applications, consider the case of $G = \text{SL}_2(\mathbb{R})$ in detail and provide elements of proofs of the general case or a complete proof of another special case beyond $\text{SL}_2(\mathbb{R})$. 
Relevant Courses


References


3. Hochschild Cohomology of Group Algebras and Crossed Products

Dr C. J. B. Brookes

Hochschild conomology is the appropriate cohomology theory for associative algebras and is closely related to deformation theory.

The initial aim of this essay would be to describe the work of Gerstenhaber [1] on the various algebraic structures on the Hochschild cohomology of an associative algebra. On the cohomology there is an associative product (the cup product) but also a Lie bracket, called the Gerstenhaber bracket, that controls the deformation of the algebra.

The essay should then consider the particular case of modular group algebras of finite groups. The abelian case was first considered by Holm [2] and Cibils and Solotar [3], and the general case was considered by Siegel and Witherspoon [3]. The cup product is relatively easy to understand but the Gerstenhaber bracket is much less tractable and has been the subject of more recent research.

If space allows it would be good to consider some of this more recent work, extending to crossed products and skew group algebras, for example the work of Shepler and Witherspoon [6]. A recent book of Witherspoon [7] provides a lot of additional material.

Relevant Courses

Useful: III Commutative Algebra, III Finite Dimensional Lie and Associative Algebras
References


4. Weyl Algebras

Dr C. J. B. Brookes

The Weyl algebras form a family of what may be thought of as non-commutative polynomial algebras. For a field $k$ the first member of the family is the $k$-algebra generated by $x$ and $y$ subject to the relation that $xy - yx = 1$. In general in characteristic zero the $n$th Weyl algebra may be represented as the ring of differential operators $k[t_1, \ldots, t_n, \partial/\partial t_1, \ldots, \partial/\partial t_n]$ of the polynomial algebra in $n$ variables and the Weyl algebras therefore provide the prototype algebras in the study of algebraic D-modules. They also arise as images of enveloping algebras of nilpotent Lie algebras and play an important role in Lie representation theory.

Read the introduction of Coutinho’s book [1] to get an idea of what is involved. Other texts are section 4.6 of Dixmier [2] and section 1.3 of [3]. Both these books have large bibliographies and lots of references to Weyl algebras. Other possible sources are [4], [5] and [6], especially for the homological properties avoided by Coutinho. There are other more recent books on D-modules but they go far beyond the Weyl algebras.

There are various possible approaches to the topic. For example, one might concentrate on the geometric aspects underlying the module theory, or think about the homological properties, taking note of the role of the non-commutativity.

Relevant Courses

Essential: Commutative Algebra

References

5. Non-smoothable Topological Manifolds  

Dr. M. Bustamante

The Kervaire invariant is an invariant of stably framed \((4k+2)\)-manifolds which detects whether such a manifold can be transformed by a sequence of surgeries into a sphere. M. Kervaire used this to produce the first example of a topological manifold, of dimension 10, which supports no smooth structure. Along the way, he also encountered an exotic 9-sphere, that is, a smooth manifold which is homeomorphic but not diffeomorphic to the standard 9-sphere.

The goal of this essay is to explain the construction of Kervaire’s non-smoothable 10-manifold, and the proof that it indeed is non-smoothable. The first part of the essay should focus on the construction of Kervaire’s manifold using a method known as plumbing \([1, \text{Section } 3.]\). You should also discuss the \(h\)-cobordism theorem and the Generalized Poincaré Conjecture \([2, \text{Chapter VIII}],[3]\), in order to recognize the boundary of a plumbing as a topological sphere. The second part of the essay should cover some of the algebraic and differential topology needed to define and analyze the Kervaire invariant \([1, \text{Sections } 1, 2]\), including quadratic forms over \(\mathbb{Z}/2\)-vector spaces, obstruction theory, stable homotopy groups of spheres, framed cobordism groups, the Pontrjagin-Thom construction, and framed surgery \([2, \text{Chapters IX, X}]\); then you should apply all this to conclude that Kervaire’s example is a non-smoothable manifold.

Relevant Courses


References


6. The Ax-Kochen-Ershov Theorem  

Dr G. Conant

In the 1960s, Ax and Kochen \([1,2]\), and independently Ershov \([3]\), showed that the elementary first-order theory of a henselian valued field of equicharacteristic 0 is completely determined the theories of its residue field and ordered value group. More precisely, suppose \(K_1\) and \(K_2\) are henselian valued fields, and let \(k_i\) and \(\Gamma_i\) be the residue field and value group of \(K_i\). Assume \(k_i\) has characteristic 0. Then the Ax-Kochen-Ershov Theorem states:
Theorem. \( K_1 \) and \( K_2 \) are elementarily equivalent (in the language of valued fields) if and only if \( k_1 \equiv k_2 \) (in the language of fields) and \( \Gamma_1 \equiv \Gamma_2 \) (in the language of ordered groups).

This theorem has surprising consequences, such as the following result for the \( p \)-adics.

Corollary. Suppose \( \phi \) is a first-order sentence in the language of valued fields. Then there is an integer \( n = n(\phi) \) such that if \( p \geq n \) is prime, then \( \mathbb{Q}_p \models \phi \) if and only if \( \mathbb{F}_p((t)) \models \phi \).

The previous corollary was used to establish an asymptotic form of a conjecture of Artin about solutions to polynomial equations in \( \mathbb{Q}_p \).

The aim of this essay is to present the proof of the Ax-Kochen-Ershov Theorem and related applications.

Relevant Courses

Essential: Model Theory, Groups, Rings and Modules (or equivalent)

Useful: Prior exposure to valued fields (e.g. the Michaelmas 2020 course on Local Fields)

References


7. Deformed Hermitian Yang-Mills Connections on Holomorphic Line Bundles .................................................................

Dr R. Dervan

The purpose of this essay is twofold. Firstly, the candidate should learn the basics of the theory of complex manifolds, which are higher dimensional analogues of Riemann surfaces; smooth complex projective varieties are a special case. The candidate should especially focus on Kähler geometry, which is the complex-geometric analogue of Riemannian geometry; Huybrechts’ book is a good reference for this material [1], though there are many.

Secondly, the candidate should study the deformed Hermitian Yang-Mills equation [2]. This is a geometric partial differential equation, defined on a complex manifold, which arises mathematically from mirror symmetry, and physically from string theory. The equation does not always admit a solution, and it is conjectured that the equation admits a solution if and only if an algebro-geometric notion of “stability” is satisfied; this is a general principle in complex geometry, which is especially concrete for this equation.

The candidate should begin by describing the equation in terms of its local eigenvalue description [2, Section 2]. The candidate should then decipher the notion of a “subsolution” for the deformed Hermitian Yang-Mills equation using this local description [3, Section 8], and use this to prove one direction of the conjecture: existence of solutions implies stability [3, Section 8]. Two other important aspects of the equation that should be studied are the “linearisation” of the equation.
[2, Section 2], and the complete proof that existence is equivalent to stability when the complex manifold has (complex) dimension two [2, Section 4]. The candidate should understand the analogous statement (but not the proof) in higher dimensions [5]. There are then different directions the candidate could proceed in: more differential-geometrically flavoured would be the moment map description of Collins-Yau [4, Section 2], and more analytically flavoured would be to show uniqueness of solutions to the equation [2, Section 3], and to study the line bundle mean curvature flow [2, Section 5]. The essay does not involve any analysis unless the flow is studied, but there is plenty of geometric analysis available for candidates with an interest in that direction.

**Relevant Courses**

*Essential:* Differential Geometry.
*Useful:* Analysis of Partial Differential Equations, depending on whether the candidate follows an analytic direction.

**References**


8. The Entropy Method in Counting Problems

Dr S. Eberhard

The so-called entropy method in combinatorics is a method used in counting problems, particularly effective for upper bounds, based principally on order randomization and entropy. Applications tend to be both sensational and beautiful. Examples include

- estimating the number of Latin squares and higher-dimensional variants [1],
- bounding the number of transversals in an arbitrary Latin square [2],
- bounding the number of solutions to the n-queens problem on a torus [3],
- bounding the number of Steiner triple systems [4] (see also [5]),
- a solution to Tomescu’s graph colouring conjecture [6].

The goal of this essay is to understand this method, particularly its strengths and weaknesses, and to illustrate with a range of applications, which may be a selection of the above or any others.
 Relevant Courses

Essential: Topics in Combinatorics. No credit can be given for material that was covered in lectures, such as Bregman’s theorem.

References

[1] Linial and Luria, An upper bound on the number of high-dimensional permutations, Combinatorica 34 (2014), 471–486

9. Metric Spaces Which are Intrinsically High-Dimensional

Dr A. Eskenazis

Let $(M, d_M)$ and $(N, d_N)$ be two metric spaces. A map $f : M \to N$ is a bi-Lipschitz embedding with distortion at most $D \geq 1$ if there exists a scaling factor $s \in (0, \infty)$ such that

$$\forall x, y \in M, \quad s d_M(x, y) \leq d_N(f(x), f(y)) \leq s D d_M(x, y).$$

For $D \geq 1$, the (distortion-$D$) metric dimension of a metric space $M$, denoted by $\dim_D(M)$, is the minimal $k \in \mathbb{N}$ for which there exists a $k$-dimensional normed space $X = (\mathbb{R}^k, \|\cdot\|)$ such that $M$ admits a bi-Lipschitz embedding into $X$ with distortion at most $D$.

In 1984, Johnson and Lindenstrauss [2] posed the following influential problem: does there exist $D = O(1)$ such that for every $n \in \mathbb{N}$, every $n$-point metric space $(M, d_M)$ has metric dimension at most $O(\log n)$? While this question was shortly answered negatively by Bourgain [1], finding the maximal metric dimension of an $n$-point metric space remained a challenge for more than a decade. In the converse direction, Johnson, Lindenstrauss and Schechtman [3] (see also [4] for a different proof) showed that there exists $C \in (0, \infty)$ such that for every $D \geq 1$, every $n$-point metric space $M$ has metric dimension $\dim_D(M) \leq C n^{\frac{1}{D}}$. Finally, in 1996, Matousek [5] devised an ingenious construction relying on ideas from real algebraic geometry to show that for every $D \geq 1$ and arbitrarily large $n \in \mathbb{N}$, there exists $n$-point metric spaces $(M_n, d_{M_n})$ satisfying $\dim_D(M_n) \geq c n^{\frac{1}{D}}$, where $c \in (0, \infty)$ is a universal constant. Recently, Naor [6] constructed a conceptually different counterexample to the Johnson–Lindenstrauss question by proving that every 3-regular expander graph sequence $\{(G_n, d_{G_n})\}_{n=1}^{\infty}$ with $|G_n| \to \infty$ equipped with the shortest path metric $d_{G_n}$ satisfies $\dim_D(G_n) \geq c |G_n|^\frac{1}{D}$ for some universal constant $c$.

A successful essay will begin by discussing the setting in which the Johnson–Lindenstrauss problem was raised, before proving in full detail the relevant positive [3,4] and negative [5,6] results.
Relevant Courses

*Useful:* Metric Embeddings. Some familiarity with tools from probability (e.g. martingales) and analysis (e.g. the complex interpolation method) will also be helpful.

References


10. Weil–Heisenberg Representations

Dr J. Fintzen

The theory of Weil–Heisenberg representations is fundamental to a plethora of results in different branches of representation theory. Let $V$ be a finite dimensional vector space over a finite field $\mathbb{F}_q$ with an odd number of elements, and equip $V$ with a skew-symmetric, nondegenerate bilinear form $\langle \cdot, \cdot \rangle$. The associated Heisenberg group $H(V)$ is the group whose underlying set is $V \times \mathbb{F}_q$ with the group operation given by $(x, a) \cdot (y, b) = (x + y, a + b + \frac{1}{2}\langle x, y \rangle)$. It turns out (and you should prove) that for each non-trivial character of the center $\mathbb{F}_q$ of $H(V)$ there exists a unique irreducible representation of $H(V)$ that restricts to the given character on the center. Let $\text{Sp}(V)$ be the subgroup of elements $g \in \text{SL}(V)$ that satisfy $(g.x, g.y) = (x, y)$ for all $x, y \in V$. Then we can form the semidirect product $\text{Sp}(V) \ltimes H(V)$, and the theory of Weil representations allows us to understand how to extend the above representations of the Heisenberg group $H(V)$ to representations of the semidirect product $\text{Sp}(V) \ltimes H(V)$.

Your essay should contain an exposition of the theory of Weil–Heisenberg representations and study some of their properties. You should include a corrected statement of Theorem 2.4.(a) and (b) of [4] formulated in your own words and provide a proof. Note that there is a typo in the theorem statement in the literature. You will work out the correct version by carefully studying the proof. For an example of a far-reaching consequence of this typo beyond the representation theory of finite groups, see [3].

Depending on your interest, and if time and space permit, you may then study further properties of Weil representations themselves (the subject of many research papers), look into recently published work that uses the theory of Weil representations, understand the Howe correspondence, or investigate other ubiquitous set-ups of extending representations from a subgroup to a larger group.
Relevant Courses

Essential: Representation Theory

References


11. Two-Descent on the Jacobian of a Hyperelliptic Curve .................

Dr T. A. Fisher

The Mordell-Weil theorem states that the group of $K$-rational points $A(K)$ on an abelian variety $A$ defined over a number field $K$, is a finitely generated abelian group. Following the proof (a “descent calculation”) gives an upper bound for the rank of $A(K)$.

This essay should begin by describing the classical “number field” or “direct” method for 2-descent on an elliptic curve. References for this include [1] and [2]. The essay should then explain how the method generalises to 2-descent on the Jacobian of a hyperelliptic curve: see [3], [4], [5], [6].

Relevant Courses

Essential: Elliptic Curves

Useful: Local Fields

References


12. The Local-Global Principle for Isogenies .................................
Dr T. A. Fisher

Let $E$ be an elliptic curve over $\mathbb{Q}$, and let $\ell$ be a prime. Suppose that the reduction of $E$ mod $p$ admits an $\ell$-isogeny defined over $\mathbb{F}_p$ for all but finitely many primes $p$. Does it follow that $E$ admits an $\ell$-isogeny defined over $\mathbb{Q}$? Sutherland [3] showed that the answer is “yes” with one specific exception. The proof involves a combination of group-theoretic arguments and the study of rational points on modular curves. The essay should also consider extensions to number fields [1], [2], [3] and to isogenies of composite degree [4].

Relevant Courses

Essential: Elliptic Curves
Useful: Modular Forms

References


13. Fraenkel-Mostowski Models for Set Theory .................................
Dr T Forster

Fraenkel-Mostowski models were developed by Fraenkel, Mostowski and Specker, originally as a means of demonstrating the independence of the Axiom of Choice from the other axioms of set theory. Later refinements were able to tease apart various choice-like principles: for example one can show that none of the implications in the chain: $\text{AC} \rightarrow \text{Every partial order can be refined to a total ordering} \rightarrow \text{every set can be totally ordered} \rightarrow \text{Every set of finite sets has a choice function}$ can be reversed.

In each case the sets of an FM model are sets that are in a suitable sense invariant under the action of a judiciously chosen group of permutations of atoms; so there is a bit of group theory involved, and of course a bit of Set Theory. These same ideas of invariance are of course in play in the “forcing” proofs of independence exhibited later by Cohen (and which are touched on in Part III “Topics in Set Theory”) but the focus there is on the forcing and the purpose of this essay is rather to study the invariance.

There is a variety of applications/aspects which the student can choose between. There are the independence proofs of course; there are recent developments in Set Theory using FM techniques (without forcing); there is a nice semantics for capture-avoiding substitution (google “Fresh ML”), and the corpus of FM work cries out for an abstract general treatment—and such a project might appeal to students who are inclined towards Category theory.
Relevant Courses

*Essential:* Part II Logic and Set Theory or equivalent.
*Useful:* Undergraduate Topology; Part III Category theory; Part III Topics in Set Theory.

References

and another by Blass(!)

https://deepblue.lib.umich.edu/bitstream/handle/2027.42/30052/0000420.pdf?sequence=1

Jech, *The Axiom of Choice*, Dover, would be a good place to start looking. A more detailed reading list can be obtained from the Essay Sponsor on application.

14. Quine’s Set Theory NF ......................................................... Dr T Forster

The set theory revealed to the world in Quine [1937] was a bit of a backwater for a very long time, largely because of unanswered questions about its consistency. Recently Randall Holmes has come up with an alleged consistency proof. The general view among the *cognoscenti* is that it is more likely correct than not; nobody actually disbelieves it, and there are people who have checked it and believe it to be correct. Given the uncertain status of this very difficult proof, it would asking too much of a Part III essay to cover it. However an essay on NF focussed on setting out the requisite historical and technical background required would not only be feasible but would also be an instructive exercise for the student and helpful to any subsequent reader wishing to master the proof when it eventually appears.

Such an essay should cover the following. Specker’s results connecting NF and Simply Typed Set Theory; Holmes’ work on tangled types and tangled cardinals arising from Jensen’s proof of Con(NFU); Rosser’s counting axiom and the refutation of AC, including a brief treatment of the relevant parts of cardinal arithmetic without choice (cardinal trees, amorphous cardinals).

That much would be core, but probably not on its own enough for an essay.

A study of the Rieger-Bernays permutation method would be a useful (but not essential) preparation for Holmes’ proof, and in any case the method is a topic of central importance in the study of NF (it will be familiar to Part II students as the method for demonstrating the independence of the axiom of foundation); a treatment of this material can be easily given and the exercise is instructive. Fraenkel-Mostowski permutation methods are relevant but that is an essay topic on its own account. Church-Oswald models for set theories with a universal set are another topic that could be covered, useful and interesting but less central to the consistency project—though germane to NF. Another interesting topic (the subject of the most recent Ph.D. done under my direction) not directly relevant to the consistency proofs but worthy of treatment is the failure of cartesian-closedness in the category of sets according to NF, and more generally a category-theoretic treatment of the world of NF sets. Finally the status of the constructive version of NF is a matter of active research. A claim has been made that it interprets Heyting Arithmetic. The allegation is the subject of a proof-checker project even as we speak.

The only comprehensive references at this stage are [1] and [2] (both beginning to show their age) but there is a wealth of other material linked from http://math.boisestate.edu/~holmes/holmes/nf.html and the setter will be happy to supply more detailed information on demand.
Relevant Courses

*Essential:* Part II Logic and Set Theory or equivalent.

References


15. The Method of Interlacing Polynomials ........................................

Professor W. T. Gowers

In 2013 Adam Marcus, Daniel Spielman and Nikhil Srivastava caused a stir by solving a famous open problem called the Kadison-Singer conjecture. The original problem is to do with $C^*$-algebras, but had equivalent formulations that were purely combinatorial. To achieve their breakthrough, Marcus, Spielman and Srivastava used so-called families of interlacing polynomials, which they had introduced a year earlier to construct bipartite Ramanujan graphs: roughly speaking, these are explicit regular graphs of bounded degree that look “as random as possible”. This essay would be about the method of interlacing polynomials and its applications. It should include a proof of the (results equivalent to the) Kadison-Singer conjecture, and perhaps results about Ramanujan graphs as well if space permits. (The precise content would be up for discussion.)

Relevant Courses

*Essential:*

It would be important to have a good grasp of linear algebra up to undergraduate level.

*Useful:*

Courses in combinatorics or functional analysis might help to make you more comfortable with the concepts in the papers.

References


16. Bounds for Ramsey Numbers ......................................................

Professor W. T. Gowers

A central result in combinatorics is Ramsey’s theorem, and in particular the fact that the diagonal Ramsey number $R(k, k)$ lies between $2^{k/2}$ and $2^{2k}$. Despite strenuous efforts by many
people, it remains a major open problem to obtain a lower bound of \( \alpha^k \) for some \( \alpha > \sqrt{2} \) or an upper bound of the form \( \beta^k \) for some \( \beta < 4 \). The current record for the upper bound is 
\[
\exp\left(-c(\log k)^2\left(\frac{2k}{k}\right)\right),
\]
due to Ashwin Sah (and proved while he was still an undergraduate), which builds on work by David Conlon, which in turn builds on work by Andrew Thomason.

This would form the centrepiece of the essay but probably not the entire content. After that, there are many directions that an essay could go, such as Ramsey numbers for hypergraphs or Ramsey numbers for more than two colours. The second and third references below are just a sample of the possibilities.

**Relevant Courses**

*Essential:*
You should be familiar with the basic concepts of graph theory.

*Useful:*
Any courses in combinatorics.

**References**


17. *D-modules, Hodge Theory, Representation Theory* .................

*Professor I. Grojnowski*

The aim of this essay is to understand the basics of the theory of D-modules, and some concrete applications to either representation theory or the topology of algebraic varieties.

Begin by learning the basic properties of holonomic D-modules on algebraic varieties—Bernstein’s lemma on \( b \)-functions, and its consequences (the formalism of the six operations). This is the basic language of algebraic geometry and modern representation theory, and has applications throughout mathematics and physics.

Then, either

1) prove the Beilinson-Bernstein theorem, describing the category of representations of a semisimple Lie algebra in terms of D-modules on the flag variety. This is an extraordinary result, which generalizes the Borel-Weil-Bott theorem (describing finite dimensional representations in terms of line bundles on the flag variety).

or

2) Study the mixed Hodge structure on D-modules, beginning by computing the Kashiwara-Malgrange filtration on vanishing cycles in some interesting cases. (If you do this, you’ll have to learn about Hodge theory, too!).
References

Many textbook expositions of D-modules now exist. The two best are by the originators of the subject—Kashiwara and Bernstein (the latter are printed notes, available on the web somewhere).

The original Beilinson-Bernstein paper is 3 pages long, it is


but there are many expositions which are probably easier to read; for background on Hodge theory there are Deligne’s extraordinary papers:


Professor I. Grojnowski

This essay is about applying the basic algebraic geometry, algebraic topology, and representation theory (Lie algebras) you’ve learnt to study the ”moment map”, or ”Springer resolution” from the cotangent bundle of the flag variety to the nilpotent cone.

Here is one possible list of topics; you may comfortably do much less. (You can also do more!).

You’ll begin by understanding the definition of the flag variety $G/B$ for a simple algebraic group $G$, that is for $SL_n$, $Sp_{2n}$, $SO_n$, and the five exceptional cases, and by computing its singular cohomology, additively, as a consequence of Bruhat decomposition.

In order to do this, you’ll need to learn the basic structure theory and representation theory of reductive algebraic groups.

Then you should compute the cohomology of line bundles on the flag variety— this is the theorem of Borel-Weil-Bott, and the Weyl character formula. There are several ways of doing this. You’ll want to understand the answer both as a consequence of general theorems (Kodaira vanishing, localisation to the fixpoints), and by hand. You may also want to understand the BGG resolution of irreducible representations by Verma modules as a Grothendieck-Cousin complex here.

Go back and rederive the additive structure of the cohomology of the flag variety by applying this to the de Rham complex. If you know Chern characters, or you know equivariant cohomology, compute the multiplicative structure of the this cohomology (with rational coefficients). Compute the cohomology of $G$, and of $BG$.

Now, define the nilpotent cone, and study the geometry of the moment map. Show the nilpotent cone is normal with rational singularities, that the fibers of the moment map are rationally connected, that the nilpotent orbits are complex symplectic varieties, that they are parameterised by homomorphisms from $sl_2$ to the Lie algebra, and describe the dimensions of the fibers.

Study the case of subregular elements, the rational double points; show the fibers are Dynkin curves.

If you’re still standing, show that the cohomology of the fibers carry a representation of the Weyl group, and all irred reps occur in this way.
IMPORTANT: If you are intending to do a PhD in representation theory, this will be a useful essay to do. However, the essay on D-modules will be even more useful!!

References

Three useful textbooks:
N. Chriss, V. Ginzburg, Representation Theory and Complex Geometry, Birkhauser
R. Steinberg, Conjugacy classes in algebraic groups, Springer LNM 366
J. Humphreys, Finite reflection groups.


Professor I. Grojnowski

This is an essay intended to ease your way into a PhD on Geometric Langlands, or geometric representation theory more generally.

The Langlands program is the description of the representations of the $F$-points of a reductive group $G$ in terms of the conjugacy classes of the ‘Langlands dual group’ $L^*G$. When the field $F$ is the function field of an algebraic curve $X$, the precise description of this correspondence is called the ‘geometric Langlands program’. It was invented by Drinfeld, Beilinson and Drinfeld, and their students and collaborators.

This is not an overview essay - there are several entry points into the subject, depending on your taste and background, and you must choose one and understand it in depth. In particular, you should be able to calculate interesting examples of whatever it is you are learning.

The part III courses on Lie algebras and algebraic geometry are essential. Otherwise, the particular background you have can dictate what order you start learning things. If you don’t have some extra background in either geometry or representation theory, the two other essays I set would be more appropriate. We will want to talk about background before you start the project, so we can work out the best way in.

Three possible entry points:

1) Geometric Satake isomorphism.

This describes the representations of a compact group (or almost equivalently, if you prefer, a semisimple Lie algebra) in terms of certain objects, perverse sheaves or D-modules, on the Langlands dual affine flag variety.

The Satake isomorphism is at heart of the Langlands program - to begin with, it offers a conceptual definition of the Langlands dual group.

The expository paper of Zhu,

$^1$Each of which is far too much for an essay, but the description gives you some ambitious goals to work towards. If you want to work on one of these, we will determine what theorems you should learn, and what to read, depending on your existing mathematical knowledge.
Zhu, X. An introduction to affine Grassmannians and the geometric Satake equivalence.
arXiv:1603.05593

is an excellent place to begin reading. If this is too algebraic geometric to be comfortable, there are other more combinatorial approaches. You could start with affine Hecke algebras, q-characters, and crystal bases for representations of semisimple groups, before proceeding to the geometry of the affine Grassmannian. (This was how the geometric Satake isomorphism was discovered, by Lusztig.)

2) Pseudo-traces and GIT, after Lafforgue

The initial goal here is to understand what trace functions can tell you about a representation. This is Lafforgue’s deep and insightful reformulation of the theorems of Procesi, following Wiles and Taylor.

Begin by understanding the definition of a pseudo-character and proposition 11.7 of


In order to do this, you will probably want to learn a bit of geometric invariant theory.

After that, it might be sensible to understand more about what pseudo-characters and GIT tells us about representation theory.

3) Geometry of G-bundles on curves, and the Verlinde formula.


Explicitly describe the stack of G-bundles on P^1 and on an elliptic curve E, and using this describe the cohomology of line bundles on the stack of bundles over these curves.

Degenerate curves to nodal rational curves, and state and prove the Verlinde formula.

Compute Pic(Bun_G), and Pic of the coarse moduli spaces.

Some references for Bun_G:


20. Bogomolov-Tian-Todorov Theorems ........................................

Professor M. Gross

A complex manifold is a manifold with coordinate charts in C^n and transition maps being holomorphic maps. Such a manifold is also a real C^∞ manifold, but the structure as complex manifold may be varied while the underlying real manifold structure is kept fixed. This led
to Kodaira and Spencer’s theory of deformations of complex manifolds, see [1]. In general, there will be a germ of a complex analytic space parameterizing deformations of the complex structure on a given complex manifold, but this analytic space need not be smooth. In this case, the deformation theory is said to be obstructed.

There is, however, a class of manifolds, called Calabi-Yau manifolds, which are complex manifolds with a nowhere vanishing top-dimensional holomorphic form, which surprisingly have unobstructed deformation theory. This was proved by Tian and Todorov, generalizing earlier results of Bogomolov. Such results are now used widely in the study of geometry of Calabi-Yau manifolds.

There are several routes this essay can take. The first route is via the theory of complex manifolds and their degenerations, leading to an exposition of the original proof (for which the best reference is Tian’s paper [2]). As the department was unable to offer a course on complex manifolds this year, this would require learning and expositing some of this material on one’s own, most importantly understanding what a Kähler manifold is and what the Hodge decomposition theorem tells us. Useful references are [3] and [4]. After that, Tian’s paper should be readable.

A second approach eschews complex manifold theory and takes a more algebro-geometric approach. Kawamata [5] gave a very short proof of the Bogomolov-Tian-Todorov theorem for non-singular projective Calabi-Yau varieties, dependent on the degeneration of the Hodge-de Rham spectral sequence, and since then there have been a number of generalizations of this type of result to cases when the varieties have some controlled classes of singularities, see e.g., [6]. The most ambitious essay writer may delve into the latest results in this direction, taking as a starting point the purely algebraic proof of the result given in [7] which uses some of the latest technology for thinking about deformation theory, and, even more ambitiously, look into the very recent [8]. However, the essay writer should be warned that to go that far, they will have to absorb a great deal of additional material in algebraic geometry.

**Relevant Courses**

*Essential:* Depending on the point of view taken, either Part III Differential Geometry or Part III Algebraic Geometry.

**References**


21. Algebraic Stacks

Algebraic stacks are a vast generalization of the notion of scheme, developed partly to describe various moduli spaces. For example, $\mathcal{M}_g$, the moduli space of algebraic curves of genus $g$, cannot be described as a scheme, but is what is known as a Deligne-Mumford stack. Morally, this is a geometric object which is locally a quotient of a scheme by a finite group, but the geometric object remembers something about this local description. If the scheme is smooth, then we obtain the algebraic-geometric equivalent of an orbifold. More generally, an Artin (or algebraic) stack allows quotients by much more complicated equivalence relations. For example, the trivial action of an algebraic group $G$ on the point has a well-defined quotient in the world of algebraic stacks, and this quotient plays the role of the classifying space $BG$ in algebraic geometry. See [2] for a very brief survey, and [3] for a longer survey.

This essay would involve internalizing the (very complicated) definition of stacks, and giving some application(s). The most obvious application is the construction of the moduli space of stable curves [1]. Other possibilities include the construction of the Chow group for Artin stacks [5], and Artin’s criterion for algebraicity of stacks [4]. The former will require delving into the theory of algebraic cycles, the latter into deformation theory.

There are several sources for the definitions. The original papers [1] and [4] give concise definitions, and [6] covers these in a more expansive way (but is in French). There are a number of online resources, (follow the links from the wikipedia page on stacks) and the Stacks Project [8], the latter being a vast compendium of most of algebraic geometry and probably not so useful for a beginner. There is also a good new book on the subject by Martin Olsson, [7].

Relevant Courses

Essential: Part III Algebraic Geometry (Michaelmas term).

References

The goal of this essay is to study the Hölder continuity for classes of energy estimates, introduced first in an elliptic framework by De Giorgi [1] to solve the 19th Hilbert problem. The 19th Hilbert problem states that the local minimisers of energy functionals of the form $E(w) = \int_\Omega F(\nabla w) dx$ are analytic. In fact the local minimisers satisfy a quasilinear equation. The idea of De Giorgi was to see this nonlinear equation as a linear one with rough coefficients i.e. of the form $-\nabla \cdot (A(x)\nabla u) = 0$ with $A$ only measurable and satisfying an ellipticity condition. Nash [7] and Moser [6] got the same result in the parabolic case with different approaches. In this essay we focus in a first place on the De Giorgi method, for which the idea of De Giorgi was to introduce relevant classes of energy estimates in the elliptic case that are satisfied by the solutions of the equations with rough coefficients and which contain all the information to deduce the Hölder continuity of the solutions. These classes have been then extended to the parabolic case in [5,2,4] with different techniques than De Giorgi and more recently in [3] with De Giorgi method.

In this essay, the purpose would be to first investigate the ideas of De Giorgi method, using the following papers [1,8,3]. It could be then interesting to have a look at the following references [5,2,4] and to understand the differences with the techniques of the De Giorgi method of the previous papers.

Relevant Courses

Essential: Analysis of Partial Differential Equations (Part III), Introduction to Non-Linear Analysis (Part III)
Useful: Linear Analysis (Part II), Analysis of Functions (Part II)

References

Locally presentable categories were introduced by Gabriel and Ulmer [1,2], and were an early attempt to capture the essential categorical structure of the category of models of a theory. The fact that they succeeded in doing just this, for a particular (very natural) class of ‘essentially algebraic’ theories, was proved by M. Coste [3]. More recently, attention has focused on the much larger class of accessible categories [4,5], which are categories of models of theories in a much broader sense; locally presentable categories are precisely those accessible categories which are complete as categories. An essay on this topic could either take as its goal the main theorem characterizing accessible categories as categories of models, or it could survey the way in which particular properties of the axiomatization of a theory are reflected in properties of its category of models. (Some examples of the latter may be found in [6].)

Relevant Courses

Essential: Category Theory

References

[1] F. Ulmer, Locally \(\alpha\)-presentable and locally \(\alpha\)-generated categories, in Reports of the Midwest Category Seminar V, Lecture Notes in Math. vol. 195 (Springer–Verlag, 1971), 230–247. (This is a summary in English of the main results of [2].)

Symplectic Ellipsoid Embeddings

A four-dimensional symplectic ellipsoid \(E(a,b)\) is determined by the lengths \(a\) and \(b\) of its minor and major axes; after symplectic rescalling, one can assume \(a = 1\). A four-dimensional symplectic ball \(B(r^2)\) is determined by its radius, say \(r\). A natural question is to find the smallest \(r\) such that \(E(1,b)\) symplectically embeds into \(B(r^2)\). The answer, in a landmark paper of McDuff and Schlenk, is both beautiful and surprisingly rich, involving an “infinite staircase” determined by the odd-index Fibonacci numbers and the Golden Mean. There has recently been considerable interest in better understanding this phenomenon for more embeddings of ellipsoids into more general symplectic four-manifolds, notably work of Cristofaro-Gardiner, Holm, Mandini and Pires, and of Casals and Vianna. The essay should start by giving an account of the constructions used by McDuff and Schlenk; it should then explain how the
bounds obtained from these constructions can be shown to be optimal (properties of Embedded Contact Homology may be taken as a ‘black box’), and/or give an account of some of the generalisations to other target manifolds.

Relevant Courses

Essential: Differential Geometry, Symplectic Topology
Useful: Algebraic Topology, Algebraic Geometry

References


25. The Hodge Decomposition Theorem ........................................

Dr A. G. Kovalev

The concept of Laplace operator $\Delta = -(\partial/\partial x_1)^2 - \cdots - (\partial/\partial x_n)^2$ for functions on the Euclidean space $\mathbb{R}^n$ can be extended to oriented Riemannian manifolds. The construction uses a certain Hodge star ‘duality’ operator, and the resulting Laplace–Beltrami operator (or Hodge Laplacian) is well-defined on differential forms. The celebrated Hodge decomposition theorem implies a natural isomorphism between the kernel of this Laplacian (i.e. the space of harmonic differential forms) and the de Rham cohomology, for a compact oriented manifold without boundary [1]. This theory admits nice extensions to compact manifolds with boundary [2] and to non-compact Riemannian manifolds with tubular ends [3]. The essay could explore some of the latter results. Interested candidates are welcome to contact me (A.G.Kovalev@dpmms) and discuss further; section 3.5 of http://www.dpmms.cam.ac.uk/~agk22/riem1.pdf could be a good preliminary reading.

Relevant Courses

Essential: Differential Geometry
Useful: Algebraic Topology

References

26. Topology of Configuration Spaces ........................................ Dr M. Krannich

The space $C_n(M)$ of $n$ distinct but indistinguishable points in a manifold $M$ has a surprisingly rich topology, even in the simplest case of Euclidian spaces, and has been object of study for algebraic topologists for several decades.

This essay focuses on different patterns in the homology of these configuration spaces. It should begin by explaining the Fadell–Neuwirth fibration [1] and the resulting connection to Artin’s braid groups. The main focus should then be a result of McDuff [2] which shows that the homology of $C_n(M)$ agrees in a range of degrees with the homology of a seemingly quite different space of sections built from the tangent bundle of $M$. This is an instance of a principle called scanning which has lead to several other striking results, even in recent years.

McDuff’s methods do not provide an estimate of how large this range of degrees actually is, but one can show via a different approach (see e.g. [3]) that the homology groups $H_k(C_n(M))$ are independent of $n$ as long as $n \geq 2k$ if $M$ has nonempty boundary. This is an example of a homological stability result, aspects of which could be discussed in a second part of the essay.

Relevant Courses

Essential: Part III Algebraic Topology

References


27. Hamiltonian Cycles and Spheres in Hypergraphs ......................... Professor I. Leader

The notion of a Hamilton cycle (a cycle through all the vertices) in a graph also makes sense for a hypergraph. There are various versions. One is that we may list the vertices cyclically in such a way that every interval of length $k$ (where the hypergraph consists of $k$-sets) belongs to the hypergraph. Is there an analogue of the well-known Dirac theorem for graphs, which states that if a graph has minimum degree at least $n/2$ then it has a Hamilton cycle?

The aim of the essay is to focus on some classical results on this question, and also on some very recent work on ‘Hamilton spheres’.

Relevant Courses

Useful: Topics in Combinatorics
References


28. Pursuit on Graphs .................................................................

Professor I. Leader

There has recently been much interest in pursuit questions on graphs. Typically, we have a number of pursuers, working as a team to catch an evader. If this takes place on a graph (so the players live on the vertices of the graph, and in each time-step they move to an adjacent vertex), how many pursuers are needed? And how does this relate to properties of the graph? Most work centres around the conjecture of Meyniel, still unproved, that the number of pursuers need be no more than about the square-root of the number of vertices.

The essay would focus on some results for general graphs, and also on specific cases of interest like random graphs.

Relevant Courses

Useful: Topics in Combinatorics

References

[1] Cops and robbers in graphs with large girth and Cayley graphs, P.Frankl


[4] On a generalization of Meyniel’s conjecture on the cops and robbers game, N.Alon and A.Mehrabian

29. Lagrangians of Hypergraphs ..................................................

Professor I. Leader

The Lagrangian of a hypergraph is a function that in some sense seems to measure how ‘tightly packed’ a subset of the hypergraph one can find. Lagrangians have beautiful properties and are of great interest, both in their own right and because they have several applications, most notably to the celebrated ‘jumping hypergraphs’ conjecture.

The main topic would be the way in which Lagrangians influence other properties, ranging from the fact, due to Motzkin, that Lagrangians provide a simple proof of Turan’s theorem, right up to the relationship between Lagrangians and ‘asymptotic density’, with the disproof by Frankl and Rodl of the Erdos conjecture that the set of possible asymptotic densities is discrete. There would also be an examination of the Frankl-Furedi conjecture on maximising the Lagrangian, taking in the recent proof of this by Tyomkyn in the ‘nice’ case and the disproof by Gruslys, Letzter and Morrison in the general case.
Relevant Courses

**Essential:** Combinatorics

References


[4] V. Gruslys, S. Letzter and N. Morrison, Hypergraph Lagrangians I: the Frankl-Furedi conjecture is false (Arxiv 1807.00793)

30. Wadge Determinacy and the Semi-linear Ordering Principle .............

Professor B. Löwe

One of the surprising consequences of the Axiom of Determinacy AD is that it implies that sets of real numbers (as usual in set theory, we are using Baire space \( \omega^\omega \) rather than the space \( \mathbb{R} \) as the “real numbers”) are semi-linearly ordered by the relation \( \leq_W \) defined by

\[
A \leq_W B \text{ if and only if there is a continuous reduction of } A \text{ to } B.
\]

Here, semi-linearly ordered means that for any \( A \) and \( B \), either \( A \leq_W B \) or \( \omega^\omega \setminus B \leq_W A \). This fact is known as Wadge’s Lemma and does not use the determinacy of all games, but only of a subclass of games known as Wadge games. In a series of papers [1,2,3], Andretta explored the relationship between the determinacy of Wadge games and the semi-linear ordering principle.

The aim of this essay is to understand and describe Andretta’s result that the determinacy of Wadge games and the Semi-Linear Ordering Principle are equivalent (e.g., [3, Theorem 27]). Time permitting, the essay could also discuss other game-related reducibilities such as Borel reductions and their corresponding semi-linear ordering principles (cf. [3, §6] and [4]).

Relevant Courses

**Essential:** Part II Logic and Set Theory (or equivalent) and Part III Infinite Games.

**Useful:** Part Ib Metric and Topological Spaces (or equivalent).

References


31. Reflection at Large Cardinals .................................

Professor B. Löwe

Let \( \Phi \) be a formula describing properties of cardinals and \( \text{LC}_\Phi \) be the statement “there is an uncountable cardinal \( \kappa \) such that \( \Phi(\kappa) \).” Usually, we think of \( \text{LC}_\Phi \) a large cardinal axiom if the property \( \Phi(\kappa) \) implies that \( \kappa \) is very large and if \( \text{LC}_\Phi \) cannot be proved in \( \text{ZFC} \).

There are different ways to measure the strength of large cardinal axioms, e.g., we say that \( \text{LC}_\Phi \) is weaker than \( \text{LC}_\Psi \) if the smallest cardinal \( \kappa \) such that \( \Phi(\kappa) \) is smaller than the smallest cardinal \( \lambda \) such that \( \Psi(\lambda) \).

The goal of this essay is to describe the general technique of reflection to prove strength inequalities: if \( U \) is a normal ultrafilter on \( \kappa \), then we consider the ultrapower of the universe by \( U \); the normality of \( U \) implies that properties of the cardinal \( \kappa \) that are preserved in the ultrapower are reflected downwards.

Many individual reflection results of this type are well documented in the textbook literature (e.g., [1]); the essay will formulate reflection as a general technique that enables us to derive the individual results as special cases.

Relevant Courses

Essential: Part II Logic and Set Theory (or equivalent), Part III Infinite Games, and Part III Model Theory.

References


32. Determinacy of Long Games .................................

Professor B. Löwe

In the course Infinite Games, we consider games of length \( \omega \). This essay will consider games of transfinite countable length: for \( \alpha < \omega_1 \), let \( \text{AD}^\alpha \) be the axiom of determinacy for games of length \( \alpha \).

For \( \alpha < \omega \cdot 2 \), the axiom \( \text{AD}^\alpha \) follows from \( \text{AD} \). However, \( \text{AD}^{\omega^2} \) implies the choice principle \( \text{AC}_R(\mathbb{R}) \) which is not a consequence of \( \text{AD} \), but a consequence of \( \text{AD}_R \). Blass proved in [1] that \( \text{AD}_R \) is equivalent to \( \text{AD}^{\omega^2} \). The main goal of the essay is to prove the following diagram of implications and non-implications:

\[
\text{AD}_R \iff \text{AD}^{\omega^2} \iff \text{AD}^{\omega^2} \iff \text{AD}^{\omega^2+n} \iff \text{AD}
\]

The non-implication requires basic properties of the model \( L(\mathbb{R}) \) (cf., e.g., [3]).

Time permitting, a result proved independently by Martin and Woodin could be included in the essay: \( \text{AD}_R \) implies \( \text{AD}^\alpha \) for all \( \alpha < \omega_1 \) [2]. If there is time left, this result could be included in the essay.
**Relevant Courses**

*Essential: Part II Logic and Set Theory (or equivalent) and Part III Infinite Games.*

**References**


**33. Polynomial Invariants of Finite Groups.**

*Dr S. Martin*

The theory of invariants of (finite or infinite) groups is a self-contained nook in the huge edifice of commutative algebra; one of its popularisers, Hermann Weyl (writing in *Invariants*, 1939, see also [8]) described it thus: ‘the theory of invariants came into existence about the middle of the 19th century somewhat like Minerva: a grown-up virgin, mailed in the shining armour of algebra, she sprang forth from Cayley’s Jovian head.’ The main problem is this: take a field $\mathbb{k}$ (in classical theory it is $\mathbb{C}$). Given a $\mathbb{k}$-representation $V$ of the group $G$, there is an induced action on the ring of polynomial functions, $k[V]$ on $V$, given by $(g \cdot f)(v) = f(g^{-1}v)$. What does the space of invariant polynomials $k[V]^G$ (those polynomials $f$ such that $g \cdot f = f$ for all $g \in G$) look like? In particular is $k[V]^G$ finitely-generated? For example if $G = \text{SL}_n$ and $V$ is the natural representation with action given by left multiplication then $k[V]^G$ is isomorphic to a polynomial algebra in one variable, generated by the determinant (so every invariant polynomial is a linear combination of powers of the determinant polynomial), hence $k[V]^G$ is finitely-generated over $k[V]$. Another example: if $V = \mathbb{k}^n$ has dimension $n$ over a field and $G$ is the symmetric group $S_n$, then the generators of $G$ act by reflections on $V$, and the ring of invariants is the polynomial algebra generated by the elementary symmetric functions. Given finite generation, one can seek a minimal basis and ask whether the syzygies (the module of polynomial relations between the basis elements) is finitely-generated over $k[V]$.

Invariant theory of finite groups is linked to Galois theory; if $k$ has positive characteristic, it is linked to modular representation theory. Invariant theory of infinite groups is linked with linear algebra, especially to the theory of quadratic forms and determinants. Representation theory of semisimple Lie groups has its origins in invariant theory.

Hilbert’s Fourteenth Problem from the 1900 ICM asks if invariants are always finitely-generated (despite his brilliance, Hilbert believed so, but actually they are not, as was shown by Nagata). For reductive algebraic groups however, finite generation does hold (Weyl used his famous unitary trick to show that such groups are linearly reductive). This fails spectacularly in characteristic $p$, though Mumford conjectured and Haboush proved that reductive groups are geometrically reductive, and then Nagata’s results proved this was enough to ensure finite generation. Subsequently there have been many developments, due to Chevalley, Shephard and Todd, Serre, Rota, Wilkerson and many others. In general, the ring of invariants of a finite group acting linearly on a complex space is *Cohen-Macaulay* so it is a finite rank free module.
over a polynomial subring: this has been generalised by many people, and consideration of related questions such as whether the ring is a UFD was done by Samuel and Nakajima.

A good survey is [2]. The texts [1], [4], [5] and [6] (and several others not mentioned here) all take different approaches, according to the whims and interests of the authors. Benson’s text [1] is short and accessible. An essay might deal with Cohen-Macaulay rings, canonical modules and the Dickson invariants. Alternatively one could concentrate on the proof of the Carlisle-Kropholler Conjecture. Applications to combinatorics appear in Stanley’s survey article [7]. Those who know about representation theory of algebraic groups should see the monumental treatment of [3].

### Relevant Courses

**Essential:** Part III Commutative algebra, Undergraduate courses in Representation theory (i.e. ordinary character theory and some idea about compact groups), ring theory and Galois theory.

**Useful:** Lie Algebras

### References


### 34. The Goodwillie–Weiss Calculus of Embeddings

**Professor O. Randal-Williams**

A basic problem in differential topology is to understand when one manifold $N$ can be smoothly embedded in another manifold $M$. More ambitiously, one can consider the space $\text{Emb}(N, M)$ of all smooth embeddings, and ask about its homotopy type. It can be very rich: understanding the set of path components of $\text{Emb}(S^1, \mathbb{R}^3)$ is the subject of knot theory.

The most powerful tool for understanding spaces of embeddings is the so-called “embedding calculus” introduced by Goodwillie and Weiss [1], which gives a sequence of “polynomial” approximations $\text{Emb}(N, M) \to T_k \text{Emb}(N, M)$. Under appropriate conditions these approximations converge to $\text{Emb}(N, M)$.

This essay should give an exposition of this subject in the special case of embeddings of $[0, 1]$ into $[0, 1] \times \mathbb{R}^{n-1}$ (with boundary conditions). You should first describe the polynomial approximations and their layers in this case, and explain the proof that it converges for $n \geq 5$ following [2] (rather than the hard theorem of Goodwillie–Klein–Weiss, which gives it for $n \geq 4$). You
should then discuss Sinha’s cosimplicial model [3] for the polynomial approximations and its associated spectral sequence, and give some details of its behaviour in low degrees following [4].

Relevant Courses

Essential: Part III Homotopy Theory

References


35. Immersions and the \( h \)-principle ..............................................

Professor O. Randal-Williams

A remarkable theorem of Smale shows that the space \( \text{Imm}(N, M) \) of immersions of one smooth manifold to another may be described in purely homotopy-theoretic terms, as long as \( N \) is noncompact or \( \dim(N) < \dim(M) \). This can be used, for example, to show that \( \text{Imm}(S^2, \mathbb{R}^3) \) is path-connected, but it is extremely non-obvious that the two embeddings \( v \mapsto \pm v : S^2 \hookrightarrow \mathbb{R}^3 \) may be joined by a path of immersions [1].

This essay will begin by explaining a vast generalisation of Smale’s theorem: Gromov’s \( h \)-principle for open partial differential relations, or even for (micro)flexible sheaves. There are many approaches one could follow for this [2,3,4], and we can choose one that fits your mathematical taste.

There are then a vast number of applications you could explain, to: immersions, foliations, symplectic or contact topology, configuration spaces, spaces of submanifolds, ... You should again discuss this with me, and we will find something that aligns with your interests.

Relevant Courses

Essential: Part III Homotopy Theory

References

This essay is an introduction to the methods of stable homotopy theory. The focus of the essay is the Nilpotence Theorem of Devinatz, Hopkins, and Smith: for a finite cell-complex $X$, a self-map $f : \Sigma^n X \to X$ is nilpotent (i.e. $\Sigma^k f \circ \cdots \circ \Sigma^n f \circ f$ is homotopic to a constant map for some $k$) if and only if its induced map on a certain generalised homology theory known as complex bordism is nilpotent. This is a fundamental result in the subject, and underlies the modern “chromatic” point of view on homotopy theory.

Your goal should be to follow the proof in [3], adding details from the original paper [2] and elsewhere as necessary. Before doing that you will need to read parts of [1] for background in stable homotopy theory. Be warned that you will not be able to cover all details, so you should carefully discuss with me what you plan to include and what you plan to assume.

Relevant Courses

*Essential:* Part III Homotopy Theory

References


Intersection theory has been a cornerstone of research in algebraic geometry for centuries. The starting point is to associate to an algebraic variety $X$ a collection of groups $A_\ast(X)$ comprising linear combinations of $\ast$-dimensional subvarieties of $X$, considered up to an equivalence. The latter is modeled on allowing “nearby” subvarieties to be identified, and the entire theory is an algebro-geometric parallel to homology groups in algebraic topology. In good circumstances, these groups can be assembled to form a ring. However, intersection theory is typically much more sensitive an invariant than homology. For example, if $X$ is an elliptic curve over the complex numbers, $A_0(X)$ is not even finitely generated.

Famous elements of the subject include Bezout’s theorem, Schwartz and MacPherson’s construction of Chern classes for singular varieties, the Grothendieck–Riemann–Roch theorem, and the modern theory of virtual fundamental classes.

This essay will focus on a single element of intersection theory, provide the necessary background, and a detailed account of this element, including concrete calculations and applications. Possible directions of exploration include the following.

- A treatment of Chern classes in algebraic geometry, focusing in particular on Chern classes for singular varieties, and the construction of the Chern–Schwarz–Macpherson class in the context of the Deligne–Grothendieck conjecture.
• A careful study of aspects of intersection theory on the moduli space of curves, including potentially the Witten–Kontsevich theorem connecting this intersection theory to 2-dimensional quantum gravity, or $\lambda_g$–conjecture.

• A discussion of the intersection theory on special classes of varieties, such as Grassmannians or toric varieties, including a foray into torus equivariant intersection theory or classical results in Schubert calculus.

• The Fourier–Mukai transform on the intersection ring of an abelian variety, and its relationship with the classical theory of curves and their Jacobians.

There are numerous other potential directions, which vary in flavour and in required background. The essay is meant to engage a substantial result or calculation in the subject rather than construct the foundations of intersection theory from the ground up.

**Relevant Courses**

*Essential*: Part III Algebraic Geometry, Part II Algebraic Topology.


**References**


**38. On Compressible Fluid Dynamics I** .................................

*Professor P. Raphael*

Singularity formation in fluid mechanics problem is the frontier of current research and a source of amazing challenges at the frontier of mechanics, physics and mathematics. The recent series of works [1], [2], [3] has opened a breach for the qualitative description of singularity formation for compressible fluids.

This essay is an introduction to the subject. The aim is in particular to understand various qualitative aspects of the phase portrait describing self similar solutions, in particular in connection with special relevant trajectories.
39. On Compressible Fluid Dynamics II .................................

Professor P. Raphael

Singularity formation in fluid mechanics problem is the frontier of current research and a source of amazing challenges at the frontier of mechanics, physics and mathematics. The recent series of works [1], [2], [3] has opened a breach for the qualitative description of singularity formation for compressible fluids.

This essay is an introduction to the subject. The aim is in particular to understand various qualitative aspects of the phase portrait describing self similar solutions, in particular through the investigation of other symmetry classes.

Relevant Courses

Introduction to Non Linear Analysis. Introduction to PDE’s.

References


40. Higher Regulators of Number Fields .................................

Professor A. J. Scholl

The analytic class number formula gives an interpretation for the residue at $s = 1$ of the Dedekind zeta function of a number field in terms of the regulator, constructed from logarithms of absolute values of units.

In 1977, A. Borel [1] found a remarkable generalisation of this result, giving an interpretation of the values of the Dedekind zeta at arbitrary integers in terms of “higher regulators”, in which
the group of units is replaced by a higher K-group of the rings of integers. The proof relies on the relation between K-theory and homology of general linear groups of rings of integers, which is then studied using Lie algebra cohomology. The goal of this essay is to understand the statement of Borel’s theorem and some of the ingredients of the proof. The courses on Local Fields/Algebraic Number Theory and Lie Algebras (and perhaps Algebraic Topology) will help in tackling the essay.

**Related courses**

Local Fields, Algebraic Number Theory, Lie Algebras, Algebraic Topology

**References**


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**41. Trisections and the Thom Conjecture**  Professor I Smith

A trisection of a smooth four-manifold, introduced by Gay and Kirby, is a way of cutting it into three standard pieces, analogous to a Heegaard splitting of a three-manifold. Recently, trisections have been used by Lambert-Cole to give a new proof of the ‘Thom conjecture’, which asserts that a smooth algebraic curve in the complex projective plane has the smallest possible genus of any smooth connected surface in its homology class (a result originally proved by Kronheimer and Mrowka using gauge theory and elliptic analysis). The new proof of the Thom conjecture reduces the 4-dimensional problem to a 3-dimensional problem of understanding certain ‘ribbon surfaces’ bounding transverse links in the standard contact 3-sphere. This essay will discuss trisections, with illustrative examples, and then explain the new proof of the Thom conjecture.

**Relevant Courses**

*Essential:* Algebraic Topology, Differential Geometry, Symplectic Topology

*Useful:* Mapping Class Groups.

**References**


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**42. Non-Orientable Lagrangian Surfaces**  Professor I Smith

A classical question in symplectic topology is to understand the Lagrangian surfaces in a fixed symplectic four-manifold. Simple cases are already surprisingly involved: a long-standing conjecture, resolved around 2008 by Shevchishin and Nemirovskii, proves that there is no Lagrangian Klein bottle in $\mathbb{R}^4$; a recent paper of Shevchishin and Smirnov shows that a Lagrangian...
real projective plane embeds into the 3-fold blow-up of the 4-ball if and only if the blow-up sizes satisfy a ‘triangle inequality;’ it is still unknown for which symplectic structures on $S^2 \times S^2$ there are Lagrangian Klein bottles, etc. This essay will discuss some of these results. Some of the arguments are elementary (in the sense of being topological in nature: using properties of intersection forms of four-manifolds, or ‘realising’ Lagrangian surfaces in almost-toric pictures), whilst others use the Gromov-Floer theory of pseudoholomorphic curves, which is built on analysis of solutions to elliptic PDEs. The foundations of this theory can be black-boxed, but the way the existence results for holomorphic curves enter into the topological constraints on surface embeddings should be explained.

**Relevant Courses**

*Essential:* Algebraic Topology, Differential Geometry, Symplectic Topology

*Useful:* Mapping Class Groups. (*Essential for some possible routes through the essay.*)

**References**

[1] Shevchishin, V. *Lagrangian embeddings of the Klein bottle and the combinatorial properties of mapping class groups.* Translation in Izv. Math. 73 (2009), no. 4, 797–859


**43. Displaceability of Lagrangian Toric Fibres**

*Dr J. Smith*

Given a Lagrangian submanifold $L$ in a symplectic manifold $(X,\omega)$, one can ask whether $L$ can be displaced from itself via a Hamiltonian flow. Such questions arise naturally in the study of dynamical systems, but the solution turns out to be extremely subtle, leading to the development of Floer theory and remarkable connections to topology and algebraic geometry.

Toric manifolds provide an important testing ground for such questions, as they can be described combinatorially (and pictorially) via polytopes, and come with a natural supply of Lagrangian submanifolds: *toric fibres*. The aim of this essay is to describe recent progress on the displaceability question in this case.

You should begin by learning the basics of toric geometry, symplectic reduction, and moment polytopes as covered in [2], and how the method of probes [3] can be used to displace many fibres. Then understand the techniques used in [1] to prove non-displaceability. This paper has many beautiful examples, including a collide-and-scatter phenomenon, and you may wish to look for novel examples of your own. Depending on your interests you could then delve further into the details of Floer theory, or explore connections to mirror symmetry or the minimal model program.
Gentle Algebras and Fukaya Categories of Surfaces

Dr J. Smith

The Fukaya category $\mathcal{F}(X)$ of a symplectic manifold $X$ encodes information about Lagrangian submanifolds in $X$ and their intersections. The morphism groups are defined using Floer theory, which involves counting holomorphic curves in $X$, but we shall restrict to the case where $X$ is a surface $\Sigma$ (of real dimension 2), where a purely combinatorial description is possible.

For each generating object $L$ in $\mathcal{F}(\Sigma)$ one obtains an equivalence between the $\mathcal{F}(\Sigma)$ and the derived category of dg-modules over the algebra $A = \text{hom}(L, L)$. Choosing different generators $L$ may give different algebras $A$, but by construction all of them have equivalent derived categories. These derived equivalences may be non-obvious from a purely algebraic perspective.

It turns out that $A$ is a gentle algebra—of significant recent interest in representation theory—and Lekili–Polishchuk [3] showed how to reverse the above construction, associating a surface to each gentle algebra. From this they gave a new sufficient condition for derived equivalence of gentle algebras, and a geometric description of the ‘AAG’ derived invariants [1].

In this essay you will understand and sketch the description of the Fukaya category in [2, Sections 3.1–3.4], explain in detail the key results of [3], and compute illustrative examples of derived equivalences and inequivalences. Technical results from [2] may be taken as black boxes. If time permits, you could then cover related topics that interest you, e.g. from [4].

Relevant Courses

Essential: Algebraic Topology

Useful: Finite Dimensional Lie and Associative Algebras (especially quiver algebras)

References


Let $f$ be a cuspidal modular form; then it admits a $q$-expansion $f(q) = q + a_2q^2 + a_3q^3 + \ldots$. Two modular forms $f, g$ are said to be congruent if the coefficients of their $q$-expansions lie in the ring of integers of a number field, and there is a prime ideal $P$ of that ring of integers such that the coefficients $a_p$ are congruent modulo $P$ for all but finitely many primes $p$.

A question of Ribet asks: suppose that $f$ is a Hecke eigenform of level $N$, and let $\ell$ be a prime not dividing $N$. When is $f$ congruent to another Hecke eigenform of level $N\ell$, which is new at $\ell$? Ribet showed that in most cases the simple necessary condition on the coefficient $a_\ell$ mod $P$ is also sufficient.

Congruences of this kind between modular forms are an important tool for studying their relation with Galois representations. The aim of this essay will be to describe the theory of newforms and to establish the existence of level-raising congruences in the case of weight $k = 2$. If time and space permit, one could continue to either treat the case of modular forms of weight $k > 2$, or discuss the existence of congruences to newforms of level $N\ell^a$ for $a > 1$.

**Relevant Courses**

*Essential:* Modular Forms  
*Useful:* Algebraic Number Theory

**References**

Serre asked how these numbers are distributed in $[-2, 2]$ as $k$ varies, and used the Eichler–Selberg trace formula to show that in fact they become equidistributed as $k \to \infty$ with respect to the density
\[
\frac{(p + 1)}{\pi} \frac{\sqrt{1 - x^2/4}}{(\sqrt{p} + \sqrt{p^{-1}})^2 - x^2}.
\]
The goal of this essay would be to give an exposition of the Eichler–Selberg trace formula for the trace of a Hecke operator $T_n$ on $S_k(\text{SL}_2(\mathbb{Z}))$, and a proof of Serre’s theorem. If time and space permit, one could treat the interpretation of the above density in terms of the Plancherel measure for $\text{GL}_2(\mathbb{Q}_p)$.

**Relevant Courses**

*Essential:* Modular Forms  
*Useful:* Local Fields

**References**


**47. Combinatorial Morse Theory .............................................**

Dr H. Wilton

Classical Morse theory (as developed in [4]) is a way of understanding the topology of a manifold $M$ via a *Morse function* $M \to \mathbb{R}$. Combinatorial Morse theory is an analogue of this theory for cell complexes, developed in parallel by Forman and Bestvina–Brady.

The latter were able to use it to provide counterexamples to various longstanding problems in topology: they constructed an infinitely presented group of type $FP_2$, and showed that at most one of the Whitehead and Eilenberg–Ganea conjectures hold [1].

The goal of this essay is to describe Bestvina–Brady’s construction, and to give a proof of their main theorem, along with appropriate background material [2, 3].

**Relevant Courses**

*Essential:* Part II Algebraic Topology  
*Useful:* Topics in Geometric Group Theory
48. Hyperbolic Groups ............................................. H. Wilton

Hyperbolic groups (as defined by Mikhail Gromov [4]) have become a powerful tool in modern group theory and low-dimensional topology. A metric space is called hyperbolic if it satisfies a coarse geometric inequality motivated by the negatively curved geometry of the hyperbolic plane. A group is called word-hyperbolic, or just hyperbolic, if it acts nicely on a hyperbolic space. The class of hyperbolic groups is so large that it is generic (in a suitable sense) among all finitely presented groups, and yet small enough that we can prove theorems about them: for instance, the word problem is solvable in hyperbolic groups.

The goal of this essay is to bring together a diverse group of sources [1, 2, 3] to develop the basic theory of these groups.

Relevant Courses

Essential: Part IB Geometry
Useful: Part II Algebraic Topology, Topics in Geometric Group Theory,

References


49. Deligne–Lusztig Theory ......................................................... Dr R. Zhou

One of the great achievements of mathematics in the last century was the classification of finite simple groups. The most interesting class of such groups are the simple groups of Lie type; these arise as the rational points of certain reductive groups over finite fields (e.g., $PGL_2(F_p)$).

Motivated by a fundamental example of Drinfeld, Deligne and Lusztig gave a beautiful geometric construction of representations of finite simple groups of Lie type by considering the $\ell$-adic cohomology of certain varieties over finite fields (now called Deligne–Lusztig varieties).

The aim of this essay is to understand the construction of Deligne–Lusztig varieties and the representations occurring in their cohomology, treating in some detail the example of the Drinfeld curve. You may want to cover some basics on algebraic groups, but the theory of $\ell$-adic cohomology can be taken as a black box.

Time permitting, the essay could also cover the following topics:

- Proof of the main theorem.
- Lusztig’s classification of irreducible characters.
- Affine Deligne–Lusztig varieties.

Relevant Courses

Essential: Part II Representation Theory (or equivalent), Algebraic Geometry.

Useful: Local fields.

References


50. Abelian Varieties over Finite Fields ......................................................... Dr R. Zhou

Abelian varieties are higher dimensional generalizations of elliptic curves and are fundamental objects of study in arithmetic geometry. A theorem of Weil tells us that the action of Frobenius on the $\ell$-adic Tate module of a simple abelian $A$ variety over $\mathbb{F}_q$ is given by a Weil $q$-integer $\gamma_A$.

Remarkably, Honda and Tate showed that the association $A \mapsto \gamma_A$ gives a bijection between isogeny classes of simple abelian varieties over $\mathbb{F}_q$ and conjugacy classes of Weil $q$-integers.

The aim of this essay is to understand the proof of the result of Honda and Tate. The essay can begin with some recollections on abelian varieties, and should cover the ingredients needed
in the proof such as Tate’s isogeny theorem and the theory of CM abelian varieties including the Shimura–Taniyama reciprocity law.

Time permitting, possible additional topics to be covered can include:

- Manin’s Problem.
- The Serre–Tate theorem and canonical lifts of ordinary abelian varieties.
- (For the very ambitious) CM liftings on Shimura varieties.

Relevant Courses

**Essential:** Algebraic Geometry, Local Fields.

**Useful:** Elliptic Curves.

References


51. Obstructions to Embeddings in the Coarse Geometry of Banach Spaces  

Dr A. Zsák

Let \((X,d_X)\) and \((Y,d_Y)\) be metric spaces. A function \(f : X \to Y\) has *compression modulus* \(\rho_f\) and *expansion modulus* \(\omega_f\) defined as \(\rho_f(t) = \inf \{d_Y(f(x), f(y)) : d_X(x, y) \geq t\}\) and \(\omega_f(t) = \sup \{d_Y(f(x), f(y)) : d_X(x, y) \leq t\}\) which satisfy \(\rho_f(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \omega_f(d_X(x, y))\) for all \(x,y \in X\). We say that \(f\) is a *uniform embedding* if \(\lim_{t \to 0^+} \omega_f(t) = 0\) and \(\rho_f(t) > 0\) for all \(t > 0\) (or, equivalently, if \(f\) is uniformly continuous, injective with uniformly continuous inverse). We say that \(f\) is a *coarse embedding* if \(\lim_{t \to \infty} \rho_f(t) = \infty\) and \(\omega_f(t) < \infty\) for all \(t > 0\).

Coarse embeddings need not be continuous or injective. Yet they play an important rôle in the nonlinear theory of Banach spaces. For example, Yu established a remarkable connection between the Novikov and Baum–Connes conjectures on the one hand and coarse embeddings into Hilbert space on the other. Kasparov and Yu generalized this to coarse embeddings into uniformly convex Banach spaces. For a while it was not known if a coarse embedding into a uniformly convex Banach space implies a coarse embedding into Hilbert space until Johnson and Randrianarivony proved that \(\ell_p\) does not coarsely embed into \(\ell_2\) for \(p > 2\) [1]. Their method uses a coarse invariant involving negative definite kernels. This result was generalized by Mendel and Naor with a complete description of coarse embeddings between \(L_p\) spaces [2]. This was one of several applications of their seminal work on metric cotype.

There is strong evidence that \(\ell_2\) is the hardest space to coarsely embed into which raised the question whether \(\ell_2\) coarsely embeds into every infinite-dimensional Banach space. Then Baudier, Lancien and Schlumprecht gave a negative answer [4]. The obstruction they used is a coarse invariant in form of a concentration inequality for Lipschitz functions. This was
inspired by a concentration inequality of Kalton and Randrianarivony used in their study of the coarse geometry of $\ell_p \oplus \ell_q$ [3]. Later Baudier, Lancien, Motakis and Schlumprecht used the concentration inequality of [4] to give the first example of a coarsely rigid unrestricted class of infinite-dimensional Banach spaces [5].

A successful essay should begin with proofs of the result in [1] and its generalization in [2]. It should then develop the concentration inequalities in [3], [4] and [5] with applications that should include proofs of the uniqueness of uniform structure on $\ell_p \oplus \ell_q$ for $1 < p < 2 < q < \infty$ [3], the coarse non-embeddability of $\ell_2$ into Tsirelson’s original space [4] and the construction of the new coarsely rigid class in [5].

**Relevant Courses**

*Useful:* Metric Embeddings and familiarity with standard results in Functional Analysis

**References**


52. **Stein Discrepancies** .................................................................

*Dr S. A. Bacallado*

In Bayesian analysis and other areas, a probability measure $\pi$ on $\mathbb{R}^d$ is often approximated with a discrete measure $\hat{\pi} = n^{-1} \sum_{i=1}^{n} \delta_{x_i}$ where $\delta_x$ is a point mass at $x \in \mathbb{R}^d$. For example $\pi$ may be a posterior distribution and $x_1, \ldots, x_n$ the output of a Markov chain Monte Carlo algorithm. In [1], Gorham and Mackey propose a method for measuring the approximation quality, which is applicable in situations where the density of $\pi$ can only be evaluated up to a constant. They introduce the notion of a Stein discrepancy, and show that under some conditions on $\pi$, it upper and lower bounds the Wasserstein-1 distance, i.e. it is capable of detecting when $\hat{\pi}$ converges to $\pi$ in Wasserstein distance, and also when it doesn’t. A vigorous literature around Stein discrepancies has evolved in the last 5 years (see [2-8]). The goal of this essay is to synthesise the ideas in 2 to 4 of the papers cited. Implementation of the methods and numerical examples are encouraged for those interested.

**Relevant Courses**

*Essential:* Modern Statistical Methods.
References


53. Bayesian Inference in Geometric Inverse Problems ......................

Dr S. A. Bacallado

Many inverse problems of scientific importance arise from measurements of a function \( \theta \) in a compact domain of Euclidean space, where \( \theta \) is the binary, characteristic function of a set with smooth boundaries. The model can be written \( Y = A\theta + \varepsilon \) where \( Y \) are the measurements, \( \theta \) is the object of interest, \( A \) is a forward operator, and \( \varepsilon \) is noise. Measurement noise makes the problem ill-posed, and one can quantify the uncertainty of a solution through Bayesian inference, which involves putting a prior on \( \theta \) and sampling the posterior distribution. Various priors for characteristic functions have been proposed in the literature; a prominent choice involves putting a prior on a latent function \( \vartheta \) and letting \( \theta(x) = \delta_{\vartheta(x) > 0} \) be the characteristic function of a level set of \( \vartheta \). Characterisations of the posterior are available for various forward operators. In particular, the property of Bayesian well-posedness has been established in many examples, which guarantees stability of the posterior and makes it possible to simulate it via Markov chain Monte Carlo. The goal of this essay is to review a subset of the results in the references provided. Numerical experiments are encouraged for those interested.

Relevant Courses

*Essential:* Modern Statistical Methods.

References


54. Singular SPDEs and their Invariant Measures  

Dr R. Bauerschmidt

The massive Gaussian free field (GFF) on the two-dimensional torus $\mathbb{T}^2$ is a Gaussian probability measure $\nu$ on the space of periodic Schwartz distributions $S' (\mathbb{T}^2)$ with covariance given by the inverse of the massive Laplace operator $-\Delta + 1$ on $\mathbb{T}^2$. The ‘$\Phi^4$ measure’ on $\mathbb{T}^2$ is formally obtained from the GFF measure $\nu$ by the weight

$$\mu(d\varphi) = \frac{1}{Z} \exp \left[ - \int_{\mathbb{T}^2} \varphi(x)^4 : dx \right] \nu(d\varphi).$$

(1)

Since $\varphi$ is a Schwartz distribution and not (almost surely) a function, the product $\varphi(x)^4$ is not defined and the notation $\varphi^4 :$ above denotes a regularisation procedure called Wick ordering. Measures of this type play a role in mathematical statistical mechanics, quantum field theory, and in PDE theory where they arise, for example, as invariant measures of singular stochastic PDEs (SSPDEs) and of non-linear Schrödinger equations. In particular, the related SSPDE is

$$d\varphi = \Delta \varphi - \varphi - 4 \varphi^3 : + \sqrt{2} dW(t),$$

(2)

where again the polynomial must be interpreted Wick ordered and, as a consequence, the interpretation of solutions is non-obvious. A successful essay could either focus on the construction and properties of the ‘$\Phi^4$ measure’ or on the construction of strong solutions to the corresponding SSPDE.

**References**


55. Using Quasi-Experimental Methods to Guide Experimental Trials  

Dr S. Burgess

Quasi-experimental methods, such as instrumental variables and regression discontinuity designs, are seen as an alternative source of information for judging causal relationships when
direct experimentation is not possible. For example, it is impractical to conduct a randomized trial to assess the causal effect of alcohol consumption on cardiovascular disease risk. Observational analyses assessing the association of alcohol consumption on cardiovascular disease risk can be conducted, but inferring causality from such analyses is unreliable. Genetic variants that plausibly satisfy the instrumental variable assumptions for alcohol consumption have been demonstrated to be associated with cardiovascular disease risk [1,2], providing evidence that alcohol consumption is a causal risk factor for many cardiovascular diseases.

The aim of this essay is to consider how quasi-experimental approaches could provide evidence to guide the design of a clinical trial. Even when a trial can be run, quasi-experimental approaches can be implemented quickly and cheaply. In some cases, quasi-experimental approaches could be used to identify subgroups of the population who would respond particularly strongly to the treatment, meaning that the subsequent trial can be made more efficient. This is an example of stratified medicine. Analyses using quasi-experimental approaches could help to define the trial endpoint, could help to clarify the mechanism of the causal effect, and could identify individuals who would respond to treatment. This essay should review the available literature on precision medicine and how it relates to quasi-experimental methods (in particular, to instrumental variables techniques). In particular, the essay could explore the use of stratification in instrumental variable analysis: what assumptions are required for investigators to validly estimate the causal effect of an exposure on an outcome within strata of a third variable?

Relevant Courses

Essential: Causal Inference, Statistics in Medical Practice.

References


56. Using the Front-Door Criterion for Causal Identification in Practice . . .

Dr S. Burgess

Pearl provides two criteria for causal identification [1]: the front-door criterion and the back-door criterion. In the vast majority of applications, the back-door criterion is used in practice. The objective of this essay is to explore the possibility of using the front-door criterion for causal identification in practice. The essay could take several routes, but should consider both theoretical (that is, what assumptions are required?, how can a causal estimate be identified and estimated?) and pragmatic considerations (that is, is it plausible that one could find a variable satisfying the front-door criterion?) [2]. Would it be possible to design a trial that uses the front-door criterion for estimation? Is there an equivalent to instrumental variables that exploits the front-door criterion?

Relevant Courses

Essential: Causal Inference, Statistics in Medical Practice.
Roughly and somewhat incorrectly speaking, for the purposes of this essay an ergodic system is the most general form of a stochastic process that satisfies the strong law of large numbers. Ergodic theory has its origins in the study of abstract dynamical systems. It is mostly analytical in flavor and it also shares many tools with probability theory and information theory. The entropy has played a central role in the development of ergodic theory, and it has provided deep and strong connections with probability and information theory.

This essay will explore and describe the foundations of the ergodic theory of discrete sample paths as outlined in the first four sections of [1], and the connections with entropy and information theory in the rest of Chapter I and in Chapter II of [1]. The material in sections I.9 and I.10 is interesting but not essential. More ambitious essays can consider discussing material from Chapter 5 of [2], Section 14 of [3], or the more recent work [4].

References


58. Robust Estimation via Robust Optimization .......................... Professor P. Loh

Many classical robust estimation procedures are based on $M$-estimators [6, 7]. Various theoretical results have been derived under the contamination setting where data are drawn i.i.d.
from a mixture distribution \((1 - \epsilon)P_\theta + \epsilon Q\), where \(P_\theta\) is the uncontaminated distribution, \(Q\) is arbitrary, and the goal is to estimate \(\theta\) with small statistical error [8].

More recently, several authors have proposed a new strategy for robust parameter estimation by introducing a robust estimation step directly into an iterative optimization procedure [5, 11, 10]. Rather than the classical approach of minimizing a robust loss function applied to individual data points, the idea is to minimize a non-robust loss function via an iterative optimization procedure which is itself robust. The proposed optimization procedure operates by replacing the gradient in successive gradient steps by a robust mean estimator for the population-level gradient vector [4, 9, 1]. Theoretically, statistical error bounds are proved which are valid even when the form of contamination is adversarial rather than in the form of a contaminated mixture.

The goal of this essay is to compare and contrast the aforementioned robust estimation techniques both empirically and theoretically. What are the tradeoffs between different procedures? How do they compare with classical M-estimation under different contamination models? Going beyond these proposals for robust gradient descent, how might other first-order optimization methods [3, 2] be made robust in an analogous manner, and what conclusions may be drawn for such procedures?

Relevant Courses

*Essential:* Robust Statistics

*Useful:* Optimisation

References


59. Conformal Removability and Schramm-Loewner Evolutions

Professor J. Miller

Suppose that $U \subseteq \mathbb{C}$ is an open set and $K \subseteq U$ is compact. Then $K$ is said to be \textit{conformally removable} if the following is true. Suppose that $V \subseteq \mathbb{C}$ is open and $\varphi: U \to V$ is a homeomorphism which is conformal on $U \setminus K$. Then $\varphi$ is conformal on all of $U$. The simplest example of a conformally removable set is a single point, but it turns out that much larger sets (such as various types of fractal curves) can be conformally removable as well. One important condition for establishing the conformal removability of a compact set is due to Jones-Smirnov [1] and its improvement due to Koskela-Nieminen [2]. One important application of conformal removability is in the context of the uniqueness of \textit{conformal weldings}, which are defined as follows. Suppose that $D_1, D_2$ are two copies of the unit disk and $\phi: \partial D_1 \to \partial D_2$ is a homeomorphism. A conformal welding of $D_1, D_2$ using the identification $\phi$ corresponds to a simple loop $\eta$ on $S^2$ and two conformal transformations $\psi_1, \psi_2$ which take $D_1, D_2$ to the two components of $S^2 \setminus \eta$ with $\phi = \psi_1 \circ \psi_2^{-1}$. Conformal removability is important in the context of conformal welding because it is related to when a given conformal welding is unique (i.e., for a given $\phi$ there is at most one $\eta$). The purpose of this essay is to review the Jones-Smirnov condition, its improvement due to Koskela-Nieminen, and its application to show that Schramm-Loewner evolution (SLE) curves are conformally removable when $\kappa < 4$ [3]. A successful essay will in addition contain discussion of the importance of the removability of SLE$_\kappa$ with $\kappa < 4$ curves in the context of Liouville quantum gravity [4].

Relevant Courses

\textit{Essential}: Complex Analysis, familiarity with Sobolev spaces.


References


Perform bond percolation on $\mathbb{Z}^d$ (which means that each edge is kept independently with probability $p$). The critical probability is defined to be the supremum of values of $p$ for which there almost surely does not exist an infinite connected component in the resulting random graph. We now fix $p > p_c$ and consider the unique infinite cluster $\mathcal{C}$ of bond percolation with parameter $p$. One way of probing the geometry of $\mathcal{C}$ is to perform a simple random walk on it, which is a process that at each time step jumps to a uniformly chosen neighbor (in the graph $\mathcal{C}$) of its current position. Barlow obtained Gaussian upper and lower bounds on the transition density for the continuous time walk and a few years later, Berger and Biskup and independently Matthieu and Piatnitski proved that for almost every percolation configuration the path of the walk suitably rescaled converges weakly to that of non-degenerate, isotropic Brownian motion. A successful essay should give an account of these developments and include proofs (or overviews of proofs) of the important results.

**Relevant Courses**

Advanced Probability, Percolation.

**References**


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**61. Convergence Guarantees for Langevin-Type MCMC Algorithms**

Professor R. Nickl & Dr. R. Altmeyer

Markov chain Monte Carlo (MCMC) algorithms are used to produce samples from typically non-standard probability distributions. They are based on specifically designed ergodic Markov chains, whose invariant measures correspond to the target distribution. MCMC algorithms are ubiquitous in many applications, for example, to compute high dimensional integrals or to sample from posterior distributions in Bayesian inference. It is therefore important to have a good understanding when these algorithms perform well and when not.

While there is a large amount of empirical works studying the convergence properties of MCMC algorithms, there are only very few theoretical guarantees. A particular challenge is the sampling from high dimensional target distributions, where the curse of dimensionality will generally lead to computational costs depending exponentially on the dimension. On the other hand, nonasymptotic convergence results with polynomial dependence on the dimension have been obtained only very recently in [2,3] for Langevin-type algorithms, a particular class of MCMC algorithms based on discretised stochastic differential equations, assuming the target distribution has a log-concave Lebesgue density. In this setting, the number of steps required to achieve a given accuracy of approximation is shown to depend polynomially on the dimension, as well.
The goal of this essay is to review the basic ideas underlying the various Langevin-type MCMC algorithms (as discussed in [6], for example), explain the theoretical results from [2] and discuss their relevance for applications. The results from [2] should also be compared either to improvements in [3,5], results for related algorithms in [4] or in the context of Bayesian nonlinear inverse problems [6]. This essay may also explore the conceptually closely related gradient descent optimisation methods, which can be found in [1].

Useful

Basic knowledge of statistics and probability.

References


62. Probabilistic Approaches to the Boltzmann Equation

Professor J. R. Norris & Dr D. Heydecker

Boltzmann introduced a partial differential equation as a model for the distribution of velocities in a homogeneous, dilute gas, and how this evolves over time. Connected to this equation, there are a number of interesting questions: (1) derivation of the equations, (2) relaxation to equilibrium in large time, and (3) well-posedness of the limiting equation.

While substantial aspects of these questions remain out of reach even now for Boltzmann’s full model, they have been the source of challenging questions even in a simplified model where the spatial location of particles is disregarded and collisions are treated via a population average.

Regarding the first two points, Kac [1] introduced a simple random model incorporating the velocities of many particles, and hoped to study the mixing time of the Markov chain in order to understand the relaxation to equilibrium of the limit equation. The third question (3) has also been investigated by probabilistic techniques: Tanaka [3] introduced an interpretation of the Boltzmann equation in terms of a nonlinear stochastic differential equation, and a probabilistic coupling method to study the limit PDE.

A good essay should give an account of probabilistic approaches to one of the problems (1-3).
and, where relevant, connections between the problems. A non-exhaustive list of papers which might form starting points for such a review is below.

**Relevant Courses**

**Essential:** Advanced Probability. Mixing Times of Markov Chains may be required for certain approaches to the subject.

**Useful:** Analysis of Partial Differential Equations.

**References**


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### 63. Scaling Limits for Markov Processes

*Professor J. R. Norris*

When a random process has a Markov structure, this allows an economical description of its dynamics. This structure also provides tools to handle limit operations, such as are of interest in stochastic models for large populations or in large random structures.

This essay is intended to offer a flexible opportunity to write an account describing some aspect of the theory or applications of scaling limits for Markov processes, to be agreed.

One approach would be to set out a selection of the methods available to establish scaling limits, where the limit objects are either differential equations, or diffusion processes, or continuous-time Markov chains, or some combination of these.

Beyond the simplest case, there are techniques to handle local averaging over inhomogeneous local structure, or variables which oscillate rapidly. It is possible also in some cases to show convergence with limit dynamics in infinite-dimensions or to obtain uniform-in-time convergence, under suitable stability hypotheses.

The methods should be illustrated by discussion of examples, perhaps with a coherent theme, such as models from mathematical biology, or large random combinatorial structures, or physical particle models.
Another approach to this essay would be to focus from the outset on a single harder example, such as convergence of the discrete Gaussian free field to a continuum limit, or convergence of some stochastic model of interacting particle dynamics in the limit of large numbers.

Relevant Courses


References


64. Optimal Transport Methods in Statistics .................................

Professor R. J. Samworth & Dr Y. Zemel

The problem of optimal transport was introduced in 1781 by Monge [8] and concerns the most efficient way to transport a distribution of objects to a distribution of locations. Its convex relaxation ([6]) gives rise to probability metrics known as Wasserstein, or optimal transport, distances. Within mathematics, these have found applications in, e.g., partial differential equations ([13]) and probability theory ([11]).

In Statistics, Wasserstein distances have been used in a myriad of setups. Classical examples are deviations from Gaussianity ([2]), goodness-of-fit testing ([9]), and concentration of measure ([7]). More recent applications include multivariate quantiles and related notions ([5]), generative adversarial networks ([1]), approximate Bayesian computation ([3]), and convergence of stochastic gradient descent ([4]). Optimal transport distances have also proven successful in data analysis of objects with complex geometric structures such as images ([12]) or point processes ([10]).

This essay could survey a small number of these topics in some depth, with a focus on the statistical aspects of the problem at hand. However, it is also a highly active current research area, and an ambitious candidate may also propose their own research question to address.

Relevant Courses

*Essential:* No specific course is necessary, but students should have mathematical and statistical maturity.

*Useful:* Topics in Statistical Theory, Probability and Measure, Advanced Probability.
Independence is a fundamental concept in both probability and statistics; it distinguishes the former from a mere branch of measure theory, and underpins both statistical theory and the way practitioners think about modelling. For statisticians, it is frequently important to ascertain whether or not assumptions of independence are realistic, both to determine whether certain theoretical properties of procedures can be expected to hold, and to assess the goodness-of-fit of a statistical model.

Recent work on independence testing has focused on nonparametric settings, and a wide variety of procedures have been proposed, including approaches based on $U$-statistics ([1], [2]), $k$-nearest neighbour methods and mutual information ([3]), distance correlation ([4]), the Hilbert–Schmidt independence criterion ([5], [6]) and multivariate ranks ([7], [8]). A significant focus of these contributions has been the determination of the minimax separation rate, i.e. the departure from independence (measured in an appropriate way) required to be detected by a test. Related lines of work include conditional independence testing ([9], [10], [11]), two-sample testing ([2]), goodness-of-fit regression problems ([3]) and graphical modelling ([12]).
One aim of this essay is to compare different approaches to independence testing, highlighting theoretically and/or empirically their relative strengths and weaknesses. There is also scope for candidates to investigate ways in which these ideas and methods can be extended to related statistical problems.

**Relevant Courses**

*Essential:* Topics in Statistical Theory

*Useful:* Information Theory

**References**


**66. Statistical Inference Using Machine Learning Methods**

*Dr R. D. Shah*

The field of machine learning has much overlap with statistics, but has been predominantly concerned with prediction problems. Here it has had great successes with random forests, boosted trees, kernel machines and neural networks, among others, enjoying spectacular predictive performance in a variety of applications.
Statisticians, on the other hand, are often also concerned with parameter estimation for particular models, and uncertainty quantification. Recently, there has been a string of work aiming to harness the predictive power of machine learning methods for these more statistical goals. These can often lead to hypothesis tests with greater power that would typically be achievable using more classical tools [1], or much more robust procedures whose validity holds across a far greater range of data-generating processes. The debiased Lasso may be viewed as one example of this, where the Lasso is used to build confidence intervals for regression coefficients in high-dimensional settings. Building on this, [2–7] consider more general procedures where machine learning methods may be leveraged to estimate nuisance parameters and thereby obtain more accurate estimates of particular parameters of interest.

Much of the work is tightly connected to the rich field of semiparametric statistics, and one option (among several) for the essay would be to review some of the papers referenced below and set some of the work within this context. Another option would be to focus more closely on a smaller subset of the papers and study some examples of the general methodology presented or propose some extensions.

**Relevant Courses**

*Essential:* Modern Statistical Methods

*Useful:* Topics in Statistical Theory, Causal Inference

**References**


**67. Activated Random Walks** .........................................................

Dr P. Sousi

Activated random walks is an interacting particle system with a conserved number of particles occupying the vertices of an infinite graph that can be in two states: active or sleeping. Active particles evolve as independent continuous time random walks and if they are alone at a vertex, then they fall asleep at rate λ. Sleeping particles wake up and become active only when an active particle steps to their location.
Suppose that at time 0, all sites have an i.i.d. Bernoulli number of particles of parameter $\mu$. If eventually all particles fall asleep, then we say that the system fixates. We define the critical density $\mu_c(\lambda)$ as follows

$$\mu_c(\lambda) = \sup\{\mu \geq 0 : \mathbb{P}(\text{ARW fixates}) = 1\}.$$  

Proving that $\mu_c(\lambda) \in (0, 1)$ for all $\lambda > 0$ is a challenging problem. It has been shown to hold for $\mathbb{Z}$ and for transitive transient graphs. However, this is still open for $\mathbb{Z}^2$. In the case of $\mathbb{Z}$ more is known; there exist positive constants $c_1$ and $c_2$ such that

$$c_2 \sqrt{\lambda} \leq \mu_c(\lambda) \leq c_1 \sqrt{\lambda}.$$

A successful essay should include detailed proofs of the main properties of this model as well as of some of the recent results.

**Relevant Courses**

*Useful:* Advanced Probability

**References**


**68. Unmeasured Confounding in High-Dimensional Data**

*Dr R. D. Shah & Dr Q. Zhao*

Unmeasured confounding is the dragon in causal inference, and can heavily bias estimates of causal effects. Unfortunately, it is difficult to avoid in any empirical scientific work using observational data.

Modern high-dimensional data, however, presents a new opportunity to potentially overcome the problem of unmeasured confounding. Intuitively, if a confounder is affecting many of the variables measured, there is a chance that it can be implicitly learned from the observed data such that approximately unbiased estimates of the causal effects can be obtained. Understanding when and how this may be possible is currently the topic of a great deal of research. Settings based on linear models have been explored in [1–4], and nonlinear models [5].

This essay could review and critique the recent developments that attempt to use aspects of high-dimensional data to overcome unmeasured confounding. Another option would be to focus more closely on some particular identification strategies and compare associated methods or slightly modify existing procedures with the aim of obtaining improved performance in certain settings.
Relevant Courses


References


69. A Century of Randomisation Inference

Dr Q. Zhao

In the 1920s, R A Fisher introduced randomisation as the “reasoned basis for inference”. This essay will try to connect the history randomisation inference to modern contexts by combining some of the following elements:

1. A review of the stimulating ideas leading to the introduction of physical randomisation [1-4].
2. A discussion on Fisher’s original proposal [5], later developments in statistics [6,7], and/or the formulation using potential outcomes [8].
3. An overview of modern variations and applications, such as conformal prediction [9] and Mendelian randomization [10,11].

Relevant Courses

Useful: Causal Inference.

References


70. **Statistics in Meta-Research** .................................................. Dr Q. Zhao

Scientific progress relies on accumulating evidence. This essay can be on any statistical aspect(s) about this process. Some potential topics include:

1. The current replication crisis in several scientific disciplines [1].

2. Statistical methods for meta-analysis [2], especially contemporary topics such as network meta-analysis [3] and selective inference [4].

3. Within-study comparisons of different study designs [5-7].

4. Bayesian evidence synthesis [8,9].

5. Estimator selection [10].

**References**


7. Christie, A. et al. (2020). Quantifying and addressing the prevalence and bias of study designs in the environmental and social sciences. *Nature Communications* (in press). (Please ask Dr Zhao for the accepted manuscript.)


### 71. Solutions to Local Anomaly Cancellation

**Prof B C Allanach**

The cancellation of local anomalies is often imposed on gauged quantum field theories. General solutions to anomaly cancellation with fixed field content often borrow techniques from algebraic geometry. The essay should explore solutions to anomaly cancellation conditions in one or more particle physics settings, for example in the minimal supersymmetric standard model, other Standard Model extensions or in $U(1)$ gauge theory with Weyl fermions

**Relevant Courses**

*Essential:* Quantum Field Theory, Standard Model, Symmetries, Fields and Particles, Advanced Quantum Field Theory

**References**


### 72. Rethinking Optical Flow via Deep Nets

**Dr A. Aviles-Rivero, Dr R. Ke & Professor C.-B. Schönlieb**

The mathematical problem of extracting motion from video data is a challenging one arising from various real-world applications. It amounts to compute a velocity field depending on space and time that captures speed and directionality of motion of underlying dynamics in the video. This problem is inherently ill-posed and computationally challenging due to the dimensionality of the data. It is also closely linked to the problem of image registration, which computes a function (which represents the motion and is often modelled as a diffeomorphism) that maps one image onto another one. Mathematical treatment for both tasks includes optical flow [1], optimal transport [2], LDDMM [3] and hyperelastic registration [4].
Recently, a new paradigm has emerged in the community, which aims to either replace or approximate these complex mathematical models through deep neural networks. The latter are computationally demanding to train but cheap at inference. Various examples exist in the recent literature (e.g. [5,6]) demonstrating the state of the art data-driven approaches and their computational efficiency. Only few of these approaches, however, attempt and eventually succeed in preserving some of the desirable mathematical properties of the inferred velocity field or registration map that support robustness and accuracy of the approach e.g. [7,8]. Some of these properties are that the registration map should be diffeomorphic, to avoid self-intersections, invertible and stable to corruptions in the data. In parallel, there are various developments in deep learning which propose neural network architectures and associated loss functions that provide mathematical guarantees [9].

Depending on the interest of the essay writer, this essay can either focus on deep learning approaches for optical flow or image registration and their mathematical properties, or investigate and review the more classical model driven approaches for motion estimation mentioned in the first paragraph. In both cases, it will be interesting to provide a critical viewpoint on the desirable mathematical and computational properties of proposed methodologies.

**Relevant Courses**

**Essential:** None

**Useful:** Background knowledge in functional and and numerical analysis is helpful, as is inverse problems and differential geometry. Some content from mathematics of machine learning may also be useful.

**References**


73. ’t Hooft Anomalies .................................................................

Dr P. Benetti Genolini

Global symmetries are crucial for the understanding of the non-perturbative dynamics of quantum field theories. One of the subtler questions that we should ask is whether we can gauge them or not. In the latter case, there is an ’t Hooft anomaly. Despite the ominous name, this is often a good feature, because these anomalies are protected and should match at small and large distances: we can compute them in the weakly-coupled description and learn about the strongly-coupled regime. As such, ’t Hooft anomalies have provided useful consistency checks of the low-energy behaviour of both supersymmetric and non-supersymmetry field theories, and constraints on their phases.

The purpose of this essay is to describe ’t Hooft anomalies in a modern viewpoint based on anomaly inflow, and survey some of their applications. It should be written at a level that would be understood by another part III student who had attended the relevant courses.

Relevant Courses

Essential: Advanced Quantum Field Theory, Symmetries, Fields and Particles
Useful: Supersymmetry

References


74. Classical Approaches to Simulating Quantum Dynamics .............

Dr B. Béri

Understanding the time evolution of complex quantum systems is required in a number of research areas including thermalisation, nonequilibrium phases of matter, or many-body quantum chaos, just to name a few. Gaining insights into the quantum dynamics is sometimes possible through the classical simulation of the system. The purpose of this essay is to describe various approaches to this and the complementary nature of the corresponding challenges, including
(but not necessarily limited to): (i) matrix-product-state methods and their limitations in terms of the systems and time scales accessible due to the entanglement content of the states of interest, and (ii) methods based on the stabiliser formalism and the corresponding restrictions on the type of efficiently simulatable dynamics.

**Relevant Courses**

*Useful:* Part II Quantum Information and Computation, Part III Quantum Computation

**References**


75. **Symmetry and Symplectic Reduction**

Dr J. N. Butterfield

Symplectic reduction is a large subject in both classical and quantum mechanics. One starts from Noether’s theorem in a classical Hamiltonian framework, and thereby the ideas of: Lie group actions; the co-adjoint representation of a Lie group $G$ on the dual $\mathfrak{g}^*$ of its Lie algebra $\mathfrak{g}$; Poisson manifolds (a mild generalization of symplectic manifolds that arise when one quotients under a symmetry); conserved quantities as momentum maps. With these ideas one can state the main theorems about symplectic reduction. Main texts for this material include [1].

The flavour of these theorems is well illustrated by the Lie-Poisson reduction theorem. It concerns the case where the natural configuration space for a system is itself a Lie group $G$. This occurs both for the pivoted rigid body and for ideal fluids. For example, take the rigid body to be pivoted, so as to set aside translational motion. This will mean that the group $G$ of symmetries defining the quotienting procedure will be the rotation group $\text{SO}(3)$. But it will also mean that the body’s configuration space is given by $G = \text{SO}(3)$, since any configuration can be labelled by the rotation that obtains it from some reference-configuration. So in this example of symplectic reduction, the symmetry group acts on itself as the configuration space. Then the theorem says: the quotient of the natural phase space (the cotangent bundle on $G$) is a Poisson manifold isomorphic to the dual $\mathfrak{g}^*$ of $G$’s Lie algebra. That is: $T^*G/G \cong \mathfrak{g}^*$. There are several ‘cousin’ theorems, such as the Marsden-Weinstein-Meyer theorem. For a philosopher’s exposition of the Lie-Poisson reduction theorem, cf. [2].

The essay should, starting from this basis, expound one or other of the following two topics. (Taking on both would be too much.)

(A): The first topic is technical and concerns the application of these classical ideas to quantum theory: more specifically, the interplay between reduction and canonical quantization. Physically, this is a large and important subject, since it applies directly to some of our fundamental theories, such as electromagnetism and Yang-Mills theories. The essay can confine itself to the
more general aspects: which are well introduced and discussed by Landsman and Belot; [3].

(B): The second topic is more philosophical. It concerns the general question under what circumstances should we take a state and its symmetry-transform to represent the same state of affairs—so that quotienting under the action of the symmetry group gives a non-redundant representation of physical possibilities? This question can be (and has been) discussed in a wholly classical setting. Indeed, the prototype example is undoubtedly the question debated between Newton (through his ammanuensis Clarke) and Leibniz: namely—in modern parlance—whether one should take a solution of, say, Newtonian gravitation for \( N \) point-particles and its transform under a Galilean transformation to represent the same state of affairs. This topic is introduced by the papers in [4]. In particular, Dewar discusses how, even when we are sure that a state and its symmetry-transform represent the same state of affairs, quotienting can have various disadvantages.

**Relevant Courses**

*Useful:* Symmetries, Fields and Particles; Philosophical Aspects of Quantum Field Theory.

**References**


This essay lies at the interface of algebraic quantum theory and the description of quantum measurement, focusing on how they each describe measurement in the setting of Minkowski spacetime. Traditionally, the algebraic approach to quantum theory has an operational flavour, with description of measurement being mostly confined to the words, and not expressed in the formalism. In particular, algebraic quantum field theory traditionally associates to each bounded region of Minkowski spacetime an algebra of operators, whose self-adjoint elements are interpreted as the physical quantities that can be measured by an operation confined to the region in question. There was some, but not much, formal analysis of this operation, e.g. in the work of Hellwig and Kraus, and of Davies and Lewis. But recently, Fewster and Verch have given a detailed analysis along these lines, which includes a theorem expressing no-signalling, i.e. “good causal behaviour”—that spacelike correlations involve no superluminal causation. For this literature, one can begin with Fewster and Verch, in [1].

On the other hand: in 1993, Sorkin argued that assuming that arbitrary field-theoretic quantities could be measured would imply superluminal signalling; (which he took as a sign that the path-integral framework was fundamental). These ideas were developed by Beckman, Gottesmann, Preskill and co-authors: at first using ideas from quantum information theory; and then applying these ideas to the measurement of Wilson loop operators in a gauge theory. A later analysis, again focussing on quantum information processing, is by Benincasa and co-authors. This literature is in [2].

Recently, two papers have addressed anew the topic of measurements that are “impossible” because they imply superluminal signalling—their results being apparently at loggerheads, even contradictory. Borsten and co-authors give a criterion for a measurement to be no-signalling, that implies strong limitations on what can be measured. Bostelmann and co-authors argue Sorkin’s argument, i.e. his protocol for signalling, does not hold good in the Fewster-Verch framework. These papers are in [3].

The aim of the essay is to adjudicate—indeed, resolve!—this debate.

Relevant Courses

Useful: Quantum Field Theory, Philosophical Aspects of Quantum Field Theory, Quantum information theory

References


77. Stratified Turbulent Mixing

Professor C. P. Caulfield

Understanding how turbulence leads to the enhanced irreversible transport of heat and other scalars (such as salt and pollutants) in density-stratified fluids is a fundamental and central problem in geophysical and environmental fluid dynamics. There is a wide range of highly important applications, not least the description and parameterization of diapycnal transport in the world’s oceans, a key area of uncertainty in climate modelling.

Recently, due not least to the proliferation of data obtained through direct observation, numerical simulation and laboratory experimentation, there has been an explosion in research activity directed at improving community understanding, modelling and parameterization of the subtle interplay between energy conversion pathways, turbulence, and irreversible mixing in density-stratified fluids. However, there are still leading order open questions and areas of profound uncertainty concerning this interesting and important research challenge.

For example, since mixing in a stratified fluid inevitably changes the potential energy of the flow, it is of great interest to understand the efficiency of the mixing, i.e. the proportion of the work done on the flow that leads to irreversible mixing, effectively the “taxation rate” of turbulence by stratification. This essay should investigate at least some of the underlying issues of the energetics, instability mechanisms and turbulent processes in high Reynolds number stratified flows leading to the existing uncertainty in this important research area, with enormous and pressing relevance to the changing behaviour of the global climate system.

Relevant Courses

Useful: Fluid Dynamics of Climate, Hydrodynamic Stability

References


Quantum Chromodynamics (QCD) is non-perturbative at the low energy scale relevant to hadron structure calculations. Lattice QCD [1,2,5], in which hadronic matrix elements are computed by discretizing the Feynman path integral, is a reliable first-principles non-perturbative approach to handle QCD. In spite of significant advances in hadron structure calculation in lattice QCD in recent years, many challenges remain such as the calculation of hadronic matrix elements at large boosts, high precision calculations of disconnected contributions to hadronic observables, controlling the excited state contaminations, in particular.

The Feynman-Hellmann (FH) theorem offers an alternative approach of calculating these challenging hadronic matrix elements with a significantly improved control over the systematics. In the first part of this essay, you would briefly review the formulation of the simplest lattice QCD action, mention the systematics arising in a standard lattice calculation, discuss the derivations of the Euclidean two-point functions for nucleon [1,2,5]. In the second part of the essay, you would define the nucleon isovector axial charge, $g_A$, detail its derivations from the traditional approach of calculating from Euclidean three-point functions, and overview its latest results from this method [3,4,5]. In the third part of the essay (major focus should be on this part), you would introduce the FH theorem in quantum mechanics (up to second order) with its proof and extend it to FH method in lattice QCD [5]. You would also detail an alternative path integral derivation of FH theorem and calculate forward matrix element [5] and henceforth, detail the calculation of nucleon $g_A$ from FH theorem with latest updates on its result and significance [5,6].

Relevant Courses

*Essential:* Quantum Field Theory  
*Useful:* The Standard Model

References

[4] Some relevant references from ref.[7].  

79. Digital Quantum Simulation of Gauge Theories  

Dr B. Chakraborty

Gauge theories provide us theoretical framework to understand our nature at its most fundamental quantum level. For example, Quantum Chromodynamics (QCD) is the gauge theory describing how quarks and gluons interact via strong interaction to form hadrons. QCD at relevant hadronic low energy scale is non-perturbative and numerical simulations of lattice QCD are
performed to compute QCD on a 4d Euclidean discretized spacetime grid. Although highly successful, the Markov-Chain Monte Carlo method in lattice field theories uses Boltzmann weight leading to crucial sign problem when the integrand is non-real positive and highly oscillating which is the case for topological terms, chemical potentials or real time dynamics. An efficient way to explore these situations is to simulate the real-time dynamics of the gauge theories, which is generally a notorious challenge for classical computational methods.

This has recently stimulated significant theoretical effort in quantum simulation of gauge theories [1-7]. In the first part of this essay, you would motivate and detail the necessity of real-time quantum computation with respect to sign problems in classical imaginary-time computation of theories like QCD [follow the introduction and its references from ref.7], and summarize the scope of quantum computation in quantum field theories [1-7]. In the second part of the essay, you would detail novel algorithms for digital quantum simulations of 1 + 1 dimensional quantum electrodynamics (the Schwinger Model) [6,7]. The Schwinger Model is an ideal prototype model for theories with both interacting fermions and gauge fields and an ideal ground for testing futuristic algorithms for next generation computing for more complicated gauge theories like QCD. An excellent essay would detail derivations of all steps in ref. [6,7] with its appendices including the improved-Suzuki Trotter method and its error.

Relevant Courses

Essential: Quantum Field Theory
Useful: Part II Quantum Information and Computation

References


80. The Nonlinear Instability of Anti-de Sitter Spacetime .................
Professor M. Dafermos

The trivial solution of the Einstein vacuum equations of general relativity is flat Minkowski space-time. This has been proven [1] to be stable under evolution by the fully nonlinear Minkowski vacuum equations: Small perturbations of Minkowski spacetime remain close to Minkowski for all time and in fact asymptote back to the Minkowski metric at late times. For the mechanism behind this phenomenon, it is essential that asymptotically flat spacetimes allow gravitational perturbations to radiate away to null infinity. A similar nonlinear stability result holds if a positive cosmological constant $\Lambda > 0$ is introduced in the Einstein equations, where the role of Minkowski space is now played by de Sitter spacetime [2].
In the case, however, where a negative cosmological constant $\Lambda < 0$ is introduced to the Einstein equations, infinity is naturally timelike, and the corresponding decay mechanism fails when reflecting boundary conditions are imposed. This leads to the possibility that so-called anti-de Sitter spacetime is nonlinearly unstable [3].

A version of this conjecture has recently been proven for suitable Einstein matter systems [4, 5], which admit non trivial dynamics in spherical symmetry. The conjecture is still open for the 3+1 dimensional vacuum, which does not admit such a symmetry. There remain many additional open problems like understanding more precise questions about the timescale of instability and the existence of so-called islands of stability, as well as understanding the nonlinear stability of AdS black holes [6, 7, 8, 9].

The purpose of this essay is to survey some part of this work. The essay could choose to focus more on the mathematical proofs [4,5] or on the heuristic work [6,7,8,9], but must include at least some discussion of both for context.

**Relevant Courses**

*Essential:* General Relativity

*Useful:* Black Holes, Introduction to Nonlinear Analysis, Analysis of Partial Differential Equations

**References**


Naked singularities are singularities which are visible to far away observers. The simplest examples are provided by negative mass Schwarzschild. More ominously, naked singularities can occur dynamically in gravitational collapse \cite{1}, even for the case of nice matter models like a self-gravitating minimally coupled scalar field under spherical symmetry \cite{2}. In fact, in recent and upcoming work, it has been shown that naked singularities can occur even in vacuum gravitational collapse \cite{3}. (These latter examples are necessarily not spherically symmetric in view of Birkhoff’s theorem.)

The presence of naked singularities poses all sorts of problems for general relativity. The weak cosmic censorship conjecture provides a possible way out for the theory. The conjecture is the statement that, while such naked singularities may occur, for generic initial data they cannot \cite{4}. This would imply that the above dynamic examples of naked singularities must all be unstable to perturbation of their initial data. The conjecture is originally due to Penrose \cite{5}.

Weak cosmic censorship has indeed been proven in the case of the Einstein-scalar field system restricted to spherical symmetry \cite{6}. Without symmetry assumptions, even for the vacuum equations, it remains one of the great open problems of general relativity.

The purpose of this essay is to survey some part of this work. The essay could choose to focus more on the existence of explicit examples of naked singularities, for instance the spherically symmetric examples \cite{1,2}, or alternatively, on Christodoulou’s proof of weak cosmic censorship in spherical symmetry \cite{4}, but should include at least some discussion of both for context.

**Relevant Courses**

*Essential*: General Relativity, Black Holes  
*Useful*: Analysis of Partial Differential Equations

**References**

[4] D. Christodoulou *On the global initial value problem and the issue of singularities* Class. Quantum Grav. 16 (1999), A23  
82. Supersymmetric Gauge Theories and Chiral Algebras

Dr N. Dorey

Symmetry plays a key role in finding exact solutions to physical systems. The larger the symmetry group, the more precise our analytic description of the system becomes. In four-dimensional field theory, standard arguments [1] preclude the existence of additional spacetime symmetries beyond those of the Poincare (and, in some cases, conformal) groups. This is not true in two dimensions where conformal field theories [3] have infinite dimensional symmetry algebras (known as chiral algebras) acting separately on left- and right-moving degrees of freedom. Supersymmetric gauge theories in four dimensions have recently been found to have infinite dimensional chiral symmetry [2,4]. However, the symmetry is respected only by a special subset of protected observables thereby evading the earlier no-go theorems mentioned above. In this essay, you will explore the correspondence between 4d SUSY theories and 2d chiral algebras.

Relevant Courses

*Essential:* Quantum Field Theory, Advanced Quantum Field Theory, Symmetries, Fields and Particles

*Useful:* Supersymmetry

References


83. Twistor Transform

Dr M. Dunajski

One of the most remarkable achievements of Penrose’s twistor program is the link it provides between solutions to certain linear and non-linear differential equations of mathematical physics and unconstrained holomorphic geometry of the twistor space. The essay would review the subject concentrating on linear massless fields four space-time dimensions. You should also explore one (or more!) of the following:

(a) Mathematical aspects of the construction such as isomorphisms between sheaf cohomology classes and massless fields.

(b) Connections with integrability.

(c) Generalisations to non-abelian gauge fields.
Relevant Courses

Applications of Differential Geometry to Physics or Differential Geometry. A firm knowledge of basic complex analysis (e.g. IB Complex Methods, or Complex Analysis) is essential.

References


84. Modelling Building Ventilation in the Light of Covid-19

Dr D. Frank & Professor S. B. Dalziel

One possible spreading mechanism of the SARS-CoV-2 virus is via airborne transmission of tiny aerosol particles that can remain suspended in the ambient air for a prolonged period of time. This potentially increases the risks of infection in enclosed indoor environments in which the existing ventilation flows are inadequate to remove the airborne contaminants in an effective manner. In fact, several clusters of the Covid-19 cases could be traced back to poorly ventilated and crowded indoor spaces such as cruise ships, restaurants and food processing facilities.

An indoor space can be ventilated naturally or ventilated mechanically by an air-conditioning system. The natural ventilation strategy relies on the existing air flows inside and outside the building that, for example, arise due to the heat emitted by sunlight, electrical appliances or the occupants, generating temperature differences between indoors and outdoors. In addition, the external wind can drive an airflow through the building. A significant advantage of the natural ventilation method is that it considerably improves the energy efficiency of buildings when compared with mechanical ventilation. In either case, the exact ventilation flow patterns developing in an indoor environment depend on a variety of factors such as the location of the air supply and exhaust vents, configuration of windows and doors, building occupancy and outdoor conditions. How the ventilation flows combine with the exhaled respiratory droplets and aerosols is currently an open research question, as is which mitigation measures can be applied to control the spread of airborne contaminants.

An essay on this topic should start with a review of well-established mathematical models for ventilation. This review should be followed by a discussion of the implications of the ventilation flow patterns for the spread of airborne contaminants. The candidate may choose to highlight open questions in this research field or discuss possible mitigation strategies for the transmission of infectious aerosols (e.g. masks, negative pressure wards or aerodynamical sealing). If the candidate wishes, they may conduct a case study of a particular ventilation problem by means of CFD numerical simulations or using reduced model equations. The candidate may decide the direction of the essay depending on their interests, although there should be a mathematical
flavour to the material presented. The references cited below offer a starting point for further reading.

**Relevant Courses**

*Essential:*
Undergraduate Fluid Dynamics

*Useful:*
Fluid Dynamics of the Environment

**References**


85. Optimisation Problems in Machine Learning: Can we Compute a Minimiser? .......................................................... Dr A. Hansen

A typical problem that arises in many areas of pure and applied mathematics is the solution of

\[ x^* \in \arg\min_{x \in D} f(x), \]

for a convex function \( f : D \to \mathbb{R} \) and a convex domain \( D \subset \mathbb{C}^n \). For example, a canonical inverse problem can be described as follows:

given (possibly noisy) measurements \( y \approx Ax \) of \( x \), recover \( x \), where \( A \in \mathbb{C}^{m \times N} \) and \( m < N \).

Here \( A \) is a matrix that represents a sampling modality such as Magnetic Resonance Imaging (MRI). Accurate signal recovery or image reconstruction from indirect and possibly undersampled data is a topic of considerable interest. The LASSO problem [3, 9]

\[ \arg\min_{x \in \mathbb{C}^N} F^A(x, y, \lambda) := \lambda \| x \|_1 + \| Ax - y \|_2^2 \]  

is a popular method for approximating the desired vector that arises in compressed sensing [4, 7]. Here \( \lambda > 0 \) acts as a regularisation parameter and the \( l^1 \) norm promotes sparsity. Solving such large-scale problems is a considerable challenge, and many first order methods have developed over the last decade or so [1, 2, 5]. Typically such methods provide convergence results on the value of the objective function. For example, under favourable circumstances fast iterative shrinkage thresholding (FISTA) [1] achieves

\[ f(x_k) - f^* = O(k^{-2}), \]

where \( x_k \) is the \( k \)th iterate and \( f^* \) is the optimal value. However, in many applications we desire convergence not of the objective function, but of **approximations to the minimisers**. This is a key difference, and a recent result shows that, perhaps surprisingly, computing the minimisers of even simple optimisation problems such as \( F^A \) is impossible, even for well-conditioned
Nevertheless, for many problems that arise in compressed sensing and machine learning, one has a bound of the form

\[ d(x, X^*)^\nu \leq \gamma (f(x) - f^*) \]  

where \( d \) is a suitable metric, \( X^* \) is the set of minimisers and \( \gamma > 0, \nu > 0 \). In this scenario it is possible to compute minimisers. Moreover, through suitable restarting, this can often be done at a much faster rate than typical first order methods [8].

This essay should give an exposition of the recent proofs of non-computability results of minimisers of convex optimisation problems that arise in compressed sensing, and now more contemporary, machine learning. This should be followed by a discussion of inequalities of the form (2) within this context.

**Relevant Courses**

*Essential:* Functional analysis/linear analysis (general analysis courses), Inverse Problems, Optimisation/Convex Analysis (or equivalent)

*Useful:* Mathematics of Machine Learning, Numerical Analysis, Automata and Formal Languages (though this is not an essay on logic or Turing machines)

**References**


The QBO is an oscillation of the longitudinal winds in the equatorial stratosphere (the layer of the atmosphere from 15km to 50km). The winds change from westward to eastward to westward with each cycle lasting about 28 months. There are continuous observations of the QBO winds since the early 1950s. (See [1].) Finding an explanation for the QBO greatly exercised leading atmospheric dynamicists in the 1950s and 1960s. It is now accepted that the QBO is the result of a two-way interaction between waves and longitudinal-mean flow. Wave propagation is associated with momentum transport and therefore potentially gives a force on the mean flow, but also depends on the structure of the mean flow.

A significant first part of an essay on this topic should be devoted to the basic dynamics of the QBO, perhaps using [2] and [3] as a starting point, but if possible taking account of more recent research and identifying points that might be changed if that review was to be updated now. Important aspects of the dynamics include the types of waves that are important, what causes any asymmetries between eastward phase and westward phase and how the temperature structure of the QBO is related to the wind structure.

A second part might go beyond the basic dynamics to address one or two further topics, such as (i) the effect of the QBO on other parts of the atmosphere [4] (and references therein), or (ii) the unexpected disruption of the QBO in early 2016 [5,6,7]. (Another disruption occurred in late 2019/early 2020 and publications on this are likely to appear very soon.)

**Relevant Courses**

*Essential:* An undergraduate course in fluid dynamics  
*Useful:* Fluid Dynamics of Climate, Fluid Dynamics of the Environment (Neither is essential.)

**References**

The tropical atmosphere is driven by radiative heating towards a state which is relatively warm at the surface and relatively cold at altitude and as a result strong convection develops, manifested by tall cumulus clouds. However the entire tropics is not in a state of active convection, but instead there is strong spatial variation at scales ranging from those of individual clouds to scales of hundreds or thousands of kilometres with large regions of active convection adjacent to large regions where convection is rare or even absent altogether.

One particular feature of the tropical atmosphere is the so-called 'Madden-Julian oscillation' (MJO). This is not a regular oscillation, but a quasi-random variation in convection and in dynamical quantities such as wind and temperature, in which a region of active convection appears over the tropical Indian Ocean, drifts eastward into the western Pacific and then diminishes in strength over the eastern Pacific. The time between successive appearances of the active convection is typically 30-60 days. Most of the global circulation models used for climate prediction give very poor simulations of the MJO, suggesting that they poorly represent the physical processes that are responsible for it, probably because it depends on quite subtle interactions between convecting and non-convecting regions and between large scales and the scales of the weather systems within which active convection is embedded.

Given the complication of the tropical atmosphere – the range of spatial and temporal scales and the importance of cloud-scale processes including interactions between clouds and radiation – it might seem that simple mathematical models would have limited relevance. However, provided the need for crude but simple representations of cloud-scale processes is accepted, relatively simply models can capture some of the important interactions between these processes and the large-scale dynamics and provide genuine insight into ways in which the representation of tropical circulations in global climate models might be improved.

One particular class of models that has been studied over the last 15 years or so, and is now being argued to provide a basis for understanding the MJO, are models in which include the simple fluid dynamical equations (e.g. as represented by the 'shallow-water equations') together with a moisture field that is transported with the fluid and affects the fluid dynamics by determining the heating. An interesting limit is when the fluid dynamics is treated as quasi-steady and the entire time evolution is controlled by the moisture field. In this limit simple wave motion is sometimes possible and these waves are described as 'moisture modes'. There is now quite a large literature on 'moisture modes' and their behaviour according to different dynamical formulations (e.g. incorporating different physical processes) and there are also several papers which discuss possible moisture-mode models for the MJO.

An essay on this topic should start by surveying some of the basic papers that have studied moisture modes in different forms, trying to present a unified summary of the important features of the behaviour and how it depends on the physical ingredients incorporated in the model. The essay might then move on to discuss in more detail the extent to which moisture modes provide an explanation for the MJO and how these simple models might be used to advance understanding of the MJO and to improve its representation in climate models. (But a student writing this essay might choose to focus on other topics, such as the way in which convection-scale processes are represented in the simple models, or the extent to which moisture modes, or simple models that allow moisture modes along other sorts of behaviour, are useful to understand other aspects of the tropical atmosphere.)

Relevant papers are listed below. The introduction to [1] provides a short overview of work on moisture modes and cites several relevant papers. [2] is an early paper that considers a relevant
simple model and identifies moisture-mode behaviour. [3] is a recent paper that attempts a systematic analysis that distinguishes clearly between moisture modes and other modes. [4] and [5] are papers that propose moisture-mode models for the MJO. [6] is a paper that argues on the other hand that the physical processes incorporated into the models described in [4] and [5] may not be relevant to the MJO in the real atmosphere and offers an alternative.

**Relevant Courses**

*Essential:* An undergraduate course in fluid dynamics

*Useful:* Fluid Dynamics of Climate (Not absolutely essential, but a any student who is considering choosing this essay and who is NOT taking this course is advised – and welcome – to discuss with the setter.).

**References**


88. Directed Percolation and Absorbing-State Phase Transitions ..............

Dr R. Jack

The theory of equilibrium phase transitions is based on probabilities of *configurations* of a system, such as an Ising model. In *dynamical phase transitions*, one concentrates instead on the *dynamical trajectories* (or paths) that a system follows, as a function of time. This opens up a range of new possibilities.

Dynamical phase transitions have their own universality classes and critical exponents. The most famous class is called *directed percolation* [1]. It is relevant in models that have absorbing states, which are states which the system can enter, but from which escape is impossible. It is believed that the directed percolation universality class (and the associated field theory) should be relevant for a wide range of systems with this feature. Given this conjecture, it may be surprising that there are very few experimental systems where evidence for directed percolation has been found. The most convincing examples so far have been found in connection to turbulent flows [2,3].
One reason for the scarce experimental realisations of directed percolation may be that it requires a complete breakdown of time-reversal symmetry (because there is no escape from the absorbing state). This situation contrasts with the reversibility of the microscopic laws of nature. Indeed, there are some model systems where an underlying reversibility allows directed percolation to be ruled out theoretically, even if one might expect it to be relevant at first glance [4].

This essay will discuss the characteristic properties of directed percolation, and how these can be detected (or not) in experimental settings and in model systems.

**Relevant Courses**

*Useful:* Statistical Field Theory, Theoretical Physics of Soft Condensed Matter

**References**


89. Elephant Random Walk .................................

Dr R. Jack

In the theory of stochastic processes, it is common to study Markov processes, in which a system’s behaviour depends only on its current state and not on its history. However, such models may be too simplistic in practical situations, for example in models of decision-making or behaviour.

The *elephant random walk* (ERW) is a very simple non-Markovian model, whose behaviour is strongly affected by long-range memory [1]. The model’s name comes from a saying that “elephants never forget”. It was defined by physicists, but a connection was discovered later to urn models in probability [2]. When the memory is strong, the ERW differs from the Markovian random walk in that its displacement does not obey a central limit theorem. The probabilities of large (rare) deviations from the mean behaviour are also strongly affected by the memory [3]. Models related to the ERW have been applied to decision making [4] and there have been numerous variants on this theme, including the *shark random swim* model [5]. This essay will discuss how memory affects these systems, identifying common features, and contrasting with Markovian models.

**Relevant Courses**

*Useful:* Theoretical Physics of Soft Condensed Matter
In the standard circuit model of quantum computing, the computational steps are unitary gates, with measurements being used only at the end to provide the final (classical) output. In contrast, in the measurement-based quantum computing (MQC) model we begin with a fixed entangled state of many qubits (a so-called cluster state, independent of the computation to be performed), and the computational steps are all just single-qubit measurements. The output of the computation is then obtained from the results of these measurements. The model was introduced in 2001 by Raussendorf and Briegel [1] and is sometimes also called the one-way model (as each 1-qubit measurement irreversibly destructively degrades the starting state).

MQC has many intriguing features, for example, the measurements upon which the output is based can always be done first at the start! And for each individual 1-qubit measurement in the whole process, the result is always uniformly random. Nevertheless this model is able to perform universal quantum computation.

The aim of this essay is to provide a coherent account of the constituents of the MQC model, and explain how it operates to achieve universal quantum computing (cf [2], [3, [4]). If time and space remain you could additionally discuss any of its many interesting features (e.g. [5],[6]). A particularly interesting such feature is an associated notion of computational depth and its special significance for Clifford circuits. There is a large literature on the MQC model and its applications, and many further references can be found by searching the electronic print archive arXiv:quant-ph at https://arxiv.org/find/quant-ph.
91. Table-Top Tests of Quantum Gravity via Entanglement

Professor A. Kent

New experimental tests of quantum gravity have recently been proposed [1,2]. These begin by creating superposition states of two mesoscopic systems, for example small metal spheres whose centre-of-mass is in a superposition of two nearby locations. They then bring these systems adjacent to one another, so that their gravitational interactions have a measurable effect on the phases of their quantum states. It is shown that this effect, if it follows standard quantum gravity intuitions (supported by perturbatively quantized general relativity), should lead to create an entangled state from these initially separate systems. Finally, ways of testing for this entanglement are described.

It is argued [1,2,3] that entanglement can only be created if the gravitational force arises from the exchange of quantum systems, i.e. if gravity is genuinely quantum.

An essay on this topic should review these arguments, the proposed experiments, and their motivation, carefully and discuss the practical difficulties that arise in implementing the experiments. Candidates should be aware of a helpful recent research review [4].

Relevant Courses

Essential: Quantum Information Theory, Quantum Field Theory
Useful: Advanced Quantum Field Theory, General Relativity

References


Data-driven methods for studying dynamical systems has taken off in the last decade due to the explosion of data available and the power of computational resources. Usually, such methods are used to derive reduced order models of systems where the governing equation is known but the dynamics is high dimensional. There is, however, a complementary effort directed at ‘learning’ the governing equations of the system if these are unknown (e.g. flows of complex fluids, biological systems, power grids etc). The approach is to use sparse regression to fit the data to an equation constructed out of an alphabet of likely terms in the equation. This essay will be about surveying some of the current literature (starting with [1-4] below) with the following issues in mind: 1. how can known physical constraints (e.g. conservation of energy) be incorporated?; 2. how practical is the approach for higher dimensional systems (e.g. turbulence of a complex fluid as opposed to a low-dimensional ODE)?; 3. how much data is needed to produce ‘useful’ results?; and 4. what types of algorithm are being used?

The exact focus of the essay can be tailored to the student (e.g. there is potential to carry out some computations). The introductions of [2] & [3] are recommended for students wanting more information on the topic.

![Diagram](image.png)

Figure 1: Pictorial summary of the approach in [3] (fig 1 in [3]).

**Relevant Courses**

*Useful:* Fluids II, Methods

**References**


Collective behaviour is prevalent in nature and human societies: ants form colonies, birds fly in flocks and human opinions evolve into parties. Mathematical modelling based on dynamical systems and partial differential equations plays an important role in the understanding of these complex phenomena in the life and social sciences.

At the microscopic level, these collective dynamics can be described as the interaction between agents according to certain rules. Common examples of frameworks for collective dynamics include the Cucker-Smale alignment model [2] for flocking and the Hegelsmann-Krause model [3] for opinion formation. One key question related to these models is their behaviour if the number of agents is large. To understand the qualitative behaviour of self-organised dynamics for large groups, it is often useful to investigate their group behaviour rather than tracing the dynamics of each of the agents.

The aim of this essay is to study models for collective behaviour and investigate its main properties in terms of emergent dynamics [1,4]. This essay will start by reviewing the aforementioned models on the microscopic level, derive the associated kinetic description and discuss its large-time dynamics which may be illustrated by computer simulations.

Relevant Courses


References


Reinforcement Learning (RL) is a branch of Machine Learning that allows an agent to behave optimally in a given environment (state space) via observation of environmental feedback [1]. In brief, the agent explores the environment by taking actions (which can be anything from moves in chess to steering in a self-driving car) and receiving positive or negative feedback accordingly. Feedback comes in the form of rewards, which, when suitably added together, make up the return associated with the overall performance. The goal of RL is to learn how to maximise this return by improving the agent’s behaviour [2].
RL has found countless applications in recent years, and it has recently started to be used in fluid mechanics. At high Reynolds number, RL has proven useful in helping control features of flow physics such as wakes and turbulence [3-5]. At low Reynolds number, applications have been motivated by biological problems in navigation [6], locomotion [7] and cloaking [8].

In this essay, candidates will review the recent literature on the application of RL to fluid mechanics. A good essay will summarise the main research papers in the field, with an emphasis on (i) the mathematical foundation of RL in Markov decision processes, (ii) the variety of RL algorithms used in fluid mechanics, (ii) the impact of RL on our understanding of flow physics.

If time permits, candidates can choose to implement one of the classical RL algorithms on a simple toy problem in fluid mechanics using Matlab (or equivalent).

References


95. Elastocapillary Coalescence ..........................................................

Professor J. R. Lister

Elastocapillary effects occur when surface-tension forces are large enough to deform elastic structures. Examples include the clumping of wet hair [1] and the deformation of micro-pillar arrays on patterned surfaces [8], and there are many other applications to biology and novel micro-fabrication technologies. The essay should review recent progress in analysing such problems, beginning with a discussion of the natural length scales arising from various balances between surface tension, gravity and elasticity. The review should then address the dynamics of capillary attraction and coalescence, and various mechanisms for determining cluster size. Alternatively, the essay could contain less review element, and instead explore ideas for novel theoretical, scaling or numerical modelling of the dynamics of capillary attraction between two flexible cylinders.

Relevant courses

Useful: Slow Viscous Flow
References


96. Modelling the Hook of a Swimming Uniflagellar Bacterium .......... Professor J. R. Lister

A simple model of a swimming uniflagellar bacterium is an axisymmetric cell body attached to a rigid helical flagellum by a short flexible link called the ‘hook’. A molecular motor in the wall of the body rotates the end of the hook, which rotates the flagellum, which provides the driving force that pushes the organism forward. A lot of modelling effort has put into detailed calculation of the resistance matrices for the body and flagellum for biologically realistic geometries, but much less into modelling the bending of the hook or understanding its dynamics.

This essay would likely take the form of a modelling and computational project to calculate the dynamics in a viscous fluid of two rigid spheres (or possibly rods) connected by an elastic filament described by Kirchoff-rod theory. Hydrodynamic interactions between the bodies would be neglected, as would the hydrodynamic resistance of the filament. The ends of the filament are attached at right-angles to the surface at a fixed points on the spheres, and exert forces and couples on them due to the bending of the filament. For 2D motion, there are just 3 degrees of freedom, corresponding to the relative position and orientation of the spheres. In the simplest system, one sphere is held fixed; more interestingly, one sphere has a force on it directed to the point of attachment. The aim is to use this simple system to provide insight into the assumptions about the hook in [1] and [2].

Alternatively, the essay could review the papers below, and try to cut through the complexity to describe what is happening.

Further guidance is available on request.

Relevant courses

Essential: Slow Viscous Flow
Symmetry is a guiding principle of modern theoretical physics. In particular, symmetries play a crucial role in how we construct and apply quantum field theories. However, if a ground state of the system does not respect all of the underlying symmetries, a low-energy observer (with access only to small perturbations about this ground state) will perceive some symmetries as (spontaneously) broken. Perhaps the most famous example of this is the Higgs mechanism, in which a vector field acquires a mass due to the expectation value of a scalar field (which “breaks” the gauge symmetry), but other kinds of symmetry breaking occur throughout particle physics and cosmology.

The goal of this essay is to investigate how such spontaneously broken symmetries can be “restored” by introducing Stückelberg fields, which provide a nonlinear realisation of the symmetry—these fields can be used to describe low-energy fluctuations about a symmetry-breaking state, and their dynamics are tightly constrained by the underlying symmetry. The essay should begin by describing the Stückelberg procedure for “restoring” gauge symmetry for a massive vector field [1,2]. It should then explore a further application of Stückelberg fields / nonlinearly realised symmetry, such as: (i) chiral perturbation theory for pions (a low-energy description of QCD [3, 4]), (ii) massive gravity (in which the diffeomorphisms of General Relativity are broken by a mass [5, 6]), (iii) the Effective Field Theory of Inflation (in which temporal diffeomorphisms are broken by the expansion of the early Universe [7, 8]).

The essay should be written at a level that would be understood by another Part III student who had attended similar Part III courses.

**References**


Dr S. Melville

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**Nonlinearly Realised Symmetries**

Symmetry is a guiding principle of modern theoretical physics. In particular, symmetries play a crucial role in how we construct and apply quantum field theories. However, if a ground state of the system does not respect all of the underlying symmetries, a low-energy observer (with access only to small perturbations about this ground state) will perceive some symmetries as (spontaneously) broken. Perhaps the most famous example of this is the Higgs mechanism, in which a vector field acquires a mass due to the expectation value of a scalar field (which “breaks” the gauge symmetry), but other kinds of symmetry breaking occur throughout particle physics and cosmology.

The goal of this essay is to investigate how such spontaneously broken symmetries can be “restored” by introducing Stückelberg fields, which provide a nonlinear realisation of the symmetry—these fields can be used to describe low-energy fluctuations about a symmetry-breaking state, and their dynamics are tightly constrained by the underlying symmetry. The essay should begin by describing the Stückelberg procedure for “restoring” gauge symmetry for a massive vector field [1,2]. It should then explore a further application of Stückelberg fields / nonlinearly realised symmetry, such as: (i) chiral perturbation theory for pions (a low-energy description of QCD [3, 4]), (ii) massive gravity (in which the diffeomorphisms of General Relativity are broken by a mass [5, 6]), (iii) the Effective Field Theory of Inflation (in which temporal diffeomorphisms are broken by the expansion of the early Universe [7, 8]).

The essay should be written at a level that would be understood by another Part III student who had attended similar Part III courses.

**Relevant Courses**

*Essential*: Quantum Field Theory; Symmetries, Fields and Particles  
*Useful*: Advanced Quantum Field Theory; Field Theory in Cosmology
The Brewer-Dobson Circulation is a large scale meridional circulation in the stratosphere (the part of the atmosphere between about 10-50km altitude) that slowly transports air upwards in the tropics, poleward and then downwards over mid and high latitudes.

The strength and variability of this circulation is important in setting the composition of the stratosphere. For example, ozone depleting substances such as chlorofluorocarbons are inert in the troposphere but in the middle and upper stratosphere are broken down by radiation into simpler molecules. Their atmospheric lifetimes, set by the average time taken for a molecule starting in the troposphere to reach the middle stratosphere, are determined by the strength of the BDC. The BDC also has an important effect on water vapour in the stratosphere. Air entering the stratosphere in the tropics, in the upward part of the BDC, encounters a cold region, the temperature of which is itself dependent on the strength of the BDC, where the dehydration of the air parcels is important in setting the concentrations of stratospheric water vapour. Although these concentrations are very low, stratospheric water vapour makes an important contribution to the 'greenhouse effect' and is also a source of hydrogen oxide radicals which control many key chemical reactions important for ozone. The water vapour signal shows multi-timescale variations from daily to decadal which are dominated by temperature variations and the tropical upwelling strength, with the BDC playing an important role in both.

The BDC cannot be explained without careful consideration of dynamics since, for example, the poleward transport requires a force to change the angular momentum, otherwise, according to the 'ballerina effect' the rotation of air relative to the Earth would increase, which is not observed.

This essay should begin by discussing the dynamical frameworks used to understand the BDC, including the Transformed Eulerian Mean framework [1] and the 'downward control principle' [2]. The essay should then review the current status of understanding the BDC and challenges in quantifying it. Possible topics to cover in the second half of the essay include discussing in more detail of one or two of the topics reviewed in [1] such as discussing the main drivers of variability in the BDC (e.g. [3]), the predicted changes to the BDC under climate change or the methods of estimating the current BDC from dynamical and chemical observations (e.g. [4]).
Relevant Courses

*Essential:* An undergraduate course in fluid dynamics

*Useful:* Fluid Dynamics of the Climate

References


99. Warped Astrophysical Discs .................................................. Professor G. Ogilvie

In a spherically symmetric gravitational potential, orbital motion is possible in any plane containing the centre of the potential. A warped astrophysical disc is a fluid flow dominated by orbital motion, in which the orbital plane varies continuously with radius and possibly with time. Both the shape of the disc and its mass distribution evolve as a result of internal torques that transport angular momentum within the disc, as well as any external torques applied to the disc.

Astrophysical discs are expected to be warped whenever a misalignment occurs in the system, as in the classic problem in which a black hole is fed with gas having an orbital angular momentum that is not parallel to the spin of the black hole; indeed, the shape adopted by the disc in this situation is a problem of considerable interest. It is also possible for a warp to arise through instability of an aligned system, as in the case of accretion on to a magnetized star.

Linear and nonlinear theories of the dynamics of warped discs have been discussed since the 1970s [1], and increasing use is being made of global numerical simulations (e.g. [2]). The aim of this essay would be to review the subject concisely, with an emphasis on recent developments of a theoretical, computational or observational nature.

An introduction to the subject and a selection of useful references can be found in Sections 1 and 2 of reference [3]. Use of the ADS archive adsabs.harvard.edu is highly recommended. Interested candidates should contact Gordon Ogilvie for further advice.

Relevant Courses

*Essential:* Dynamics of Astrophysical Discs

*Useful:* Astrophysical Fluid Dynamics
References


100. Wave Attractors in Rotating and Stratified Fluids

Professor G. Ogilvie

Internal waves can propagate in rotating and/or stably stratified fluids (as often occur in astrophysical and geophysical settings) as a result of Coriolis and/or buoyancy forces. Their properties are radically different from those of acoustic or electromagnetic waves. The frequency of an internal wave depends on the direction of the wavevector but not on its magnitude. Waves of a given frequency follow characteristic paths through the fluid and reflect from its boundaries. In many cases the rays typically converge towards limit cycles known as wave attractors. One application of this finding is to tidally forced fluids in astrophysical and geophysical settings. If tidal disturbances are focused towards a wave attractor, this can lead to efficient tidal dissipation that in some cases is independent of the small-scale diffusive processes.

This essay should review the subject of internal wave attractors, including some of the more recent developments. Some simple explicit examples should be provided, which could involve original calculations. Topics which might be covered include:

1. The behaviour of rays for pure inertial waves in a uniformly rotating spherical shell.
2. The relation, if any, between the propagation of rays within a container and the existence of inviscid normal modes.
3. The consequences of a wave attractor for the decay rate of a free oscillation mode, or the dissipation rate of a forced disturbance, in the presence of a small viscosity.
4. The roles of nonlinearity and instability in wave attractors.
5. The relevance of wave attractors to tidal dissipation in astrophysical systems.

Interested candidates should contact Gordon Ogilvie for further advice.

Relevant Courses

Useful: Astrophysical Fluid Dynamics

References

In most cases of phenomenological interest, Quantum Field Theory cannot be solved exactly and perturbative methods are needed to make predictions for observables. The path from fields and Lagrangians to amplitudes and correlators passes by the well-developed machinery of Feynman diagrams. It has long been realised that this prescription for calculating observables obscures much of the structure of the final result. The poster child of this phenomenon are amplitudes for massless fields with spin, which can be computed very effectively referring only to on-shell quantities such as helicities and momenta, as opposed to needing degrees of freedom that are removed by constraints or gauge redundancy. Many “on-shell methods” for amplitudes are known for Lorentz invariant theories and new structures are being discovered in this very active research field. Conversely, much less is known about observables in curved spacetime and in particular correlations functions in cosmology, where the expansion of the universe spontaneously break time translations and boosts. New ideas to address this shortcoming and very promising results have emerged in the past couple of years. It has become clear that it is important to focus on the well-defined observables in a theory of quantum gravity, which live at the conformal boundary of quasi de Sitter spacetime. From this “boundary” perspective, it is possible to “bootstrap” these observables without recurring to any explicit model-dependent “bulk” calculation. Instead, the idea is to use only symmetries and general principles such as unitarity and locality to constrain the form that correlators can take in any theory.

In this essay, one starts by reviewing the calculation of cosmological correlators using the in-in formalism (following [1,2,3] or the notes of Field Theory in Cosmology) and the calculation of the wavefunction of the universe (e.g. [4,5,4]). Then one learns about the consequences of unitarity, one of the pillars of quantum mechanics, for amplitudes in flat spacetime [7,8] and for correlators in quasi de Sitter spacetime [5]. Then one studies how symmetries and locality can be used to “bootstrap” three-point correlation functions in quasi de Sitter spacetime following [9]. Finally, in a more advanced part of the project, one learns about the spinor helicity formalism and on-shell methods for amplitudes [10,11,12] and studies a possible way to adapt it to cosmology [13]. The ambitious student might attempt to develop alternative helicity variables that are more appropriate to the cosmological/de Sitter setting. Inspirational readings on the subject might be found in [14,15].

### Relevant Courses

**Essential:** Quantum Field Theory, General Relativity.

**Useful:** Cosmology, Field Theory in Cosmology

### References


Stochastic thermodynamics provides a powerful framework to find thermodynamic laws that hold beyond equilibrium and for small, fluctuating systems [1, 2]. A major recent development in this field is the formulation of the thermodynamic uncertainty relation (TUR), which states that the product of the entropic cost for driving a system and the relative uncertainty of any fluctuating current produced by the system is always greater than 2k_B [3]. Many variants and refinements of this relation have been derived in recent years [4]. Nonetheless, the original TUR has so far only been proven for systems whose variables are even under time-reversal. Whether it holds for systems with variables that are odd under time-reversal (most importantly systems with inertia) remains an open question. In addition to reviewing the literature on TUR’s, the student would be encouraged to do original research on this question. In a simple model system (e.g. for a clock), can we exploit inertia to be more precise than the limit set by the TUR?

### Relevant Courses

**Essential:** Statistical Physics (Part II)

**Useful:** Theoretical Physics of Soft Condensed Matter (Part III)
References


103. Instantaneous Nonlocal Measurements and Quantum Location Authentication .........................................................

Dr D. Pitalúa-García

An instantaneous nonlocal measurement consists in the measurement of a quantum observable of a quantum state that is distributed between two or more locations, in such a way that the outcome of the measurement is obtained from the classical outcomes of local measurements that are performed at spacelike separated regions [1,2]. Its consistency with relativistic causality requires the satisfaction of some conditions [3]. Instantaneous nonlocal measurements can be implemented with a sufficient amount of entanglement [4], using methods [5,6] based on different quantum teleportation protocols, for example.

Quantum location authentication is a cryptographic task with the goal of authenticating the location of an object using quantum systems and exploiting that information cannot travel faster than the speed of light [7]. The existence of instantaneous nonlocal measurements implies that some protocols for quantum location authentication are insecure: an adversary with enough distributed entanglement can implement instantaneous nonlocal measurements and in this way break the security of protocols for quantum location authentication [5,6,8]. However, this impossibility result can be avoided by restricting the adversary’s power [9].

An ideal essay will review the literature on this research area, showing a clear understanding of the main physical ideas and mathematical concepts. You are not expected to cover all papers. For example, you might choose to focus on instantaneous nonlocal measurements by covering Refs. [1–6], or to focus on quantum location authentication by covering Refs. [5–9].

Relevant Courses

Useful: Part II Principles of Quantum Mechanics, Part II Quantum Information and Computation

References


Many inverse problems in mathematical physics can be formally expressed as

\[ Ax = y , \]  

where \( A \) is an operator from a normed vector space \( X \) into a normed vector space \( Y \), \( y \in Y \) is given data, often measured data, and \( x \in X \) is the unknown. Usually such problems are ill-posed, and various methods are used to ensure existence and uniqueness, as well as stability of the solution, which is referred to as ‘regularisation’.

Regularisation can in some cases be achieved by projection onto finite-dimensional subspaces \( A_n \subset X \). Krylov subspace methods are iterative methods in which the solution is sought by successive approximations \( x_n \in K_n \), where \( K_n \) is the Krylov subspace \( \text{span}\{d, Bd, \cdots B^{n-1}d\} \), with \( B \) and \( d \) dependent on \( A \) and \( y \). The Conjugate Gradient method and its variants are examples of Krylov subspace methods, and other have also been used, sometimes together with Tikhonov regularisation.

This essay should explore the regularising properties of Krylov subspace methods, focusing on some particular issues according to personal interests and background. This could also be, for example, applications of Krylov methods to practical inverse problems.

A few example references are given below, and more will be provided depending on the choice of focus.

**Relevant Courses**

*Essential:* Basic knowledge of linear analysis, from any course.

*Useful:* The Part III courses Inverse Problems.
The propagation of a scalar wave $\phi$ in any medium is described by the wave equation
\begin{equation}
\nabla^2 \phi(x,t) - \frac{1}{c^2} \frac{\partial^2 \phi(x,t)}{\partial t^2} = F(x,t), \quad x \in \mathbb{R}^3, t \in \mathbb{R}
\end{equation}
with or without a source term $F(x,t)$. In this essay, you should only consider the case $F(x,t) = 0$, or possibly a source localised in the plane $z = 0$: $F(x,t) = \delta(z) \exp(-i\omega t)$. For time-harmonic waves $\phi(x,t) = \psi(x) \exp(-i\omega t)$ and $F(x,t) = f(x) \exp(-i\omega t)$, equation (1) reduces to the time-independent Helmholtz equation, which in the case $F = 0$ is
\begin{equation}
\nabla^2 \psi + k^2 \psi = 0,
\end{equation}
where $k = \frac{\omega}{c} = k_0 n(x)$, and $n(x)$ is the refractive index of the medium.

Most realistic media vary in a complicated way, often with fast variations on a spatial scale which is small compared with the propagation distance, as well as slower variations on a much larger scale, that can often be ignored. The spatial variations of the medium are described by the refractive index, and the fast variations are usually random in nature. In many cases, such as propagation through turbulent atmosphere, or the ocean, or biological tissue, this can be described by a refractive index $n(x) = 1 + w(x)$, where $w(x)$ is a zero-mean stationary random process. When the index of refraction is a random process, the wave field is itself a random process and we are interested in how the statistics of the random medium affects the statistics of the wave field, i.e. the so-called ‘moments’ of the field. Finding solutions for these quantities is a difficult problem. When the wave varies slowly in the direction of propagation, the paraxial approximation applies. It turns out that, in this case, it is possible to derive and solve equations for some of the key moments of the field, including the mean intensity.

This essay should focus on deriving the equations for the key moments of the field, providing analytical solutions where possible, and explaining their range of validity and their basic physical significance. Then you could choose to focus on numerical implementations of propagation in random media, particularly for quantities where an exact solution is not available, commenting on the numerical model and issues of efficiencies or comparison to experimental results (from the literature: no numerical implementation of your own is required). Or you could choose to focus on properties of existence, uniqueness, and continuity of solutions for some cases. Or you could choose to focus on one (or two) particular applications to physical problems. The choice should depend on your interests and background.

A few possible references are given below, but more specific references depending on the choice of focus will be provided.
Relevant Courses

Essential: Knowledge of the wave equation and basic concepts in wave propagation, from any course.

References


106. Topological Censorship .................................................................

Professor H. S. Reall

The topological censorship theorem [1,2] in General Relativity states that if a spacetime satisfies the Einstein equation with matter satisfying a certain energy condition, then any causal curve which starts and ends at infinity can be deformed to a curve that “remains near infinity”. This means that if the spacetime has non-trivial topology, such as a wormhole, then a distant observer cannot send a signal through that topology. Roughly speaking, non-trivial topology collapses too rapidly for light to cross it.

This essay should explain the proof of the topological censorship theorem at a level that would be accessible to another Part 3 student who had attended the Black Holes course. This will involve carefully explaining various results about causal structure (e.g. in [3]) in more detail than covered in the Black Holes course. The essay should go on to describe the application of topological censorship to understanding the topology of black holes [4,5].

Relevant Courses

Essential: General Relativity, Black Holes.

References

The basic principles underlying string theory as a quantum theory of gravity have still not yet been identified and, as such, much of our current understanding of the theory comes from the study of string theory on particular backgrounds. String theory is still only well-understood on a relatively small number of backgrounds. Group manifolds are an important class of backgrounds which allow for some of the more novel aspects of string theory to be explored without the theory becoming intractable. Of particular interest is the case where the group is $SL(2;\mathbb{R})$, a non-compact group, where the worldsheet theory describes a string propagating in three-dimensional anti-de Sitter space ($AdS_3$); a manifold with constant negative curvature. The study of string theory on such backgrounds provides deeper insight into string theory on curved backgrounds more generally and has direct relevance to the AdS/CFT correspondence.

It is envisaged that the essay would review the general features of bosonic string theory in a background that includes an $AdS_3$ part. The classical theory and general issues that arise in its quantisation, with care taken to understand the spectrum of the theory would also be covered. There are a number of features that make the spectrum of the theory very different from the bosonic string on flat space and these features, such as the presence of discrete and continuous representations and the role of spectral flow, would be studied. Space and time permitting, an ambitious essay could then develop in a number of possible directions which include, but are not restricted to; a study of the superstring in $AdS_3 \times M_7$, where $M_7$ is a seven-dimensional compact manifold (typical choices for $M_7$ include $S^3 \times T^4$, $S^3 \times S^3 \times S^1$, or $S^3 \times K3$), a review of different ways of describing the superstring theory (such as the Green-Schwarz or hybrid formalisms), or the study of some simple correlation functions.

**Relevant Courses**

*Essential:* Quantum Field Theory, String Theory

*Useful:* Advanced Quantum Field Theory, Supersymmetry

**References**


108. Hawking Radiation from AdS/CFT .................................

Dr J E Santos

Understanding the behaviour of Quantum Field Theories (QFTs) in curved spacetime is an important problem, not least because we know that the universe does contain regions of very large curvature. A key discovery was Hawking’s calculation demonstrating particle production in black hole backgrounds. These particles have a thermal spectrum, confirming that black
holes should properly be thought of as thermodynamic objects. A general argument based on the Euclidean time formalism shows that for any QFT, an equilibrium state on a black hole background (the so-called Hartle-Hawking state) should be thermal. However most of what is known about QFTs in curved spacetime comes from calculations involving free or weakly interacting theories. Little is known about the case when the QFT is strongly coupled.

Gauge/Gravity Duality provides a new way of probing the behaviour of certain strongly coupled QFTs in curved backgrounds. In its most precise and well motivated form, it is the claim that Type IIB Superstring theory on $\text{AdS}_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory with gauge group $SU(N)$ on the $(3 + 1)$ dimensional conformal boundary of $\text{AdS}_5$. In the large $N$ strong coupling limit of the boundary gauge theory, the bulk string theory becomes weakly coupled and the string length scale becomes small. In principle this should allow us to study quantum effects in the strongly coupled theory, such as Hawking radiation, by solving classical gravitational equations of motion in the bulk. This technique was explored in [1-4].

The essay should do a thorough review of the technique described above and should be written in a language accessible to other Part III students taking similar courses.

**Relevant Courses**

*Essential:* General Relativity, Quantum Field Theory, Black Holes

*Useful:* Advanced Quantum Field Theory and Gauge/Gravity Duality

**References**


109. The Aubin–Lions Lemma and Applications to Evolution Equations ...  
Dr S. M. Schulz

A common strategy for solving a complicated partial differential equation is to solve a sequence of simpler approximate problems, with the hope that these approximate solutions converge to a solution of the original equation. Crucial to this approach is the existence of appropriate compactness theorems for the space of functions that the approximate solutions belong to. When considering evolution equations, which involve time, it is natural to consider spaces of the form $L^p(0, T; B)$, where $T > 0$ and $B$ is some Banach space. The classical Aubin–Lions lemma (cf. [2], and later [3]) was the first compactness criterion for subsets of $L^p(0, T; B)$, with later generalisations due to Jacques Simon (cf. [5]) and others (see [4] and the references contained therein).

A successful essay will both survey the compactness results contained in [2,3,4,5], and illustrate the application of such results in solving evolution equations (see [4] and the references therein for example applications, and Section 3.4 of [1]). Depending on the preference of the student,
more emphasis can be given to the functional analytic details of the compactness results, or to the applications to partial differential equations.

**Relevant Courses**

*Essential:* Part II Linear Analysis, Part II Probability and Measure.

*Useful:* Part III An Introduction to Non Linear Analysis, Part III Analysis of PDEs.

**References**


110. The Search for CMB B-mode Polarization from Inflationary Gravitational Waves .................................................................

Dr B. D. Sherwin

Our most promising theory for the early universe involves a phase of cosmic inflation, which not only rapidly expands and flattens the universe, but also generates the primordial density perturbations from quantum fluctuations in the inflaton field. While we have good evidence for inflation, e.g. from the Gaussianity, adiabaticity and near-scale invariance of the scalar density perturbations, one prediction of inflation has not yet been found: many inflationary models produce a stochastic background of primordial gravitational waves. A detection of this background would not only provide a definitive confirmation of inflation, but could also give new insights into the microphysics of inflation and, more broadly, physics at the highest energies.

The best current way of finding this gravitational wave background is to search for a characteristic pattern in the polarization of the Cosmic Microwave Background (CMB), the B-mode polarization. This essay should explain the physics underlying the search for this B-mode polarization pattern, which is currently a major area of research in cosmology.

The essay should first review the calculation of the gravitational wave background produced by standard single-field slow-roll inflation, a standard result described in past Part III lecture notes as well as a comprehensive review of the field (Kamionkowski & Kovetz 2016, henceforth KK16). The essay should also explain why the strength of the gravitational wave background (together with the scalar spectral index) can provide powerful constraints on the properties of inflation, such as the potential shape, energy scale, and field excursion (CMB-S4 2016, KK16).

Drawing on KK16, CMB-S4 2016, past lecture notes and other resources, the essay should provide a (brief) review of the basics of CMB polarization, describe what the CMB B-mode polarization is, and explain why it is a powerful probe of inflationary gravitational waves.
The remaining parts of the essay can, to some extent, be tailored to the student’s interests. One option is to explain in detail the major observational challenges in B-mode searches for inflationary gravitational waves, discussing the problems of foregrounds (Bicep/Keck/Planck 2015) and gravitational lensing as well as mitigation methods such as multifrequency cleaning and delensing (Smith et al. 2012). Another option is to focus more on the theoretical background, describing in detail different classes of inflationary models and what these generically predict for B-mode polarization (CMB-S4 2016 and references therein). Students may also discuss a combination of both observational and theoretical aspects.

Relevant Courses

*Essential:* Cosmology

*Useful:* Advanced Cosmology, Quantum Field Theory, General Relativity

References


111. Two-Dimensional Yang-Mills Theory .............................................

Dr D. Skinner

In four dimensions Yang-Mills theory is a typically intractable strong-coupled quantum field theory, but in two dimensions it is exactly solvable. Its partition function can be studied from many points of view, from lattice techniques introduced by Migdal, to canonical quantization of the space of class functions on the gauge group, to non-Abelian localization and topological field theory. In the large $N$ limit, it is believed to provide a realization of ’t Hooft’s insight that gauge theories should be equivalent to a string theory. This essay will explore 2d YM from one or more of these perspectives.

Relevant Courses

*Essential:* Advanced Quantum Field Theory and Symmetries, Fields & Particles

References

Modern quantum algorithms require computational resources which are currently beyond the reach of state of the art implementations. But even minimal quantum resources can be made useful if we use them in conjunction with powerful classical optimization methods. This approach has been exploited in Variational Quantum Eigensolver (VQE) [1-3]. It makes use of Ritz’s variational principle to prepare approximations to the ground state and its energy. In this algorithm, the quantum computer is used to prepare a class of variational ‘trial’ states which are characterized by a set of parameters. Then, the expectation value of the energy is estimated and used by a classical optimizer to generate a new set of improved parameters which are then used to prepare the next iteration of trial states. The advantage of VQE over purely classical simulation techniques is that it is able to prepare trial states that cannot be generated by efficient classical algorithms.

An important ingredient when simulating fermionic quantum systems is the encoding which maps fermionic modes to qubits [4-6]. The key property an encoding is its ability to preserve the locality of fermionic operators in the qubit picture. In general, no encoding succeeds in this perfectly [7].

This essay should discuss the algorithm and various fermion to qubit encodings [4-6].

Relevant Courses

Part III Quantum Computing (M16) is recommended.

References


Determining when it is possible to transform one set of quantum states into another allows us to better understand the utility of a quantum resource. Given two bipartite states $|\psi_{AB}\rangle$ and
\[ |\phi_{AB}\rangle, \text{ the possibility of transforming } |\psi_{AB}\rangle \rightarrow |\phi_{AB}\rangle \text{ by means of local quantum operations and classical communication is completely characterized by majorization relations [1]. Sometimes, when it is not possible to convert between states directly, the parties can nevertheless perform the transformation } |\psi\rangle \otimes |\omega\rangle \rightarrow |\phi\rangle \otimes |\omega\rangle, \text{ if they have access to the catalyst state } |\omega\rangle [2,3]. \]

This task becomes difficult when we the task is to transform one set of non-orthogonal states into another. Surprisingly, when the initial set consists of pure states and the final set consists of arbitrary states, there exist conditions which tell us when such transformation is possible [4].

Recently, a connection between unitary matrices and complex analysis opened up new possibilities for deriving necessary and sufficient conditions for interconverting quantum states [5].

This essay should discuss conditions for converting between individual states, sets of states. Optionally, you may discuss and recent work [5] in the context of state transformations.

**Relevant Courses**

Part III Quantum Information Theory (L24) is recommended.

**References**


**114. The Inequalities of Quantum Information Theory ................. Dr S. Strelchuk**

Strong subadditivity of the von Neumann entropy is one of the most fundamental results with numerous applications in Quantum Information Processing [1]. It states that for a tripartite quantum state \(\rho_{CRB}\):

\[ S(CB)\rho + S(RB)\rho \geq S(CRB)\rho + S(B)\rho, \]

where \(S(A)\rho = -\text{Tr}\rho \log \rho\) denotes the von Neumann entropy of the designated system. It is the only entropic inequality that holds for three parties with no known inequalities of this kind for four or more parties that hold without additional constraints.

In contrast, Classical Information Theory has a large number of the so-called ‘non-Shannon’ inequalities for entropies of probability distributions on four and more parties which cannot be reduced to inequalities with fewer parties [2-4]. Some of them still hold in the quantum case if we impose a certain set of constraints on quantum correlations [5,6].

This essay should discuss classical non-Shannon inequalities and their constrained quantum counterparts.
Relevant Courses

Part III Quantum Information Theory (M24) and Information Theory (L16) are recommended.

Useful:
The following textbooks may be used for guidance:

References


115. Submesoscale Ocean Dynamics ........................................

Dr J.R. Taylor

In recent years, high resolution satellite imagery and numerical models have revealed a wealth of relatively small (1-10km) features in the upper ocean, collectively called the ‘submesoscale’. These features include fronts, filaments, eddies, all of which also have larger ‘mesoscale’ counterparts. However, the dynamics of submesoscales are distinguished by a Rossby number of order unity with non-geostrophic effects felt at leading order. Submesoscales are drawn considerable attention from the physical oceanography community, partly due to their ability to increase the stable stratification in the mixed layer, while inducing large vertical velocities, aiding the exchange of water between the ocean surface and interior.

Submesoscale motions are generally thought to arise due to instabilities associated with vertical shear and horizontal density gradients. Some of these instabilities are finite Rossby number flavors of mesoscale phenomena. Others are distinct to the submesoscale. This essay should start with a discussion of submesoscale features and the instabilities that lead to their formation. This should include a discussion of previous work in this area.

Although there are a ‘zoo’ of processes that are active at the submesoscale, many of the existing studies of submesoscales have focused on a specific instabilities or processes. The essay should provide context for the relative importance of various processes under different conditions, perhaps with the aid of regime diagrams. Another possible approach would be to identify a basic state that supports several submesoscale instabilities and perform a linear stability analysis to identify the nature of the fastest growing modes.
Quantum chromodynamics (QCD) is a quantum field theory that exhibits many interesting phenomena such as asymptotic freedom and confinement. Moreover, it describes the strong interaction of particle physics, i.e. how quarks and gluons interact and give rise to the non-trivial structure and dynamics of hadrons. Recent observations of a number of puzzling and ‘exotic’ structures have generated a lot of interest and hadrons are currently the subject of many theoretical and experimental investigations.

Computing the masses and other properties of hadrons within QCD is a long-standing challenge because the QCD coupling is strong in the relevant low-energy regime. Lattice QCD is a non-perturbative technique that enables first-principles computations of the properties of hadrons using Monte Carlo methods. This essay should give a brief introduction to relevant aspects of lattice QCD and then discuss a topic related to hadron spectroscopy in detail. For example, how the many finite-volume energy eigenstates needed to study excited hadrons are computed, or how information on scattering amplitudes, resonances and bound states can be extracted from lattice calculations.
As early as in 1949, L. Onsager conjectured that a weak solution of ideal incompressible flow (Euler equations) will conserve energy if the velocity field is Hölder continuous with exponent greater than $1/3$. His conjecture, which was based on Kolmogorov’s 1941 theory of turbulence and was taken up by mathematicians only in the 1990s, when Eyink [6] proved it for Hölder continuous functions with exponent larger than $1/2$, and Constantin, E and Titi [4] independently gave a complete proof of a “stronger” version of the conjecture in the context of Besov spaces for exponent larger $1/3$. In [5] Duchon and Robert used similar ideas as in [4] to establish, under similar regularity assumption of the weak solutions, the local conservation of energy in the Euler equations. Notably, all the above results were established in the absence of physical boundaries, that is, in the whole space or subject to periodic boundary conditions. In 2018 Bardos and Titi [1] extended the above results and proved the Onsager conjecture for the conservation of energy of the Euler equations in domains with physical boundaries. Moreover, a similar result was established for the conservation of generalized entropy for general system conservation laws [2], hence asserting the universality of the Onsager’s $1/3$ exponent.

The exact focus of the essay can be tailored to the candidate, e.g. by concentrating on either of the papers [1] or [5].

In the inviscid Leray–$\alpha$ [3] and the Euler–$\alpha$ (inviscid Camassa-Holm or inviscid Navier-Stokes–$\alpha$) [7] models of incompressible flows, the corresponding advection nonlinear terms are smoothed in non-local manners. A similar analysis to above should be possible for these models in order to deduce what would be the analogue of the Onsager conjecture in these cases. If time permits, a candidate might outline an approach to this analysis.

**Relevant Courses**

*Essential:* Introduction to Nonlinear Analysis.


**References**


118. **Standard Model and Higgs Effective Field Theories**

Dr M. Ubiali

The use of an effective field theory (EFT) approach to the study physics beyond the Standard Model (BSM) is well motivated given that new particles might well be heavier than the energy scale probed at the Large Hadron Collider (LHC). The use of EFTs offers a model-independent approach (complementary to the study of any particular BSM scenarios). Moreover it gives strong indications about the kind of new interactions beyond the SM and of the mass scales of the new particles that might be discovered in Nature beyond the energy scale that is currently probed by the LHC.

The purpose of this essay is to present two particularly promising EFTs, in the context of the characterisation of the recently discovered Higgs boson, namely the Standard Model Effective Field Theory (SMEFT) and the Higgs Effective Field Theory (HEFT).

The first half of the essay will set the scene in terms of experimental data for the Higgs boson discovery, and the calculation of the contributions to the Higgs Lagrangian coming from dimension-6 operators. The second half should be devoted to the description of the similarities and of the differences between SMEFT and HEFT, both in terms of the dimension-6 operators that are relevant in parametrising the interaction of the Higgs boson with itself and with the other SM particles and in terms of their interpretations at the energy scale of new physics.

**Relevant Courses**

*Essential:* Quantum Field Theory, Standard Model, Particles and Symmetries, Advanced Quantum Field Theory

**References**


*(and references therein)*
119. The Induced Gravity Scenario ..............................................

Dr A. C. Wall

In 1968, A.D. Sakharov proposed that the Einstein-Hilbert term in the gravitational action might arise entirely from quantum loop corrections. (Note that since the coefficient in front of the EH term is proportional to 1/G, this means that the bare Newton’s constant is *infinite*.) This proposal is known as the “Induced Gravity Scenario”.

This idea may seem esoteric, but it turns out to be closely related to a common belief in quantum gravity research: that the Bekenstein-Hawking entropy of black holes has a state-counting interpretation in terms of microscopic quantum degrees of freedom. One could therefore argue that any good theory of quantum gravity (e.g. string theory) should be understood as being an induced gravity theory.

Your essay should clearly describe this relationship between induced gravity and black hole thermodynamics. To succeed, you will need to give a careful account of renormalization theory in curved spacetime, and its relation to black holes, while paying close attention to the physical significance of any ultraviolet cutoffs that are used.

The references below are some of the key original articles, but you are encouraged to bring them into dialogue with other research on related topics. (Note that since quantum gravity is a speculative subject, you are not required to agree with any given author’s point of view, so long as you can provide evidence for your own viewpoint.)

**References**


120. Linear Fields in Anti-de Sitter Spacetimes ..............................

Dr C. M. Warnick

Spacetimes with negative cosmological constant, known as anti-de Sitter spacetimes, attract much interest in theoretical physics owing to the conjectured AdS/CFT correspondence. A feature of these spacetimes is a lack of global hyperbolicity connected to the existence of a timelike conformal boundary. To study linear fields in these backgrounds, one is required to specify boundary conditions ‘at infinity’. The goal of this essay is to study the mathematical questions this raises, and their consequences for physics, in particular of black holes. A good
essay will include a discussion of the issue of boundary conditions for fields in anti-de Sitter, the consequences for well-posedness of the equations, and a review of results for linear fields on AdS black hole backgrounds.

**Relevant Courses**

*Essential:* General Relativity

*Useful:* Analysis of PDE; Black Holes

**References**


**121. The Einstein–Klein-Gordon Equations**

Dr Z. Wyatt

The Einstein–Klein-Gordon system describes the interaction between an unknown Lorentzian spacetime \((M,g)\) and a massive scalar field \(\phi\). The system admits fascinating gravitational phenomena, such as non-decaying soliton-like solutions called boson stars. Another striking result is that solutions to the Klein-Gordon equation on a fixed subextremal Kerr geometry can grow exponentially in time, even from smooth initial data, yet this does not hold true for the zero-mass Klein-Gordon (i.e. wave) equation. The goal of this essay is to understand some of the mathematical and physical results arising from the Einstein–Klein-Gordon equations. It should include a presentation of one of the following topics:

(1) **Time-Periodic Solutions.** An introduction to mini-boson stars, including reference to numerical and mathematically rigorous results [1, 5] and perhaps referring to Birkhoff’s and Derrick’s theorems; other time-periodic solutions containing black holes [4, 2].

(2) **Superradiant Instabilities.** Mode solutions of the Klein-Gordon equation on a subextremal Kerr black hole; superradiance and energy extraction from the black hole [7]; growing mode solutions via heuristic [3] and analytic methods [6]; comparison with results for wave equations and relevance to the Cauchy problem of general relativity.

**Relevant Courses**

*Essential:* General Relativity

*Useful:* Black Holes, Analysis of PDEs
References


122. NIP Theories and Related Topics ...............................  
Dr G. Conant

While much of early model theory was motivated by the study of stable theories, many recent developments in model theory have focused on the wider class of NIP or dependent theories. This class is defined as follows. A complete \( \mathcal{L} \)-theory \( T \) is NIP if there does not exist a model \( M \models T \), an \( \mathcal{L} \)-formula \( \varphi(x_1, \ldots, x_m, y_1, \ldots, y_n) \), and subsets \( \{ \bar{a}_i : i \in \mathbb{N} \} \subseteq M^m \) and \( \{ \bar{b}_X : X \subseteq \mathbb{N} \} \subseteq M^n \), such that \( M \models \varphi(\bar{a}_i, \bar{b}_X) \) if and only if \( i \in X \).

The first aim of this essay is to introduce NIP theories and their properties. The second aim is one of the following specialized topics (chosen by the essay writer).

1. \( \alpha \)-minimality. Let \( M \) be a first-order structure in a language containing (at least) a symbol \( < \) interpreted as a linear order. Then \( M \) is called \( \alpha \)-minimal if every definable subset of \( M \) is a union of finitely many intervals with endpoints in \( M \). This topic will cover the basics of \( \alpha \)-minimality, with examples, and then present a proof that the complete theory of any \( \alpha \)-minimal structure is NIP.

2. Borel definability of invariant types in NIP theories. Let \( T \) be a complete \( \mathcal{L} \)-theory, and let \( M \models T \) be \( \kappa \)-saturated, where \( \kappa \) is sufficiently large. Fix \( A \subset M \) with \( |A| < \kappa \). A type \( p \in S_n(M) \) is \( A \)-invariant if, for any \( \mathcal{L} \)-formula \( \varphi(x_1, \ldots, x_n, y_1, \ldots, y_m) \) and tuples \( \bar{b}, \bar{c} \in M^m \), if \( tp(\bar{b}/A) = tp(\bar{c}/A) \) then \( \varphi(\bar{x}, \bar{b}) \in p \) if and only if \( \varphi(\bar{x}, \bar{c}) \in p \). Given an \( A \)-invariant type \( p \in S_n(M) \) and a formula \( \varphi(\bar{x}, \bar{y}) \), one obtains a “\( \varphi \)-definition for \( p \)”, namely, the set of types \( q \in S_m(A) \) such that \( \varphi(\bar{x}, \bar{b}) \in p \) for some/any \( \bar{b} \models q \). This topic will present the proof that, if \( T \) is NIP and \( p \in S_n(M) \) is \( A \)-invariant, then the \( \varphi \)-definition for \( p \) is a Borel set (in the topology on the space of complete types over \( A \)).

Regardless of the topic chosen, the author should strive to present the material in their own way (rather than following the order of a textbook or other standard source). A good essay will connect the chosen topic to related results from stability theory, and also discuss some applications and/or further directions related to the topic.

Relevant Courses

A first course in model theory is essential for this essay; and some familiarity with stability theory is strongly recommended.
References


123. Nonlinear Stability of Planar Shear Flows

Professor C. G. A. Mouhot

This essay is concerned with understanding a conceptual counterpart of Landau damping in two-dimensional incompressible fluid mechanics. The “damping” discovered by Landau in 1946 at the linearized level is a stability mechanism in plasmas due to phase mixing. It was later extended to galactic dynamics by Lynden-Bell. Recently it was proved that Landau damping implies nonlinear global-in-time stability for analytic or Gevrey solutions [2,3].

The equivalent stability phenomenon in fluid mechanics is called inviscid damping, and the beautiful paper [1], inspired by [2] but also introducing new interesting ideas, proved global-in-time stability of Gevrey-regular perturbations around planar shear flows for the 2D incompressible Euler equation.

The first goal of the essay is to read and understand the paper [1] and write a synthetic, precise and readable global sketch of the proof. Second, the essay should (i) identify and discuss what is the counterpart of “plasma echoes” (i.e. resonances) in this setting and how the latter are controlled, and (ii) should identify and discuss why the proof requires Gevrey regularity for the perturbation (rather than a control on a finite number of derivatives).

Relevant Courses


*Useful:* Any introductory course to a class of hyperbolic PDEs.

References

In this essay you will study the wavefront set of a distribution and its main properties. For a distribution \( u \) on a manifold \( X \), the wavefront set \( \text{WF}(u) \) is a subset of \( T^*X \setminus \{0\} \) which is conic with respect to multiplication by positive scalars in the fibres of the cotangent bundle \( T^*X \). It describes not only the position of the singularities of \( u \) but also their directions. The notion is fundamental for defining operations with distributions (calculus of wavefront sets), particularly products.

The main objective of this essay is to describe in detail the calculus of wavefront sets, including in particular Hörmander’s condition for the product of distributions. Once this is established, some options will arise, like a discussion of Hörmander’s theorem for the propagation of singularities of solutions of partial differential equations with real principal symbol (with a general outline of its proof) or a presentation of the use of wavefront sets on Feynman propagators (with motivation and background).

Chapter 11 of [3] and [2] are very friendly introductions to the topic.

** Relevant Courses **

*Differential Geometry, Analysis of Partial Differential Equations, Distribution Theory and Applications*

**References**


125. **Cohen-Lenstra-Martinet Heuristics** .........................

**Professor J. A. Thorne**

In 1983, Cohen and Lenstra introduced what many mathematicians now refer to as “Cohen-Lenstra-Martinet heuristics”, which aim to give a meaning to the statement that the ideal class group of a number field is a “random finite abelian group”. The first goal of this essay would be to give an introduction to the Cohen-Lenstra-Martinet heuristics. The direction taken after this would be up to the candidate, but some possible directions include:
• Generalizations, by Dummit, Voight and others, of the CLM heuristics to describe the joint
distribution of narrow class groups and unit signature groups, and theoretical evidence
for such generalizations;
• the reformulation, by Bartel and Lenstra, of the CLM heuristics in terms of Arakelov class
groups;
• unconditional progress towards the CLM heuristics in the setting of global function fields.

References

[2] B. Breen, I. Varma, J. Voight. On unit signatures and narrow class groups of odd abelian
[4] D. S. Dummit, J. Voight. The 2-selmer group of a number field and heuristics for narrow
class groups and signature ranks of units. *Proceedings of the London Mathematical Society*,

126. Computing the Volume in High-Dimensions .......................... Dr V. Jog

Computing the volume of a convex body is an age-old fundamental problem in computer sci-
ence with a four decades-long history that is central to several domains, including integral
calculus, thermodynamics, and fluid dynamics. It is also a challenging problem! Indeed, even
approximating the volume within an exponential factor in the dimension was shown to be im-
possible using deterministic polynomial-time algorithms in [1] and [2]. Under this backdrop
came the groundbreaking result of Dyer, Frieze, and Kannan [3], who developed a randomized
polynomial-time algorithm to estimate the volume up to arbitrary relative accuracy.

The complexity bound of the algorithm in Dyer et al. [3] had a very high power of 23 in the
dimension $n$. Subsequent works spanning the past three decades have reduced this power from
$n^{23}$ to $n^{16}$ [4], $n^{10}$ [5, 6], $n^8$ [7], $n^7$ [8], $n^5$ [9], $n^4$ [10], and most recently, $n^{3.5}$ [11]. Each
improvement in the power of $n$ required radically new ideas that revealed novel facets of the
volume computation problem. Some examples include the study of isoperimetric inequalities,
analysis of the mixing times of geometric random walks, and isotropic positions of convex sets.

The goal of this essay will be to understand the various theoretical developments that have
enabled increasingly efficient volume computation algorithms. Why are deterministic algorithms
unsuccessful? What were the key contributions that reduced the complexity of randomized
algorithms from $n^{23}$ to $n^{3.5}$? What are the bottlenecks in further reducing the complexity? Are
state-of-the-art algorithms practical? What are some of the scientific applications of efficient
volume estimation algorithms?
References


127. Stochastic Series Expansions for Gaussian processes

Professor J R Norris

Given a centered Gaussian process $(Z_t)_{t \in [0,1]}$ with continuous covariance function $R(s,t)$ and almost surely continuous trajectories, it is possible to represent it as a series of the form

$$Z_t = \sum_{k=1}^{\infty} \sqrt{\lambda_k} I_k e_k(t)$$

where $e_k(t)$ is the eigenfunction of the covariance operator acting on $L^2[0,1]$ corresponding to the eigenvalue $\lambda_k$, and $(I_k)$ is a sequence of independent standard Gaussian random variables. Such a series converges uniformly in $t$ in mean square and almost surely. This expansion is known as Karhunen–Loeve decomposition of the process $Z_t$.

The standard Karhunen–Loeve decomposition described above is known explicitly and widely used for certain processes ([1]). Apart from those, more general series representations are also of interest. For instance, one might consider a weighted $L^2$-space in place of the standard one, to derive a polynomial series expansion for Brownian motion ([2]); alternatively, one might derive an explicit series representation for fractional Brownian motion which is not Karhunen-Loeve.
Another optimal representation involves wavelet series approximation ([5]).

The essay should include a coherent introduction to the subject, including a review of the literature. It should further address one or more advanced aspects of the theory. Examples of such include but are not limited to the following: numerical applications of series expansions for stochastic differential equations; comparison of different notions of optimality; applications of such representations in statistics.

References


128. Statistical Properties of Stochastic Gradient Descent .................

Prof R. J. Samworth and Dr R. D. Shah

Stochastic Gradient Descent (SGD) is an iterative stochastic optimisation method for smooth functions ([2]), which can be seen as a stochastic approximation of gradient descent. It is widely used in applications, especially in large-scale settings such as neural networks ([3],[7]), as it reduces the computational cost, although this often means a slower rate of convergence ([1]). Its origins can be traced back to the Robbins–Monro algorithm ([6]), and multiple extensions exist, such as the averaged SGD ([5]).

As a stochastic optimisation method, the statistical properties of SGD are of considerable interest. The question of the optimality (in certain settings) of the procedure is studied in [4]. Both asymptotic and finite-sample properties are studied in [8]. A detailed analysis of the case of Generalized Linear Models is presented in [9].

This essay should focus on statistical aspects of SGD in a number of different settings. It could also be interesting to compare SGD to other methods or to provide empirical evidence related to the theoretical results. A motivated candidate may also provide their own ideas, as this is an active field of research.

Relevant Courses

Essential: Topics in Statistical Theory
Convective Instabilities in Galaxy Clusters

Galaxy clusters are gravitationally bound astrophysical structures comprising hundreds, even thousands, of galaxies. Well known examples ‘near’ us include the Virgo, Coma, and Hercules clusters. A characteristic feature of these structures is the extremely hot (∼10^7 K) ionised gas that permeates the space between the galaxies. This gas, referred to as the intracluster medium (ICM), is weakly collisional, and as a consequence the conduction of heat and momentum is anisotropic, aligning itself with the local magnetic field. The anisotropy of the heat transport, in particular, is associated with two unusual ‘convective’ instabilities: the magnetothermal instability and the heat-flux buoyancy instability (MTI and HBI) (see references [1], [2], and [3]). Currently, researchers are testing how these (and the disordered flows they initiate) influence the global structure and properties of galaxy clusters.

In this essay you should discuss the basic physics of weakly collisional and magnetised plasma, and then review the linear theory of the MTI and HBI, paying attention to how the familiar convective stability results are altered by the anisotropic heat flux. You may then survey the nonlinear simulations ([4],[5], [6]), their potential role in conundrums such as the ‘cooling flow problem’ ([7]), or you could have a look at ‘micro-instabilities’ caused by the plasma’s pressure anisotropy ([8]).

**Relevant Courses**

*Essential:* Astrophysical Fluid Dynamics

*Useful:* Structure and Evolution of Stars
The rings of Saturn are perhaps the most familiar and beautiful objects in astrophysics; they also exhibit some of its most puzzling phenomena ([1]). Complex patterns and waves have been carved into the disk by a myriad of processes: viscous and gravitational instabilities, impacts with micrometeoroids, and the gravitational influence of external and internal moons, to name but a few. The patterns extend over a vast range of lengthscales (from 100 metres to 100 kilometres) and are only partially understood. For a review of their observations see [2], for a review of their theory see [3], and for a general review of planetary rings see [4].

Your essay could either (a) survey the observations of these structures and the relevant physics in each case, or (b) concentrate on just one class of structure and go into some mathematical/physical detail. Specific topics that could be discussed include:

(i) How to model the rings. The kinetic theory of cold and dense granular flow.
(ii) Gravitational instability and canted self-gravity wakes in the A and B-rings.
(iii) Viscous overstability and periodic microstructure on 100m scales in the A and B-rings.
(iv) The ballistic transport process: sharp inner ring edges, and 100-1000km structure in the C-ring and inner B-ring.
(v) Spiral density wave launching in the A-ring by external satellites.
(vi) Embedded 100m moonlets in the A-ring (‘propellers’).
(vii) The bizarre dynamics and structures of the braided F-ring.

Relevant Courses

*Essential:* Astrophysical Fluid Dynamics, Dynamics of Astrophysical Discs
Effective field theories (EFTs) are at the heart of modern physics, helping us to describe physical phenomena on a variety of energy scales. A particularly powerful feature of EFTs is that they make the regime of validity of these descriptions manifest. This is especially important in the context of gravitational physics, where we generically encounter non-renormalisable interactions. As a consequence gravitational theories come with a cutoff that effectively marks the largest energy scale where such theories can reliably be applied; for General Relativity this is the Planck scale. One of the current most pressing issues in gravitational physics is to better understand the nature of dark energy, i.e. the agent responsible for the currently observed accelerated expansion of the Universe on very large scales. In this context, applying EFT techniques has proved particularly fruitful, recently leading to a systematic, robust and comparatively model-independent understanding of how precisely dark energy may manifest itself in a cosmological setting and how we can probe it further.

The goal of this essay is to provide a clear account of how EFTs of dark energy are constructed [1,2] and how they can be used to probe and constrain the nature of dark energy. This will initially involve understanding how dark energy degrees of freedom can be thought of as Goldstone bosons of spontaneously broken time translations (also see [3]) and how to construct a general ansatz for linear dark energy interactions in a 3+1 decomposition of space-time [1,2]. The essay should further discuss what the regimes of validity of the resulting EFTs are and how these are related to dark energy phenomenology on cosmological scales. Of particular interest will be what this implies regarding which types of observations can be used to self-consistently test an EFT of dark energy. An ambitious student might in addition explore one of the following options: I) What qualitative constraints current observations (solar system tests, binary pulsars, large scale structure, CMB) allow us to place on the interactions present in EFTs of dark energy. II) How the above approach can be extended beyond the linear interactions [4]. III) What this implies for testing dark energy with current and near-future gravitational wave observations [5,6].

**References**


**Relevant Courses**

*Essential:* Quantum Field Theory, General Relativity.

*Useful:* Cosmology, Field Theory in Cosmology

**References**

132. Fermion Boson Correspondence .............................................. Dr D M A Stuart

This concerns a correspondence between apparently different quantum field theories describing fermions on one hand and bosons on the other hand. The general idea arose in the early days of quantum theory in work of Jordan on the neutrino theory of light; early work is explained in the article of Born and Nagendra Nath, of some historical interest. Later work both made the original insight more precise, generalized it and related it to developments in infinite dimensional Lie algebras. An essay could explain the correspondence in a basic case and offer a development of the correspondence in perhaps a specific field theory, or its use in representation theory or other later developments.

Relevant Courses

Useful: Quantum Field Theory, Advanced Quantum Field Theory, Symmetries, Fields and Particles.

References


133. Euler Systems ................................................................. Professor A J Scholl

In 1988, Viktor Kolyvagin stunned the number theory world with a proof of the finiteness of the Tate–Shafarevich group for a wide class of elliptic curves over Q. Hitherto, there was no known example of an elliptic curve with finite Tate–Shafarevich group. Subsequently, Rubin used Kolyvagin’s ideas to give a simple proof of the so-called “Main Conjecture” for the class groups of cyclotomic fields — a result proved earlier by Mazur and Wiles using a lot of difficult algebraic geometry. Kolyvagin’s proof relied on what he called “Euler systems”, certain families of Galois cohomology classes which can be used to annihilate Selmer groups and class groups.

A good essay will explain what Euler systems are, what they are good for, give an overview of the work of Kolyvagin and Rubin, and highlight the differences between them.
The adventurous may go further and discuss Kato’s Euler system or more recent developments.

**Relevant Courses**

Elliptic curves, Algebraic Number Theory

**References**


The books [1,3,4] are available for free download to Cambridge users via the University Library website. Rubin’s appendix [2] can only be found in the 1990 edition of Lang’s book.

**134. Kadanoff-Wilson Renormalisation Group Approach to the Dynamics of Epidemics with Applications to the Covid-19 Pandemic**

Dr R. Adhikari

The transmission of contagion is a problem of multiple spatial and temporal scales. While contagion is passed on from one individual to another, it is the aggregate of the number of infected individuals that is of relevance for public health. Aggregation over a population of individuals has a natural analogue to the coarse-graining of microscopic degrees of freedom that is the foundation of the Kadanoff-Wilson picture of the renormalisation group. Drawing on Wilson’s conception of the renormalisation group as a tool for studying problems involving multiple scales, this essay invites the application of the renormalisation group ideas to examine the dynamics of epidemics over large networks of connected individuals. Formulating contagion as a stochastic process over a contact network of individual, what dynamics results when the network is aggregated, say, over spatial regions, or age groups? The resulting Doi-Peliti “field theory” would necessarily violate detailed balance and, hence, be non-Hermitean. How does this breakdown of detailed balance effect the renormalisatio group flow in the space of “coupling constants”, in particular the probability of infection on contact? How would these impact the fitting of aggregated epidemiological models to surveillance data? How would one translate this understanding of the coarse-grained microscopic dynamics to evaluate the macroscopic effects of microscopic non-pharmeceutical interventions (such as physical distancing, wearing masks, etc)? These are questions that this essay should explore.

**References**


