Mathematical Tripos Part III Lecture Courses in 2022-2023

Department of Pure Mathematics and Mathematical Statistics

Department of Applied Mathematics and Theoretical Physics

1 Algebra

Lie Algebras and their representations (L24)

Ian Grojnowski

This course is an introduction to the basic properties of finite dimensional complex Lie algebras and of their representations.

Lie algebras are 'infinitesimal symmetries'; linearisations of groups. They are ubiquitous in many branches of mathematics: in topology, in arithmetic and algebraic geometry, and in theoretical physics (string theory, exactly solvable models in statistical mechanics,...)

One reason for their importance is that the finite dimensional complex representations of the simple Lie algebras are exactly the same as those of the corresponding groups. So instead of needing to study the topology and geometry of the simple Lie groups, or the algebraic geometry of the simple algebraic groups, we can use nothing other than linear algebra and still completely describe these representations.

(Later you will want to study the topology and geometry as well, of course!)

We will cover the following topics:

- Definitions, motivations, and basic structure theory.
- Root systems, Weyl groups, the finite simple Lie algebras.
- Classification of finite dimensional representations, Verma modules, Weyl character formula.
- Crystals, Littelmann paths.
- if there is time, we will finish by discussing affine Lie algebras, the basic representation, Boson-Fermion correspondence, and theta functions.

Prerequisites

None other than linear algebra, but the part II course on representation theory (or equivalent) will be useful as background.

Literature

1. V. Kac, *Infinite dimensional Lie algebras.* Cambridge University Press
2. M. Kashiwara, *On crystal bases,* Representations of groups. (Banff, AB, 1994). 155–197, CMS Conf. Proc., 16,
3. N. Jacobson, *Lie algebras*

Additional support

Exercises will be provided in class, four examples classes will be given. There will be a one-hour revision class in the Easter Term.

Introduction to modular representation theory (M16)

Non-Examinable (Part III Level)

Stuart Martin

Modular representation theory, namely, the study of representations of finite groups over fields of characteristic other than zero, was initiated by L.E. Dickson (1902). Such representations arise naturally in number theory, the theory of error-correcting codes, combinatorics and in topology.

The first really major developments came with Richard Brauer, who from 1935 onwards exploited this rich and virtually untapped area of mathematics. Brauer's methods were mainly character-theoretic. What is now called the theory of Brauer characters determines the composition factors of the representation but, unlike the situation over $\mathbf{C}$, not the equivalence type. One of Brauer's main goals was finding numerical constraints on the orders and internal structure of finite simple groups, and his methods were later used by Glauberman and others in the Classification program.

The next revolution in the theory came with Sandy Green, who considered the modules themselves. His techniques were completely different to those of Brauer, and his main goals lay in understanding the modules, rather than Brauer, who worked mostly with characters and blocks (a set of equivalence classes of characters). The aim of this course is to give a flavour of both classical techniques

of Brauer and the more modern techniques of Green. We'll discuss Brauer characters, defect groups, blocks, decomposition numbers as well as projective and injective modules (as time allows).

I'll aim to cover some of the following:

- Review of assumed and basic material: group algebras, representations and modules, reducibility and decomposability, Maschke's theorem, tenors and homs, Frobenius reciprocity, Mackey's theorem, maximal and primitive ideals, Jacobson radical, complete reducibility, semisimple rings, Artin-Wedderburn structure theorem.
- Modular character theory: p-singular and p-regular elements, Brauer characters, Grothendieck groups.
- Irreducible, projective and injective modules: DVRs, (splitting) p-modular systems, forms, decomposition numbers and the decomposition matrix, counting modular irreducible modules.
- Projective indecomposable modules and their characters: the Higman criterion, primitive orthogonal idempotents, idempotent refinement.
- Block theory: the socle and the head, Cartan matrix, lifting projectives, central idempotents.
- Central characters: the central character of an ordinary irreducible mod p determines the block.
- (If time permits) Further block theory: defect groups, relative projectivity and blocks of defect 0. (Statement of) the Alperin conjecture, some comments on local–global conjectures.

Prerequisites

Basic group theory; ordinary representations/character theory from the Part II (or equivalent) course; Sylow theory for finite groups; commutative algebra from the relevant Part III course (rings, ideals, completions, local rings, primality, maximality, the Artin-Wedderburn theorem); some categorical nonsense. The Part III course on the representations of symmetric groups is complementary to this course and provides a good source of examples.

Preliminary Reading

1. C.W. Curtis, *Pioneers of representation theory: Frobenius, Burnside, Schur and Brauer.* AMS, 1999
2. P. Etingof et al, *Introduction to representation theory.* AMS, 2011

3. I.M. Isaacs, *Character theory of finite groups.* Dover reprint, 1994

4. G.D. James & M.W. Liebeck, *Representations and characters of groups* 2nd edition. CUP, 2001

5. W. Ledermann, *Introduction to group characters* 2nd edition. CUP, 1987

Literature

1. J.L. Alperin, *Local representation theory.* CUP, 1986

2. D.J. Benson, *Representations and cohomology* Volume 1. CUP, 1991

3. D.A. Craven, *Representation theory of finite groups: a guidebook.* Springer, 2019 - it has 574 references!

4. P. Schneider, *Modular representation theory of finite groups.* Springer, 2013

5. P. J. Webb, *A course in finite group representation theory.* CUP, 2016

Group Cohomology (L16)

Christopher Brookes

Group cohomology uses methods motivated by algebraic topology to study groups and their representations.

Applications appear throughout mathematics, in representation theory, topology, geometry and number theory (Galois cohomology).

I hope to cover all/most of the following:

- Definitions and resolutions.
- Low degree cohomology, derivations and group extensions.
- Group homology, the Schur multiplier (or multiplicator) and central extensions.
- Crossed products, division algebras and Brauer groups of fields.
- Cup products and cohomology rings.
- Lyndon-Hochschild-Serre spectral sequence, inflation-restriction exact sequence.
- Cohomological dimension, finiteness conditions F_n and FP_n.

Prerequisites

My approach will be purely algebraic but it will be useful to have some acquaintance with (co-)homology from other contexts. Many of the examples will be of groups arising in geometry and topology.

You should know your way about groups, rings and modules. For this, the III course on Commutative Algebra will provide good experience. Some Representation Theory would also be useful.

I shall assume acquaintance with tensor products (of vector spaces/modules and algebras), and, for the section on Brauer groups of fields, I shall also assume that you have met Galois groups (though you won't in fact need much more than the definition).

Literature

1. D.J. Benson, *Representations and Cohomdlogy, volume II, Cohomology of Groups and Modules.* Cambridge University Press, 1991.
2. K.S. Brown, *Cohomology of Groups.* Springer, 1982.
3. L. Evens. *The Cohomology of Groups.* Oxford University Press, 1991.
4. R.S. Pierce, *Associative Algebras.* Springer, 1982.
5. D.J.S. Robinson, *A Course in the Theory of Groups.* Springer, 1982.
6. J.J. Rotman, *An Introduction to Homological Algebra.* Springer, 2009.
7. C.A. Weibel, *An Introduction to Homological Algebra.* Cambridge University Press, 1994.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Commutative Algebra (M24)

Oren Becker

This course will provide an introduction to the theory of commutative rings and modules over these rings. It should be viewed as a foundational course for Algebraic Geometry and Algebraic Number Theory.

The course will cover a selection of topics including:

- Unital commutative rings, ideals, localisations and fields of fractions, prime ideals and the topological space Spec(A)
- Modules, Nakayama's Lemma, tensor products, flatness
- Integral extensions, going up and down theorems, Noether normalisation and Hilbert Nullstellensatz, valuation rings
- Noetherian and Artinian rings and modules, Hilbert basis theorem, primary decomposition in Noetherian rings, basic theory of Dedekind domains
- Dimension theory and Hilbert functions
- A brief introduction to homological algebra

Prerequisites

You will have attended a first course in ring theory, such as the IB course Groups, Rings and Modules. Experience of more advanced material such as Part II courses Galois Theory, Representation Theory, Algebraic Geometry or Number Fields is desirable (contributing to your level of algebraic maturity) but not essential.

Literature

1. M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
2. D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Spring-Verlag, 1995.
3. H. Matsumura, *Commutative Algebra*, second edition, The Benjamin/Cummings publishing co., 1980.
4. H. Matsumura, *Commutative Ring Theory*, Cambridge Studies 8, CUP, 1989.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Representation Theory of Symmetric Groups (M24)

Stacey Law

The representation theory of symmetric groups is a classical subject with many connections across mathematics, computer science and physics. The topic continues to lie at the forefront of modern research, due to the rich interplay between algebra and combinatorics which has led to the development of a deep yet versatile theory.

In this course, we will develop some of the techniques underpinning both the proof of the McKay Conjecture for symmetric groups, as well as other key results in the area from the last few years. Time permitting, we will cover:

- Combinatorics of partitions: Young diagrams and tableaux.
- Specht modules and simple modules for symmetric groups.
- Standard basis for Specht modules.
- Hook length formula for dimensions.
- The determinantal form of irreducible characters.
- Young's Rule, Branching Rule, Murnaghan–Nakayama Rule.
- Further combinatorics of partitions: James' abacus, cores, quotients and towers.
- McKay numbers for symmetric groups.

Prerequisites

Part IB Groups, Rings and Modules and Part II Representation Theory (or equivalent). Relevant concepts from Part IB Linear Algebra will be reviewed. Although certain sections of the course will be quite combinatorial, there are no specific combinatorial prerequisites.

Literature

1. G. D. James, *The Representation Theory of the Symmetric Groups.* Lecture Notes in Mathematics **682**, Springer, Berlin, 1978.
2. G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group.* Encyclopedia of Mathematics and its applications vol. 16, Addison-Wesley, 1981.

3. J. Olsson, *Combinatorics and Representations of Finite Groups*, Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen, Heft 20, 1994.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in Easter Term.

2 Algebraic Geometry

Algebraic Geometry (M24)

Dhruv Ranganathan

The course will be an introduction to modern algebraic geometry, focusing on schemes, morphisms, and sheaves. The course will develop the basic toolkit used by practicing algebraic geometers, and will also be appropriate for anyone with an interest in acquiring a working knowledge of algebraic geometry, including those in nearby fields such as differential geometry, topology, and number theory.

The course will begin with discussion of a range of examples from the theory of algebraic varieties, to recall their basic properties and exhibit the phenomena that motivate the theory of schemes. We will then turn to an introduction to schemes and morphisms, and eventually move towards coherent sheaves and their cohomology.

Prerequisites

The basic theory of rings and modules and elementary topology will both be assumed. It is recommended that students have had a previous course in Commutative Algebra, take one in parallel, or familiarize themselves with the concepts from Atiyah–MacDonald's text on the subject. A parallel course in commutative algebra is both sufficient and standard The key ideas from algebra that we will need are ideals, modules, tensor products, and localization. A prior course in algebraic varieties is not logically necessary, but students should be prepared to occasionally consult with resources on these more classical aspects for constructing and working with examples. Some intuition is provided from topology and the theory of manifolds, and this may be helpful but is not essential.

Literature

The following books complement the course material. The first and second are roughly interchangeable. The final text provides valuable intuition for the many unfamiliar concepts in scheme theory.

1. R. Hartshorne, *Algebraic Geometry*, Springer (1977).
2. R. Vakil, *The rising sea. Foundations of algebraic geometry.* Available online.
3. D. Eisenbud and J. Harris, *Geometry of schemes*, Springer (2000).

Additional support

Four examples sheets will be provided and four associated examples classes will be given; these will be essential. Additional problems and examples may be provided to clarify more difficult concepts. There will be a revision class in the Easter Term.

Abelian varieties (L16)

Anthony J Scholl

An abelian variety is an irreducible projective variety over a field which is also a group. The simplest example is an elliptic curve. Other examples include the Jacobian variety of a (smooth projective) curve, whose points parametrise divisor classes of degree zero. There are beautiful analytic and algebraic theories of such objects. They also crop up in other areas of geometry and in number theory. This course will cover some of the basic theory of abelian varieties. Topics to be covered may include:

- The Jacobian of a curve (analytic and algebraic).
- Abelian varieties, line bundles and polarisations.
- Abelian varieties over finite fields. The Weil conjectures.

Prerequisites

For the algebraic part of the course, the Michaelmas Part III Algebraic Geometry course (or equivalent). For the analytic theory, basic complex analysis, algebraic topology (covering spaces and the fundamental group) and Riemann surfaces.

Literature

1. David Mumford: *Curves and their Jacobians.* University of Michigan Press, 1977.
2. David Mumford: *Abelian varieties.* Second edition. Tata Institute of Fundamental Research, 1974.
3. Christina Birkenhake, Herbert Lang: *Complex abelian varieties.* Springer-Verlag, 2004.
4. H. P. F. Swinnerton-Dyer: *Analytic theory of abelian varieties.* LMS Lecture Nots 14. Cambridge University Press, 1977.

There are many books on abelian varieties (and copious online resources). The books [1,2] by Mumford are classics, although [2] is quite condensed in places. The books [3,4] deal with the analytic theory (the first 5 chapters of [3] give a very detailed account). There is a draft book by Edixhoven, Moonen and Van Der Geer on the algebraic theory which is available at https://www.math.ru.nl/ bmoonen/research.htmlbookabvar which does everything from a modern scheme-theoretic point of view.

Additional support

3 examples sheets will be provided and 3 associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

3 Analysis and PDEs

Functional Analysis (L24)

András Zsák

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. We will cover the following topics:

Hahn–Banach Theorems on the extension of linear functionals. Locally convex spaces.

Duals of the spaces $L_p(\mu)$ and $C(K)$. The Radon–Nikodym Theorem and the Riesz Representation Theorem.

Weak and weak-* topologies. Theorems of Mazur, Goldstine, Banach–Alaoglu. Reflexivity and local reflexivity.

Hahn–Banach Theorems on separation of convex sets. Extreme points and the Krein–Milman theorem. Partial converse and the Banach–Stone Theorem.

Banach algebras, elementary spectral theory. Commutative Banach algebras and the Gelfand representation theorem. Holomorphic functional calculus.

Hilbert space operators, C^*-algebras. The Gelfand–Naimark theorem. Spectral theorem for commutative C^*-algebras. Spectral theorem and Borel functional calculus for normal operators.

Some additional topics time permitting. For example, uniform convexity and smoothness, ultraproducts, the Fréchet–Kolmogorov Theorem, weakly compact subsets of $L_1(\mu)$, the Eberlein–Šmulian and the Krein–Šmulian theorems, the Gelfand–Naimark–Segal construction.

Prerequisites

Thorough grounding in basic topology and analysis. Some knowledge of basic functional analysis and basic measure theory (much of which will be recalled either in lectures or via handouts). In Spectral Theory we will make use of basic complex analysis. For example, Cauchy's Theorem, Cauchy's Integral Formula and the Maximum Modulus Principle.

Literature

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales).* Oxford University Press, 2011.

2. Bollobás, Béla *Linear analysis: an introductory course.* Cambridge University Press, 1990.

3. Murphy, Gerard J. *C^*-Algebras and Operator Theory.* Academic Press, Inc., 1990.

4. Rudin, Walter *Real & Complex Analysis.* McGraw-Hill, 1987.

5. Rudin, Walter *Functional Analysis.* McGraw-Hill, 1990.

6. Taylor, S. J. *Introduction to measure and integration.* Cambridge University Press 1973.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be some material as well as examples sheets and announcements available at `www.dpmms.cam.ac.uk/~az10000/`

Analysis of Partial Differential Equations (M24)

Prof. M. Dafermos

This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is presently a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics. The course will mostly focus on developing the theory and methods of the modern approach to PDE theory. Emphasis will be given to functional analytic techniques, relying on a priori estimates rather than explicit solutions. The course will primarily focus on approaches to linear, elliptic and evolutionary problems through energy estimates, with the prototypical examples being Laplace's equation, the heat equation, the wave equation and the Schrödinger equation.

The following concepts will be studied: well-posedness; the Cauchy problem for general (non- linear) PDE; characteristics; Sobolev spaces; elliptic boundary value problems (solvability and regularity); evolutionary problems (hyperbolic, parabolic and dispersive PDE).

Prerequisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self- contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Literature

1. Some lecture notes and example sheets from previous lecturers of the course are available online at:

   http://cmouhot.wordpress.com/teachings/

   https://www.dpmms.cam.ac.uk/ cmw50/Teaching/downloads/page10.html

   The following textbooks are excellent references:

2. Evans, L. C., *Partial Differential Equations*, Springer, 2010

3. Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.

4. John, F., *Partial Differential Equations*, Springer, 1991

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

An introduction to non linear analysis (M24)

Pierre Raphaël

The 6th Clay problem has an elementary formulation: may incompressible fluids form singularities? This question and the description of singularities, at the heart of the mathematical study of non linear waves and their partial differential equations, is relevant in various areas of mathematical physics (astrophysics, non linear optics, electromagnetism ...) and has attracted an immense amount of mathematics in the last fourty years with inputs from very different fields: harmonic analysis, dynamical systems, variational methods.

This class is an introduction to the mathematical study of non linear waves. We will focus onto a canonical model, the non linear Schrödinger equation. We will introduce in a self contained way using very little prerequisites the basic concepts that have emerged in the last fourty years: Cauchy theory and scattering à la Ginibre and Velo (1983), stability of solitary waves (Cazenave-Lions 1983) through variational methods, and an introduction to singularity formation and the minimal mass Theorem (Merle 1992).

Prerequisites

Basic notions of functional analysis (Hilbert and Banach spaces). Basis notion of distributions theory ($\mathcal{S}(R^d)$ and $\mathcal{S}'(R^d)$). Basic notion of continuous Fourier transform (Plancherel).

Literature

1. J.-M. Bony, *Intégration et analyse hilbertienne, cours de l'École Polytechnique*, 2006.

2. J.-M. Bony, *Cours d'analyse, théorie des distributions et analyse de Fourier, Éditions de l'École Polytechnique.*

3. H. Brézis, *Analyse fonctionnelle. Théorie et applications*, Masson, 1984.

4. J. Ginibre et G. Velo, *On a class of nonlinear Schrödinger equations. I. The Cauchy problem, general case,* Journal of Functional Analysis, **32**, 1979, pages 1–32.

5. P.-L. Lions, *The concentration-compactness principle in the calculus of variations. The locally compact case. I.* Ann. Inst. H. Poincaré Anal. Non Linéaire, **1**, 1984, pages 109–145.

6. F. Merle, *Determination of blow-up solutions with minimal mass for nonlinear Schrödinger equations with critical power,* Duke Math. Journal, **69**(2), 1993, pages 427–454.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Elliptic Partial Differential Equations (L24)

Greg Taujanskas & Neshan Wickramasekera

This course will provide an introduction to the theory of linear second order elliptic partial differential equations. Second order elliptic equations play a fundamental role in many areas of mathematics including Calculus of Variations, Riemannian Geometry and Mathematical Physics. Linear elliptic theory provides the foundation for studying a number of non-linear problems arising in these fields, such as minimal submanifolds, harmonic maps and evolution problems in Geometry and Mathematical Physics.

The course will provide a rigorous treatment, based on a priori estimates, of both classical and weak solutions to linear elliptic equations, focusing on the

question of existence and uniqueness of solutions to the Dirichlet problem and the question of regularity of solutions. Specific topics include:

- harmonic functions
- maximum principles for general second order equations
- Schauder estimates (via L. Simon's scaling argument)
- the continuity method for existence of solutions
- solvability of the Dirichlet problem in balls
- Perron's method, divergence form operators
- De Giorgi–Nash–Moser estimates and the Harnack theory
- (as time permits) a brief discussion of the quasilinear theory centred around the prototypical example of the Minimal Surface Equation

Prerequisites

Part III *Analysis of PDEs.*

Literature

1. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order.*
2. L. Simon, *Schauder estimates by scaling.* Calc. Var. & PDE, **5**, (1997), 391–407.
3. P. Minter, Lecture notes for the courses *Elliptic PDEs* and *The Minimal Surface Equation and Related Topics*, https://minterscompactness.wordpress.com/lecture-notes/

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

4 Applied and Computational Analysis

Unbounded Operators and Semigroups (M16)

David Stuart

Scientific applications of functional analysis typically involve operators whose domain is a subset of a Hilbert space, and which are not necessarily continuous; differential operators are examples. Development of an analytic theory for appropriate classes of such operators was intimately connected with the development of quantum mechanics: Stone's theorem on generators of strongly continuous one parameter unitary groups, von Neumann's spectral theory for self-adjoint operators and the Stone von-Neumann uniqueness theorem on representations of the Heisenberg relations are early highlights. Later developments led to applications to PDEs, stochastic processes and quantum field theory.

- Examples of unbounded operators on Hilbert spaces. Closed operators; adjoints and self-adjointness. Semigroups, generators and $\dot{u} = Au$ for unbounded A: the Hille-Yosida and Stone theorems.
- Examples and further properties: Trotter product formula, PDEs, Markov processes, Hypercontractive estimates and semigroups.
- Perturbation theory and convergence of unbounded operators and semigroups.
- Inhomogeneous equations $\dot{u} = Au + f(t)$; Kato's construction of evolution operators for nonautonomous equations $\dot{u} = A(t)u + f(t)$ with applications.
- Monotone operators. Applications to nonlinear problems.

Prerequisites

The part II courses *Linear Analysis*, *Analysis of Functions* and also *Probability and Measure* are most relevant.

Literature

- General text books:
  1. J. Goldstein, *Semigroups of linear operators and applications.* OUP, 1985.
  2. M. Reed and B. Simon, *Methods of Modern Mathematical Physics I: Functional Analysis.* Academic Press, 2nd edition, 1980.

3. M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis and self-adjointness.* Academic Press, 1975.
4. K. Yosida, *Functional Analysis.* Springer, 1980.

- More specialized monographs:

1. T. Kato, *Quasi-linear equations of evolution, with applications to partial differential equations.* Springer LNM 448, 1974.
2. H. Brezis, *Operateurs Maximaux Monotones et Semi-Groupes De Contractions Dans Les Espaces De Hilbert.* North Holland, 1973.

Additional support

Weekly office hour. Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Nonlinear Functional Analysis and Applications (M24)

Non-Examinable (Part III Level)

Edriss S. Titi

In these self-contained lectures we will introduce and develop some of the basic analytical tools in nonlinear functional analysis. The main focus will be on the implementation of these methods in investigating and solving certain nonlinear problems, with emphasize on nonlinear partial differential equations. The course will be useful to all students with interest in nonlinear analysis, applied and computational mathematics, and differential geometry.

The course will cover a selection of topics including:

- Review of Basic Function Spaces.
- Some Fixed Point Theorems.
- Galerkin Method.
- Monotone Iterations and Monotone Operators.
- Differential and Integral Calculus in Banach Spaces.
- Variational Methods.
- Palais-Smale Condition and Moutain Pass Lemma.
- Pohozaev's Identity.

Prerequisites

This course assumes the students to know Real Analysis and basic Linear Functional Analysis.

Literature

1. E. Zeidler, *Nonlinear Functional Analysis and its Applications, Parts I, II, III.* Springer–Verlag.
2. S. Kesavan, *Nonlinear Functional Analysis and its Applications: A First Course.* Hindustan Book Agency (India), 2004.
3. Additional various sources from the mathematical literature.

Additional support

This will be non-examinable graduate course. There will be no examples sheets associated with this course. Nonetheless, some problems will be given once in a while for those who are interested to have hands on the material. Moreover, students will have the opportunity to discuss their solutions.

Numerical solution of differential equations (M24)

Arieh Iserles

The goal of this lecture course is to present and analyse efficient numerical methods for ordinary and partial differential equations. The exposition is based on few basic ideas from approximation theory, complex analysis, theory of differential equations and linear algebra, leading in a natural progression to a wide range of numerical methods and computational strategies. The emphasis is on algorithms and their mathematical analysis, rather than on applications.

The course consists of three parts: methods for *ordinary differential equations* (with an emphasis on initial-value problems and a thorough treatment of stability issues and stiff equations), numerical schemes for *partial differential equations* (both boundary and initial-boundary value problems, featuring finite differences and finite elements) and, time allowing, *numerical algebra of sparse systems* (inclusive of fast Poisson solvers, sparse Gaussian elimination and iterative methods). We start from the very basics, analysing approximation of differential operators in a finite-dimensional framework, and proceed to the design of state-of-the-art numerical algorithms.

Desirable Previous Knowledge

Good preparation for this course assumes relatively little in numerical mathematics *per se,* except for basic understanding of elementary computational techniques in linear algebra and approximation theory. Prior knowledge of numerical methods for differential equations will neither be assumed nor is necessarily an advantage. Experience with programming and application of computational techniques will obviously aid comprehension but is not a prerequisite.

Fluency in linear algebra and decent understanding of mathematical analysis are a must. Thus, linear spaces (inner products, norms, basic theory of function spaces and differential operators), complex analysis (analytic functions, complex integrals, the Cauchy formula), Fourier series, basic facts about dynamical systems and, needless to say, elements from the theory of differential equations.

There are several undergraduate textbooks on numerical analysis. The following present material at a reasonable level of sophistication. Often they present material well in excess of the requirements for the course in computational differential equations, yet their contents (even the bits that have nothing to do with the course) will help you acquire valuable background in numerical techniques:

Introductory Reading

1. S. Conte & C. de Boor, *Elementary Numerical Analysis,* McGraw–Hill, New York, 1980.
2. G.H. Golub & C.F. van Loan, *Matrix Computations,* 3rd edition. Johns Hopkins Press 1996.
3. M.J.D. Powell, *Approximation Theory and Methods,* Cambridge University Press, Cambridge, 1981.
4. G. Strang, *Introduction to Linear Algebra,* Wellesley-Cambridge Press, Cambridge (Mass.), 3rd ed. 2003..

Reading to complement course material

1. U. M. Ascher, *Numerical Methods for Evolutionary Differential Equations,* SIAM, Philadelphia, 2008.
2. E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer-Verlag, Berlin, 2nd ed. 1993.
3. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems,* Springer-Verlag, Berlin, 2nd ed. 1996.

4. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University Press, Cambridge, 2nd ed. 2008.

5. K.W. Morton & D.F. Mayers, *Numerical Solution of Partial Differential Equations: An Introduction,* Cambridge University Press, Cambridge, 2005.

6. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Wellesley–Cambridge Press, Cambridge (Mass.), 2nd ed. 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in Mathematics for Deep Learning (E16)

Non-Examinable (Part III Level)

Carlos Esteve Yagüe

This non-examinable course is intended to be an introduction to the mathematical theory of Deep Learning. The course is aimed at part-III students with a mathematical background, with the goal of attracting their attention to this new research field which is nowadays experiencing a huge development.

The great success of the application of Deep Learning techniques to a wide range of real-life problems has raised a number of mathematical questions, some of them being still unanswered, or partially answered. These questions concern the actual approximation and generalization power of deep neural networks as well as the success of non-convex optimization techniques used during training.

During the course, we will give an introduction to the mathematical tools and the notation used to describe and analyse Deep Learning techniques. Then, we will present the main problems and challenges arising in the field and review some of the existing results.

The course will cover a selection of topics including:

- **Introduction to Deep Learning architectures:** Multi-Layer Perceptron, Residual Networks, Convolutional Neural Networks, etc.
- **Expressivity of Neural Networks:** results on approximation of functions by means of one-layer neural networks, the role of the depth, etc.
- **Optimization of Deep Neural Networks:** convergence of stochastic gradient descent, loss landscape analysis, implicit regularization, etc.

- **Generalization of overparametrized Neural Networks:** Neural tangent kernels, double descent phenomenon, etc.
- **Geometrical properties of NN architectures:** convolutional NNs, scattering transform, equivariant NNs, etc.
- **DL for sciences:** Physics Informed NNs, DL for inverse problems, etc.

Prerequisites

This course is addressed to graduate students in mathematics. No specific knowledge is required, but a background in mathematical analysis, statistics and/or optimization are welcome.

Literature

1. J. Berner, P. Grohs, G. Kutyniok, and P. Petersen, *The modern mathematics of deep learning*, arXiv preprint arXiv:2105.04026 https://arxiv.org/pdf/2105.04026.pdf.

Topics in Convex Optimisation (L16)

Dr H. Fawzi

Mathematical optimisation problems arise in many areas of science and engineering, including statistics, machine learning, robotics, signal/image processing, and others. This course will cover algorithms for convex optimization problems, i.e., algorithms to solve problems of the form

$$\min_{x \in X} f(x)$$

where $f : R^n \to R$ is a convex function, and $X \subset R^n$ is a convex set. The course will start by reviewing some convex analysis and basic optimization methods (e.g., gradient method), after which we will move to more advanced methods for smooth, nonsmooth, and structured optimization problems. Topics include subgradient methods, proximal and operator splitting methods, accelerated methods, duality, interior-point path-following methods, among others. We will discuss and prove convergence guarantees for the different algorithms, and illustrate them with applications in different areas of applied mathematics.

Prerequisites

This course assumes basic knowledge in linear algebra and analysis. Some knowledge of convex analysis will be useful.

Literature

1. S. Boyd, L. Vandenberghe, http://web.stanford.edu/ boyd/cvxbook/*Convex Optimization*, Cambridge University Press, 2004.

2. Y. Nesterov, https://doi.org/10.1007/978-1-4419-8853-9*Introductory Lectures on Convex Optimization*, Springer, 2004.

3. S. Bubeck, https://arxiv.org/abs/1405.4980*Convex Optimization: Algorithms and Complexity*, Foundations and Trends® in Machine Learning, 2014.

4. N. Parikh and S. Boyd, https://web.stanford.edu/ boyd/papers/pdf/prox$_a lgs.pdf Proximalalgorithms, F$

5. A. Ben-Tal and A. Nemirovski, http://dx.doi.org/10.1137/1.9780898718829*Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

5 Astrophysics

Structure and Evolution of Stars (M24)

Anna N Żytkow

The only way we know how to form heavy elements is through stellar processes. Thus stars are central in bulding the world around us. Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These equations have to be supplemented by an account of the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed.

Approximate solutions of the equations will be described. Polytropic gas spheres, homology principles and the virial theorem will be presented. The evolution of stars will be discussed, starting from their formation, evolution along the main-sequence, followed by the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

There will be a brief discussion of helioseismology, stellar rotation and mass loss from stars.

The only way in which we may test stellar structure and evolution theory is through the comparison of theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

Prerequisites

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, although a detailed knowledge of all of these is not expected.

Preliminary Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Literature

1. Eldridge, J.J. and Tout, C.A. *The Structure and Evolution of Stars*, World Scientific, 2019.
2. Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution, Second Edition*, Springer-Verlag, 2012.
3. Iben, I. *Stellar Evolution Physics,* Vol. 1 and 2, Cambridge University Press, 2013.
4. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
5. Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

Additional support

There will be four example sheets, each of which will be discussed in an examples class. There will be a revision class in the Easter Term.

Astrophysical Fluid Dynamics (M24)

Roman Rafikov

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs.

The course will cover a selection of topics including:

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- MHD equilibria, Grad-Shafranov equation.
- Conservation laws, stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, astrophysical shocks (supernova explosions) and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.

Prerequisites

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of vector calculus, fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas*. Cambridge University Press.

2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.

3. Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows*. Cambridge University Press. Available as an e-book from

   http://ebooks.cambridge.org

4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics*. University Science Books.

5. Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics*. Imperial College Press.

6. Ogilvie, G. I. (2016). *Lecture Notes: Astrophysical Fluid Dynamics*. J. Plasma Phys. **82**, 205820301.

Additional support

Four example sheets will be provided and four associated classes will be given. Extended notes supporting the lecture course are available from reference 6 in the list above. There will be a revision class in Easter Term.

Extrasolar Planets: Atmospheres and Interiors (M24)

Nikku Madhusudhan

The field of extrasolar planets (or 'exoplanets') is one of the most dynamic frontiers of modern astronomy. Exoplanets are planets orbiting stars beyond the solar system. Thousands of exoplanets are now known with a wide range of masses, sizes, temperatures, and orbital parameters, spanning gas giants, ice giants, rocky planets and more. The field is now moving into a new era of Exoplanet Characterization, which involves understanding the atmospheres, interiors, and formation mechanisms of exoplanets, and ultimately finding potential biosignatures in the atmospheres of rocky exoplanets. These efforts are aided by both high-precision spectroscopic observations as well as detailed theoretical models of exoplanets.

The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical

processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and thermal inversions, equilibrium/non-equilibrium chemistry, atmospheric dynamics and clouds/hazes) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, parametric models and retrieval techniques) (3) exoplanetary interiors (equations of state, mass-radius relations, and internal structures of giant planets, super-Earths/mini-Neptunes, and rocky exoplanets), and (4) habitability and biosignatures. Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed. The course will also include a discussion on detecting biosignatures in rocky exoplanets, the relevant theoretical constructs and observational prospects with current and future facilities.

Prerequisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self-contained and basic concepts will be introduced as required.

Literature

1. Seager, S., Exoplanet Atmospheres: Physical Processes, Princeton Series in Astrophysics (2010).
2. Exoplanets, University of Arizona Press (2011), ed. S. Seager.
3. de Pater, I. and Lissauer J., Planetary Sciences, Cambridge University Press (2010).
4. Chapters on exoplanetary atmospheres and interiors in the book Protostars and Planets VI, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning. Available publicly on astro-ph arXiv (e.g., arXiv:1402.1169, arXiv:1401.4738).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Modern stellar dynamics (M16)

Eugene Vasiliev

This course covers various aspects of structure and evolution of self-gravitating stellar systems. We start with a review of classical stellar dynamics, including potentials, orbits, equilibrium models, continue with more advanced concepts such as relaxation and kinetic theory, and touch upon some frontier research topics such as galactic archeology. The course is mostly theoretical, but will make connections with state-of-the-art observational facilities and recent discoveries. In the examples classes, we will consider both textbook problems and more open-ended exercises drawn from contemporary astronomical literature.

Fundamental concepts: potential–density pairs, Poisson equation and its solution. Continuum limit, distribution functions. Orbits and integrals of motion.

Equilibrium models of stellar systems: Jeans theorem, collisionless Boltzmann equation and its moments, Jeans equations. Models based on distribution functions. Particle- and orbit-based models. Observational constraints for equilibrium models.

Dynamical evolution of self-gravitating systems: collisional Boltzmann equation, Fokker–Planck approximation, two-body relaxation. Gravitational thermodynamics, evolution of star clusters. Galaxy mergers and tidal disruptions, dynamical friction, formation of streams and shells, galactic archeology.

Prerequisites

The course relies extensively on standard mathematical tools such as calculus, ODEs and PDEs, classical mechanics, etc. Some knowledge of astrophysics, in particular, the Part II course "Stellar dynamics and structure of galaxies" would be a valuable asset. The Part III course "Life and death of galaxies" in Lent is also recommended and covers a complementary range of topics.

Literature

1. Binney, J., & Tremaine, S. *Galactic dynamics* (2nd edition), Princeton University Press, 2008
2. Bovy, J., *Dynamics and Astrophysics of Galaxies*, Princeton University Press (in preparation), current draft is available at https://galaxiesbook.org.
3. Saslaw, W., *Gravitational physics of stellar systems*, Cambridge University Press, 1985

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The Life and Death of Galaxies (L24)

Vasily Belokurov

This course will provide the observational perspective on the evolution of galaxies.

The course will cover a selection of topics including:

- **SEDs.** Panchromatic view of galaxies. Dependance of the galactic structure on wavelength. Components of the galactic SED. Stars. Dust. Nebular emission. Stromgren sphere. Spectral lines. Initial mass function: its shape and observational constraints. Links between SEDs (continuum, absorption and emission lines) and star-formation history, the current (recent) star-formation, stellar and gas contents. Age-metallicity degeneracy.
- **Star-formation.** Star-formation rates. Metal-enrichment. Yield. Closed box, instantaneous mixing, one zone model. G-dwarf problem. Inflows and Outflows. Mass-metallicity relation. Star-Formation Law. Cloud collapse and fragmentation. Star formation on sub-galactic scales. Quenching. Star-formation history of the Universe. Integrated light: Look back vs Archaeology. Halo and galaxy assembly in the Cold Dark Matter Universe. Galaxy formation efficiency as a function of galaxy mass. SFH with resolved stellar populations. Stellar Feedback. Dwarf galaxies.
- **Dynamics.** Dynamics of elliptical galaxies. Distribution Functions. Collisionless Boltzman Equation (CBE). Integrals of CBE. Jeans Equations. Virial Theorem. Poisson Equation. Slow and Fast rotators. Dynamical Friction. Fundamental and Mass planes. Relaxation and phase mixing. Violent relaxation. Dynamics of spiral galaxies. Rotation curves. Dark matter. Navarro-Freak-White density profiles. Galactic halos. Mass measurements with gravitational lensing. Dark matter structure and substructure in the Local Group. Dwarf galaxies as dark matter laboratories.
- **Structure and evolution.** Structure of galaxies. Two types of galaxies: dead and alive. Light distribution. Density de-projection. Sersic profile and its modifications. Detailed structure of elliptical galaxies. Cusp scouring. 3D shapes of ellipticals. Galaxy luminosity function. Schechter function. Connection between galaxy type/structure and environment. Mergers. Tides. Stellar streams. Accretion signatures in the Local Group. Formation and evolution of elliptical galaxies.

- **Active Galactic Nuclei and Quasars.** Evidence for (S)MBHs in the centres of galaxies. Energy production. BH growth. Co-evolution of central BHs and galaxies. M-sigma relation. AGN feedback.

Prerequisites

Knowledge of Galactic Dynamics (e.g. Part II Astrophysics Course "Stellar dynamics and Structure of Galaxies") would come in handy.

Literature

1. Binney, J., and Tremaine, S. *Galactic Dynamics* Second Edition. Princeton University Press, 2008
2. Mo, H., van den Bosch, F., and White, S. *Galaxy Formation and Evolution* Cambridge University Press, 2010
3. Sparke, L., and Gallagher, J. *Galaxies in the Universe* Second Edition. Cambridge University Press, 2007
4. Longair, M. *Galaxy Formation* Second Edition. Springer, 2008

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Astrophysical black holes (L16)

Debora Sijacki

Black holes are one of the most fascinating objects lying at the interface of mathematics, physics and astronomy. From the astrophysical stand point they give rise to extremely rich and complex phenomena occurring from sub-parsec to Mega-parsec scales and covering the full electromagnetic spectrum. A large body of state-of-the-art current and upcoming observational facilities and theoretical models is aimed at investigating black hole properties and their link with the larger scale environment, making this an exciting and fast paced research field. With the recent gravitational wave detections of merging black hole binaries the field has experienced further stimulus, as black holes have become unique multi-messengers to explore cosmology, gravity in the strong regime, high energy phenomena and complex (magneto)hydrodynamic flows.

This course will cover a range of concepts pertinent to astrophysical (supermassive) black holes highlighting both observational and theoretical advances in the field. The aim of the course is to give an overview of the possible formation and growth channels of these objects and to discuss various mechanisms through which black holes interact with their surroundings, with the synopsis as follows:

- Basic concepts; observational evidence for dormant and non-dormant objects (SgrA*)
- AGN properties and classification
- Formation pathways for supermassive black holes
- Black hole growth overview
- Fuelling mechanisms from kpc to sub-pc scales
- Bondi-Hoyle solution and limitations
- Accretion disk models: thin, slim and thick discs
- Outflows: basic concepts; collimated wind and jet phenomena
- Energy-, momentum- and radiation pressure-driven outflow solutions
- Impact of outflows on host properties

Prerequisites

Good knowledge of material covered in Part II courses: Astrophysical fluid dynamics, stellar dynamics and structure of galaxies, structure and evolution of stars is required. Some knowledge of (thermo)dynamics, electromagnetism and galaxy formation is advantageous but not strictly necessary.

Literature

1. Frank, J., King, A. R., & Raine, D., *Accretion Power in Astrophysics*, Cambridge University Press, 2002.
2. Netzer, H., *The Physics and Evolution of Active Galactic Nuclei*, Cambridge University Press, 2013.
3. Misner C. W., Thorne K. S., & Wheeler J. A., *Gravitation*, W. H. Freeman and company, 1973.
4. Pringle, J. E., & King, A. R., *Astrophysical Flows*, Cambridge University Press, 2007.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Dynamics of Astrophysical Discs (L16)

Gordon Ogilvie

Discs are ubiquitous in astrophysics and participate in some of its most important processes. Most, but not all, feed a central mass: by facilitating the transfer of angular momentum, they permit the accretion of material that would otherwise remain in orbit. As a consequence, discs are essential to star, planet and satellite formation. They also regulate the growth of supermassive black holes at the centres of galaxies. Although astrophysical discs can vary by some ten orders of magnitude in size and vary hugely in composition, all share the same basic dynamics and many physical phenomena.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular-momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained activity and angular-momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of exoplanets.

Provisional synopsis:

- Occurrence of discs in various astronomical systems; basic physical and observational properties.
- Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.
- Global and local views of an astrophysical disc.
- Evolution of an accretion disc.
- Vertical disc structure, scaling relations and timescales, thin-disc approximations, thermal and viscous stability and outbursts.
- Shearing sheet, symmetries, shearing waves.

- Hydrodynamic waves and instabilities; gravitational instability and turbulence.
- Vortices, zonal flows and dust dynamics.
- Satellite–disc interaction: tidal potential, resonant torques, migration and gap-opening.
- Magnetic fields and magnetorotational instability.

Prerequisites

Newtonian mechanics and basic fluid dynamics. Some previous knowledge of basic magnetohydrodynamics is helpful for the magnetorotational instability, but self-contained notes on this topic will be available.

Literature

Much information on the astrophysical background is contained in [1]. Some of the basic theory of accretion discs is described in review articles [2,3].

1. J. Frank, A. King and D. Raine, *Accretion Power in Astrophysics*. 3rd edition. Cambridge, 2002.
2. J. E. Pringle, Annu. Rev. Astron. Astrophys. **19**, 137 (1981).
3. J. C. B. Papaloizou and D. N. C. Lin, Annu. Rev. Astron. Astrophys. **33**, 505 (1995).

Additional support

Three example sheets will be provided and three associated example classes will be given. There will be a revision class in the Easter Term.

6 Combinatorics

Combinatorics (M16)

Béla Bollobás

I hope to cover the topics below, in the order indicated. If time remains, there are plenty of further exciting topics to present.

Shadows and the Kruskal–Katona Theorem. Intersecting and ℓ-intersecting uniform set systems; the Erdős–Ko–Rado Theorems. Modular intersections obtained by simple algebraic methods.

Alon's Combinatorial Nullstellensatz and its applications, including the Cauchy–Davenport Theorem and the Erdős–Heilbronn Conjecture.

The Kakeya Problem over finite fields; Dvir's Theorem. Arithmetic progressions in vector spaces over finite fields; cap sets in $\mathbf{F}_3^n$; the Ellenberg–Gijswijt Theorem.

The Erdős–Szekeres Cup-Cap Theorem.
The Pór–Valtr Theorem on convex clusters.
Suk's Theorem on Convex Polygons in sets of points in general position, greatly strengthening the Erdős–Szekeres Theorem.

Pre-requisites

Mathematical maturity and love of combinatorial arguments would be welcome. Familiarity with the Part II Graph Theory course is an asset, but not necessary.

Literature

For the 'classical' results in extremal combinatorics, we shall use the old book below, soon to be updated and enlarged with Imre Leader. For the more recent results in the course, most of the original papers are listed below.

1. Alon, N., Combinatorial Nullstellensatz, in *Recent Trends in Combinatorics* (Mátraháza, 1995), *Combin. Probab. Comput.* **8** (1999), 7–29.

2. Bárány, I., and P. Valtr, A positive fraction Erdős–Szekeres theorem, Dedicated to the memory of Paul Erdős, *Discrete Comput. Geom.* **19** (1998) 335–342.

3. Bateman, M., and N.H. Katz, New bounds on cap sets, *J. Amer. Math. Soc.* **25** (2012), 585–613.

4. Bollobás, B., *Combinatorics – Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability*, Cambridge University Press, Cambridge, 1986. xii+177 pp.

5. Croot, E., V.F. Lev, and P.P. Pach, Progression-free sets in $\mathbf{Z}_n^4$ are exponentially small, *Ann. of Math.* **185** (2017) 331–337.

6. Dvir, Z., On the size of Kakeya sets in finite fields, *J. Amer. Math. Soc.* **22** (2009) 1093–1097.

7. Dvir, Z., and B.L. Edelman, Matrix rigidity and the Croot-Lev-Pach lemma, *Theory Comput.* **15** (2019), Paper No. 8, 7 pp.

8. Ellenberg, J.S., and D. Gijswijt, On large subsets of $\mathbf{F}_q^n$ with no three-term arithmetic progression, *Ann. of Math.* (2) **185** (2017) 339–343.

9. Erdős, P., and G. Szekeres, A combinatorial problem in geometry, *Compositio Math.* **2** (1935) 463–470.

10. Pór, A., and P. Valtr, The partitioned version of the Erdős–Szekeres theorem, *Discrete Comput. Geom.* **28** (2002) 625–637.

11. Suk, A., On the Erdős–Szekeres convex polygon problem, *J. Amer. Math. Soc.* **30** (2017) 1047–1053.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extremal and Probabilistic Combinatorics (M16)

Julian Sahasrabudhe

This is a first course in modern extremal and probabilistic combinatorics with the main objective of introducing several of the most fundamental techniques in the area. Using these techniques we will study three classical and interconnected objects:

1. The binomial random graph $G(n,p)$ – Where we study the question "what does a 'typical' graph look like?"

2. The extremal numbers $ex(H,n)$ – "How many edges can a graph on n vertices have before we are forced to have a copy of the graph H?"

3. The Ramsey numbers $r(H)$ – "How large can a complete graph be before we are forced to have a monochromatic copy of H in every two colouring of its edges?"

Of course, the techniques we cover are *much* more broadly applicable than this, but we shall mainly limit ourselves to the study of these objects. In this course I hope to cover almost all of

Szemerédi's regularity lemma
Basic statement and philosophy, Erdős-Stone, triangle removal lemma, Roth's theorem, Ramsey numbers of bounded degree graphs, the number of triangle-free graphs.

Dependent random choice
Extremal numbers of bipartite graphs, Ramsey numbers of hypercubes.

The local lemma
Basic lemma of Erdős and Lovász, lower bounds on the diagonal Ramsey numbers $R(k)$, lower bounds on the off-diagonal Ramsey numbers $R(3, k)$.

The binomal random graph
Thresholds for connectivity, Hamilton cycles, appearance of small subgraphs, thresholds for monotone properties; the Bollobás theorem on the chromatic number; The Rődl–Ruciński theorem and Ramsey properties of $G(n, p)$.

Other techniques
FKG inequality; Janson inequalities and applications; Rödl nibble; the hardcore model and upper bounds on $R(3, k)$.

Prerequisites

Familiarity with the topics and flavor of Part II graph theory (or an equivalent course) is useful and recommended although not strictly required.

Literature

This course draws from various sources, but many of the topics in this course are now "standard" and so one can easily dig up other lecture notes online. Here are a couple sources that will cover the topics in this course (and *much much* more).

1. N. Alon and J. Spenser *The Probabilistic Method*, Wiley (any edition).
2. R. Morris and R.I. Oliveira *Extremal and Probabilistic Combinatorics*, available at https://impa.br/wp-content/uploads/2017/04/28CBM$_0$4.*pdf*.1993.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Ramsey Theory (L16)

Imre Leader

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. A typical example is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions.

The flavour of the course is combinatorial. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. We shall cover a number of 'classical' Ramsey theorems, such as Gallai's theorem and the Hales-Jewett theorem, as well as some more recent developments. There will also be several indications of open problems.

We hope to cover the following material.

Monochromatic Systems

Ramsey's theorem (finite and infinite). Canonical Ramsey theorems. Colourings of the natural numbers; focusing and van der Waerden's theorem. Combinatorial lines and the Hales-Jewett theorem. Applications, including Gallai's theorem.

Partition Regular Equations

Definitions and examples. The columns property; Rado's theorem. Applications. (m, p, c)-sets and Deuber's theorem. Ultrafilters; the Stone-Čech compactification. Idempotent ultrafilters and Hindman's theorem.

Euclidean Ramsey Theory

Basic definitions; Ramsey sets and spherical sets. Triangles are Ramsey. Regular polygons are Ramsey. Transitive sets; the block sets conjecture.

Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey's theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness.

Literature

R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, John Wiley, 1990.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Introduction to computational complexity (L16)

Timothy Gowers

Computational complexity is the study of the computational resources, such as time and storage space, needed to perform various tasks. It contains a host of interesting open problems, of which the most famous is the question of whether P = NP, but also a number of very surprising results, some of them unconditional and some of them conditional on hardness assumptions. The aim of the course will be to introduce you to the area and to give some idea of what is out there if you want to pursue it further.

The course will cover a selection of topics including most or all of the following. The first part of the course will focus on hardness of problems, and the second part on unexpectedly efficient algorithms.

- The definition of the complexity classes P and NP and the theory of NP-completeness.
- Counting problems and the #P-completeness of computing the permanent.
- Circuit complexity. Lower bounds for restricted circuit models.
- Why it is hard to prove lower bounds for circuit complexity.
- Randomized algorithms: polynomial identity testing, existence of perfect matchings, approximating the number of perfect matchings in a dense graph, testing for primality.
- Strassen's algorithm. Fast integer multiplication.

Prerequisites

The course will from time to time appeal to definitions and results from other branches of mathematics such as combinatorics, probability, and number theory, but these will not be very advanced and should be covered in a typical undergraduate course.

Literature

The first book listed below contains proofs, while the second is more of a gentle overview.

1. Boaz Barak and Sanjeev Arora *Computational complexity: a modern approach.* 1st edition. CUP, 2009. Also available in draft form at https://theory.cs.princeton.edu/complexity
2. Avi Wigderson *Mathematics and computation: a theory revolutionizing technology and science.* 1st edition. Princeton University Press, 2019. Also available in draft form at https://www.math.ias.edu/files/Book-online-Aug0619.pdf.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

7 Continuum Mechanics

Perturbation Methods (M16)

Stephen J. Cowley

This course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in some cases can provide accurate predictions when the parameter or coordinate has only moderately large or small values.

Some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. The advantage of uniformly valid expansions for comparison with experiment and numerical solutions will be covered. [7]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. Further examples will be given of asymptotics beyond all orders. This section will include a brief introduction to the summation of [divergent] series, e.g. covering Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, and Domb-Sykes plots. [6]

- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKB[JLG]' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied, potentially including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium). [3]

Prerequisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward differential equations and partial differential equations and evaluate simple integrals.

Literature

Relevant Textbooks

1. Bender, C.M. & Orszag, S., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *This is probably the most compre-*

hensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to 'Stokes' lines as 'anti-Stokes' lines, and vice versa. The course will use Stokes' convention.

2. Hinch, E.J., *Perturbation Methods*, Cambridge University Press (1991). *This is the book of the course; some view it as somewhat terse.*

3. Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975). *This is the original book on perturbation methods; somewhat dated, but still a useful read.*

Reading to Complement Course Material

1. Berry, M.V., *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).

2. Boyd, J.P., https://deepblue.lib.umich.edu/bitstream/handle/2027.42/41670/10440$_2$004$_A$rticle$_1$93995.pd 1$isAllowed = yThe\ Devil's\ invention:\ asymptotic,\ superasymptotic\ and\ hyperasymptotic\ series, ActaApp$ 98(1999).$Alsoavailableat$

http://hdl.handle.net/2027.42/41670 and
http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf.

Kevorkian, J. & Cole, J.D., *Perturbation Methods in Applied Mathematics*, Springer (1981).

Additional support

In addition to the lectures, three examples sheets will be provided and three associated examples classes will run in parallel to the course. There will be a revision class in the Easter Term.

Slow Viscous Flow (M24)

John R Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a

viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Prerequisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Preliminary Reading

1. D.J. Acheson, *Elementary Fluid Dynamics.* OUP, 1990. Chapter 7
2. G.K. Batchelor, *An Introduction to Fluid Dynamics.* CUP, 1970. pp.216–255.
3. L.G. Leal, *Laminar flow and convective transport processes.* Butterworth, 1992. Chapters 4 & 5.

Literature

1. J. Happel & H. Brenner, *Low Reynolds Number Hydrodynamics.* Kluwer, 1965.
2. S. Kim & J. Karrila, *Microhydrodynamics: Principles and Selected Applications.* Butterworth-Heinemann, 1993
3. C. Pozrikidis, *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP, 1992.
4. O.M. Phillips, *Flow and Reactions in Permeable Rocks.* CUP, 1991.

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Fluid Dynamics of Climate (M24)

Peter Haynes and John Taylor

Understanding the Earth's climate and predicting its future evolution is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the climate system, helping to make the planet hospitable for life. The dynamical complexity of this fluid motion and the wide range of space and time scales involved is one of the most difficult aspects of climate prediction.

This course, focusing on the large-scale behaviour of stratified and rotating flows, provides an introduction to the fluid dynamics necessary to build mathematical models of the climate system. The course begins by considering flows which evolve on a timescale which is long compared with a day, where the Earth's rotation plays an important role. The rotation is felt through the Coriolis force (a fictitious force arising from use of a frame of reference rotating with the Earth) which causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics, particularly in combination with stable density stratification. Canonical models are introduced and studied to illustrate phenomena such as adjustment to a state of geostrophic balance, where Coriolis force balances pressure gradient, new wave modes that can communicate dynamical information on both regional and global scales, and new hydrodynamic instabilities that lead to atmospheric weather systems and ocean eddies.

The course then moves on apply these basic ideas to important aspects of the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the structure and hence the effects of ocean eddies and atmospheric weather systems, the dynamics of the meridional (north/south) circulation in the ocean and atmosphere and the associated transport of heat and of chemical and biological tracers. The final part of the course will consider the special dynamics of tropical regions which give rise to phenomena such as El Nino, the Walker circulation and the Madden-Julian Oscillation. Simple models of the important two-way coupling between dynamics and moisture (which provides a dynamical forcing through condensational heating) will be introduced and analysed.

Prerequisites

Undergraduate fluid dynamics

Literature

The following provide some introductory reading and will also complement the detailed presentation of material given in lectures.

1. Vallis, G.K., *Atmospheric and Oceanic Fluid Dynamics*, 2nd edition. Cambridge University Press, 2017.
2. Gill, A.E., *Atmosphere-Ocean Dynamics*. Academic Press, 1982.
3. Marshall, J. and R.A. Plumb, *Atmosphere, Ocean, and Climate Dynamics*. Academic Press, 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Introduction to Computational Fluid Dynamics (M16)

Non-Examinable (Part III Level)

Emmanuel Dormy

Numerical simulation is playing an expanding role in the study of fluid dynamics and often complements experiments and theory. In this course, we will introduce and analyse the various methods available to solve the partial differential equations relevant to fluid dynamics. We will study their application to a wide variety of problems and highlight the effects of discretisation strategies. The objective of the course is to gain a practical knowledge, but also a general view of the existing methods and the ability to decide on the best suited choice for a given problem.

Some or all of the following topics will be covered:

- Finite differences / finite volumes / finite elements / spectral methods
- transport schemes / semi-lagrangian methods

- splitting algorithms
- numerical diffusion/dispersion/anisotropy
- compressible flows
- turbulent flows
- boundary conditions

Prerequisites

An elementary knowledge of fluid dynamics will be assumed.

Literature

We will not follow any particular book, as most of them are specalised on a particular method. A few references are

1. D. S. Burnett *Finite Element Analysis: From Concepts to Applications.* Addison Wesley, 1987.
2. C. Canutto, M. Y. Hussaini, A. Quarteroni, T.A. Zang, *Spectral Methods: Evolution to Complex Geometries And Applications to Fluid Dynamics.*, Springer 2007.
3. L. Kadanoff. *Excellence in Computer Simulation.* In : Comp. in Sci. & Eng. 6, 57–67, 2004.

Additional support

The course will cover both the theoretical aspects and practical implementations (here in Python).

Hydrodynamic Stability (L16)

Rich Kerswell

Developing an understanding of how "small" perturbations grow, saturate and modify fluid flows is central to addressing many challenges of interest in fluid mechanics. Furthermore, many applied mathematical tools of much broader relevance have been developed to solve hydrodynamic stability problems, and hydrodynamic stability theory remains an exceptionally active area of research, with several exciting new developments being reported over the last few years.

In this course, an overview of some of these recent developments will be presented. After a brief introduction to the general concepts of flow instability, presenting a range of examples, the major content of this course will be focussed on the broad class of flow instabilities where velocity "shear" and fluid inertia play key dynamical roles. Such flows, typically characterised by sufficiently "high" Reynolds number Ud/ν, where U and d are characteristic velocity and length scales of the flow, and ν is the kinematic viscosity of the fluid, are central to modelling flows in the environment and industry.

A hierarchy of mathematical approaches will be discussed to address a range of stability problems, from more classical concepts of normal mode growth on laminar parallel shear flows and their subsequent weakly nonlinear behaviour, to transient but significant growth of infinitesimal perturbations and finite amplitude instabilities.

Prerequisites

Undergraduate fluid mechanics, linear algebra, complex analysis and asymptotic methods.

Literature

1. F. Charru, *Hydrodynamic Instabilities,* CUP, 2011.
2. P. G. Drazin & W. H. Reid, *Hydrodynamic Stability* 2nd edition. CUP, 2004.
3. P. J. Schmid & D. S. Henningson, *Stability and transition in shear flows,* Springer, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be an hour and a half revision class in the Easter Term.

Non-Newtonian Fluid Mechanics (L16)

Eric Lauga

Standard courses in fluid mechanics are concerned with the dynamics of Newtonian flows. In the Newtonian limit, viscous stresses depend linearly on the instantaneous deformation rate of the fluid. However, in many instances relevant to industry as well as natural and physical sciences, a wide variety of

fluids display non-Newtonian behaviour. In fact, we are all familiar with these fluids in our daily life. For example, in the kitchen, while water and olive oil are Newtonian, mayonnaise and ketchup are non-Newtonian fluids. Similarly, in the bathroom, toothpaste, shampoo and shaving cream are materials which can be made to flow like liquids but also share many properties with elastic solids. Most biological fluids are also non-Newtonian, in particular blood and mucus.

In this course, we give an introduction to the mathematical modelling of flowing Non-Newtonian fluids. After introducing the experimental phenomenology of non-Newtonian flows, we will present the mathematical frameworks to tackle:
(i) Generalised Newtonian fluids with instantaneous but nonlinear stress-deformation responses;
(ii) Linear viscoelastic fluids that have a memory of their past deformation;
(iii) Nonlinear viscoelastic fluids displaying normal stress differences and resistance to extension;
(iv) Yield-stress fluids that can only deform if applied stresses exceed critical values;
(v) Viscoelastic instabilities.

Throughout the course, mathematical modelling will be motivated and compared with experiments. At the end of the course, students will be equipped with the necessary skills to carry out independent research in complex fluids and rheology relevant to a wide range of scientific problems, for both fundamental research and industry.

Prerequisites

Undergraduate fluid dynamics, vector calculus and mathematical methods.

Literature

1. National Committee for Fluid Mechanics Film on "Rheological Behavior of Fluids" at: `http://web.mit.edu/hml/ncfmf.html`
2. F. A. Morrison (2001) Understanding Rheology, Oxford University Press.
3. R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager (1987) Dynamics of Polymeric Liquids, Vol. 1: Fluid Mechanics, 2nd ed, Wiley.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

Direct and Inverse Scattering of Waves (L16)

Orsola Rath Spivack

The study of wave scattering is concerned with how the propagation of waves is affected by objects, and has a variety of applications in many fields, from environmental science to seismology, medicine, telecommunications, materials science, military applications, and many others. If we know the nature of the objects and we want to find how an incident wave is scattered, we call this a 'direct scattering problem' and practical applications will include for example underwater sound propagation, light transmission through the atmosphere, or the effect of noise in built-up areas. If we measure and know the scattered field produced by an incident wave, but we do not know the nature of the objects that have scattered it, we call this an 'inverse scattering problem' and applications will include for example non-destructive testing of materials, remote sensing with radar or lidar, or medical imaging.

This course will provide the basic theory of wave propagation and scattering and an overview of the main mathematical methods and approximations, with particular emphasis on inhomogeneous and random media, and on the regularisation of inverse scattering problems. It is intended as a kind of bridge between the Applied and Computational Analysis courses and the Continuum Mechanics courses and should appeal to students interested in waves as well as in inverse problems.

The course will cover a selection of topics including:

- Boundary value problems and the integral form of the wave equation.
- The parabolic equation and Born and Rytov approximations for the scattering problem.
- Scattering by randomly rough surfaces and propagation in inhomogeneous media.
- Ill-posedness of the inverse scattering problem, and the Moore-Penrose generalised inverse.
- Regularisation methods and methods for solving some inverse scattering problems.
- Time reversal and focusing in inhomogeneous media.

Prerequisites

This course assumes basic knowledge of PDEs, and of Fourier transforms. Some familiarity with linear algebra and with basic concepts in functional analysis is helpful, though by no means necessary.

The Part III course Inverse Problems is a useful complement to part of this course, although not a prerequisite.

Preliminary Reading

These can provide some useful background:

1. C.W. Groetsch *Inverse Problems in the Mathematical Sciences.* Braunschweig 1993
2. L.D. Landau and E.M. Lifschitz *Fluid Dynamics.* Pergamon 1987 [Chapter 8]. Also available at

   users-phys.au.dk/srf/hydro/Landau+Lifschitz.pdf

Literature

1. D. Colton and R. Kress *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, 1992.
2. D.G. Crighton et al, *Modern Methods in Analytical Acoustics.* Springer, 1992.
3. H.W. Engl, M. Hanke and A. Neubauer, *Regularization of inverse problems.* Kluwer, 2000.
4. A. Ishimaru, *Wave Propagation and Scattering in Random Media.* Academic Press, 1978.
5. A. Kirsch, *An introduction to the mathematical theory of inverse problems.* Springer, 1996.
6. B. Uscinski, *The elements of wave propagation in random media.* McGraw-Hill, 1977.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Demonstrations in Fluid Mechanics (L8)

Non-Examinable (Part III Level)

Stuart B Dalziel, Jerome A Neufeld

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The main purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed. Although the course is entitled 'demonstrations', the course has a history of including some novel experiments that have not, to our knowledge, been performed anywhere previously. Some of these have subsequently led to PhD projects. The course thus offers insight into how experiments can provide insight into previously unexplored (or undiscovered) phenomena.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- flows in porous media and fluid flows relating to carbon sequestration;
- fluid flow and elastic deformation;

- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account, although participation is encouraged.

Prerequisites

Undergraduate Fluid Dynamics.

Literature

1. M. Van Dyke, *An Album of Fluid Motion.* Parabolic Press.
2. G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen, *Multimedia Fluid Mechanics,* Multilingual Version CD-ROM. CUP.
3. M. Samimy, K. Breuer, P. Steen, & L. G. Leal, *A Gallery of Fluid Motion.* CUP.

Fluid Dynamics of the Built Environment (E8)

Rajesh Bhagat, Daria Frank

More than half of the world's population today live in cities and this number is projected to increase to more than two thirds over the next decades. This rapid growth presents challenges in creating a sustainable and clean urban environment. On the forefront of the issues with the progressing urbanisation is the difficult task of improving the urban air quality, blackboth outdoors and indoors, which is a crucial factor for human health, well-being and productivity. blackFurthermore, humans spend about 90% of their times indoors, and in order to maintain comfortable indoor environment (heating, cooling etc.) buildings consume about 30% of the total energy; imperatively, reducing energy consumption is crucial to limit the climate crisis. Covid-19 has exposed our vulnerability to poor indoor environment. The current pandemic is not a onetime exception but pandemics are ever-present and the next pandemic is matter of 'when' and not 'if'.

This course aims at discussing air flows occurring in an urban environment and the associated challenges that exist across different length scales. By their nature, urban air flows are turbulent flows at a high Reynolds number. They are often buoyancy- or wind-driven and are accompanied by a transport of contaminants or mixing.

We shall start by introducing the basic concepts of the theory of turbulent flows. In particular, we will discuss here different methods for turbulence modelling as well as the law of the wall and how it applies to large-scale flows above the cities. We will then turn our attention to air flows on a smaller street level. Here, flows in street canyons as well as pollution dispersion in cities shall be considered. We will proceed by examining air flows around an individual building including pressure distribution on the building facade and vortex shedding. External conditions around a building directly influence the indoor ventilation patterns. Hence, we will finish the first part of the course by discussing the coupling between the indoor and outdoor environments.

blackIn the second part of the course, we will discuss the fluid dynamics of indoor environments. We will start with basic theories of plumes and jets. We will carry out scaling analyses to identify the dominant forces in action in indoor environments. Subsequently, we will examine the modelling approaches for building ventilation; in particular, we will look into buoyancy-driven displacement, and mixing ventilation. We will also investigate the relative effect of wind and buoyancy. We will then turn our attention to the question of indoor airborne disease transmission. We will discuss the fluid mechanical aspects of coughing and sneezing, and the origin of the two-meter rule – advised by the governments across the globe – in order to prevent close-range droplet transmission, and the famous Wells Riley Model of airborne disease transmission and their limitations. Finally, we will end the course by drawing a summary of the course and the known unknowns of the field.

Pre-requisites

Undergraduate fluid dynamics is desirable for some topics.

Literature

Reading to complement course material:

- D. Etheridge, M. Sandberg, "Building Ventilation: Theory and Measurements"
- D. C. Wilcox, "Turbulence modelling for CFD"
- T. Chenvidyakarn, "Buoyancy Effects on Natural Ventilation"

Further reading suggestions will be given during the lectures.

8 Soft Matter and Biological Physics

Theoretical Physics of Soft Condensed Matter (M16)

Mike Cates

Soft Condensed Matter refers to liquid crystals, emulsions, colloidal suspensions and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, generally with noise so that the equations of motion are stochastic PDEs. The form of these equations is helpfully constrained by a requirement that the Boltzmann distribution is regained in the steady state. Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a 'smectic liquid crystal' (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field (and several steps short of the renormalization group, which we leave to the Statistical Field Theory course).

Important models of soft matter include diffusive ϕ^4 field theory ('Model B'), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates 'Model H', a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth. We will also explore the continuum theory of nematic and polar liquid crystals, which spontaneously break rotational but not translational symmetry, focusing on topological defects and their associated mathematical structure such as homotopy classes.

Finally, the course will briefly analyse soft-matter systems whose microscopic dynamics involves the continuous dissipation of energy, such as self-

propelled colloidal swimmers. We will discuss how their absence of time-reversal symmetry leads to qualitative changes in dynamical behaviour.

Prerequisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. It is not necessary to attend any other particular part III courses but there is potential synergy with Statistical Field Theory, Slow Viscous Flow, and to a lesser extent some other courses too. In previous years the audience has included a mix of students whose main specialism is either fluid dynamics or field theory. People with these differing backgrounds may find different parts of the course easier or harder, but the intention is to create a roughly level playing field so that all can enjoy learning about this interdisciplinary field.

Literature

(a) D. Tong *Lectures on Statistical Physics*
http://www.damtp.cam.ac.uk/user/tong/statphys.html

Before embarking on this course you need to understand the equation $F = -k_B T \ln Z$ and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Acquaintance with the Landau theory of phase transitions is helpful. David Tong's lecture notes are an excellent resource for revising and reviewing the key material.

(b) M. E. Cates and E. Tjhung *Theories of binary fluid mixtures: from phase-separation kinetics to active emulsions.* J. Fluid Mech. (2018), **836**, pp 1-66.
https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/theories-of-binary-fluid-mixtures-from-phaseseparation-kinetics-to-active-emulsions/5BD133CB20D89F47E724D77C296FEF80/share/106fd30f307db12134745de39fd568fbbaa

This JFM perspectives article has significant overlap (perhaps 40%) with the course but takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics as needed. It gives a flavour of the types of problem we will address and some of the methodologies involved. However we will not have time to cover much of the material it contains on active systems.

(c) P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics.* Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, and topological defects all

feature in the description of condensed matter systems at $\hbar = 0$. This is more for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

(d) Unofficial lecture notes were taken several years ago:
https://dec41.user.srcf.net/h/III$_L$/$theoretical_physics_of_soft_condensed_matter$

These notes are far from perfect but are mentioned here so that all students are aware of their existence.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Note on past papers

Starting in the 2016/7 academic year, this course was given in 16-lecture format for 3 years, then as a 24-lecture course for 2 years, before reverting to 16 lectures in the 2021/2 academic year. This means the summer 2021 tripos paper examined some material that lies beyond the current syllabus. (There was no honours paper in 2020.)

Biological Physics and Fluid Dynamics (M24)

R.E. Goldstein

This course will provide an overview of the physics and fluid dynamics of living systems, with an emphasis on their mathematical description. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology.

The topics to be covered span the range of length scales from molecular to ecological, with emphasis on key paradigms. They include: *Microscopic Physics* – van der Waals forces, screened electrostatics, interactions of extended bodies; *Fluctuations and Brownian Motion* – Equipartition, fluctuating continuous objects, diffusion, polymers and entropic forces; *Biological Filaments* – differential geometry, elasticity, constraints, variational principles, elastohydrodynamics, instabilities; *Membranes* – differential geometry, Helfrich model, fluctuations, shapes and shape transformations;

Cellular Motion – cell swimming, flagella and low-Re locomotion, hydrodynamic interactions; *Biological Pattern Formation* – chemical kinetics, reaction-diffusion systems, Turing instability, chemotaxis.

Prerequisites

Statistical physics, fluid mechanics, electromagnetism, and dynamical systems constitute the appropriate background for this course.

Literature

(a) P. Nelson. *Biological Physics.* W.H. Freeman (2007).

(b) J.D. Murray. *Mathematical Biology I. & II.* Springer (2007, 2008).

(c) J.N. Israelachvili. *Intermolecular and Surface Forces.* 2nd edition. Academic Press (1992).

(d) E.J.W. Verwey and J.Th.G. Overbeek. *Theory of the Stability of Lyophobic Colloids.* Elsevier (1948).

(e) M. Doi and S.F. Edwards. *The Theory of Polymer Dynamics.* OUP (1986).

(f) E. Lauga. *The Fluid Dynamics of Cell Motility.* CUP (2020).

(g) D. Bray. *Cell Movements.* Garland (2000).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Stochastic Processes in Biology (L16)

Maria Bruna

Stochastic processes arise in many applications in chemistry, biology and ecology, including chemical reactions, cell proliferation and motility and the spatio-temporal evolution of large groups of animals. This course will cover stochastic simulation algorithms and mathematical methods for the analysis of continuous-time stochastic models, including:

- Chemical reaction networks.
- Slow-fast systems and reduction to effective models.
- Random walks, Brownian motion, and reaction-diffusion processes.

- Metastability and exit problems.
- Interacting diffusion processes and collective behaviour.

For each of the topics, we will introduce the relevant stochastic simulation algorithms and how are they implemented in practice (there will be live coding examples and computational problems in the example sheets). We will also present the corresponding probabilistic descriptions in the form of systems of ODEs (chemical master equations) and PDEs (Fokker–Planck equation, backward Kolmogorov equation). We will highlight the connections and differences between modelling approaches (deterministic vs stochastic, discrete vs continuous).

Prerequisites

This course assumes basic knowledge of probability, ODEs and PDEs. While the computational component of the course is not assessed, some basic proficiency in programming will be useful.

Literature

(a) L. J. S. Allen, *An Introduction to Stochastic Processes with Applications to Biology.* CRC Press, 2010.

(b) R. Erban & S. J. Chapman, *Stochastic Modelling of Reaction-Diffusion Processes.* Cambridge University Press, 2019.

(c) C. W. Gardiner, *Handbook of Stochastic Methods.* Springer, 1985.

(d) D. J. Higham & P. E. Kloeden, *An Introduction to the Numerical Simulation of Stochastic Differential Equations.* SIAM, 2021.

(e) G. A. Pavliotis, *Stochastic Processes and Applications.* Springer, 2015.

Additional support

Three examples sheets will be provided, and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will also be one drop-off coding session in the first week.

9 Differential Geometry and Topology

Differential Geometry (M24)

Jack Smith

Differential geometry is the study of manifolds—smooth spaces locally modelled on Euclidean space—and structures that live on or in them. This course will introduce the main ideas from a rigorous pure mathematical perspective whilst developing in parallel the concrete 'in coordinates' language needed for performing calculations. This will lay the foundation for future study in geometry and topology, and provide the language for modern theoretical physics. Throughout the emphasis will be on building up geometric intuition. Topics will include:

- Manifolds, smooth maps, tangent spaces, submanifolds, foliations, and transversality.
- Vector bundles and operations on them. The (co)tangent bundle and tensors.
- Differential forms, the exterior derivative, and de Rham cohomology. Integration and Stokes's theorem.
- Connections on vector bundles. Parallel transport and curvature.
- Flows and Lie derivatives. Frobenius integrability.
- Lie groups and Lie algebras. The exponential map. Lie group actions.
- Principal bundles, reduction of structure group, and associated vector bundles. Connections and curvature revisited.
- Riemannian metrics, the Levi-Civita connection, and the Riemann tensor. The Laplace–Beltrami operator and statement of the Hodge decomposition.

Prerequisites

Familiarity with point set topology and multi-variable calculus at the level of Part IB Analysis and Topology will be assumed, as will Part IB Linear Algebra. No previous formal exposure to geometry is necessary.

Literature

(a) Liviu I. Nicolaescu, *Lectures on the geometry of manifolds.* 2nd edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

(b) John M. Lee, *Introduction to smooth manifolds.* 2nd edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013.

(c) Stephen Bruce Sontz, *Principal bundles. The classical case.* Universitext. Springer, Cham, 2015.

(d) (For a physics perspective) Mikio Nakahara, *Geometry, topology and physics.* 2nd edition. Graduate Student Series in Physics. Institute of Physics, Bristol, 2003.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Algebraic Topology (M24)

Jacob Rasmussen

Algebraic topology assigns algebraic invariants (groups and homomorphisms) to topological spaces and continuous maps between them. The most important example of such an invariant is ordinary homology theory, which is part of the basic language of geometry today. This course will cover homology and cohomology, together with applications to the topology of manifolds and vector bundles. The emphasis will be on learning to compute and use these invariants in a variety of examples. A tentative syllabus is as follows:

- *Homology.* Singular homology and cohomology. Eilenberg-Steenrod axioms and cellular homology. The Hurewicz homomorphism.
- *Cohomology and products.* Cohomology and the universal coefficient theorem. The Künneth theorem and cup products.
- *Vector Bundles.* Vector bundles and principal bundles. The long exact sequence of a fibration. The Euler class and the Thom isomorphism.
- *Topology of Manifolds.* Handlebodies. The fundamental class of an oriented manifold. Poincare duality. The Leftshetz fixed-point theorem.

Prerequisites

The only required background is basic point-set topology, but those who have not had a previous course in algebraic topology should expect to work

hard to keep on top of the material in the first eight lectures, which will be quite rapid. The material in the Michaelmas term Differential Geometry course will be useful as well.

Literature

(a) A. Hatcher, *Algebraic Topology*, CUP (2002).

(b) D. Fuchs and A. Fomenko, *Homotopical Topology*, (2nd ed.) Springer, (2016).

(c) J.P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press (1999).

(d) R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer (1982).

(e) J.W. Vick, *Homology Theory*, Springer (1994).

Hatcher is the standard text, and most closely matches the course syllabus. Vick is a more terse alternative, May more advanced. Bott and Tu and Fuchs and Fomenko offer different perspectives on some of the topics in the course.

Additional support

Four examples sheets will be provided, each with an associated examples class. There will be a two-hour revision class in the Easter Term.

Stationary measures for random walks on homogeneous spaces (L16)

Non-Examinable (Part III Level)

Timothée Bénard

This course mixes dynamics, Lie groups, and probability theory. The aim is to explain the classification of invariant measures for a random walk on a compact homogeneous space, following the very influential work of Benoist-Quint [2]. The proof will be carried out in the case where the homogeneous space is a quotient $X = G/\Lambda$ with $G = PSL_2()$ and $\Lambda \subseteq G$ a cocompact discrete subgroup. Given a probability measure μ on G, we consider the random walk on X whose transitional probability measure at a point $x \in X$ is given by $p(x, gx) = \mu(g)$ for $g \in G$. The asymptotic behavior of the μ-random walk on X is described by its invariant measures, also called stationary measures. Our goal is to classify those measures:

[Benoist-Quint] Assume the support of μ is compact and generates a Zariski dense subgroup of G. Then every μ-stationary non-atomic probability measure on X is G-invariant.

The proof relies on an exponential drift argument reminiscent of Ratner's approach to show the rigidity of unipotent flows acting on a general homogeneous space. Benoist-Quint's method has also inspired new ways to study self-similar fractals [4], as well as the proof of a celebrated result by Eskin-Mirzakhani [3] concerning the regularity of $SL_2()$-orbits on moduli spaces.

Prerequisites

Some knowledge about Lie groups, Lie algebras, ergodic theory, and probability theory (notably martingale convergence theorems).

Literature

(a) Y. Benoist and J.-F. Quint, *Introduction to random walks on homogeneous spaces*, Japan. J. Math. **7** (2012), 135-166.

(b) Y. Benoist and J.-F. Quint, *Stationary measures and invariant subsets of homogeneous spaces (II)*, Journal of the AMS **26** (2013), $n°3$, 659-734.

(c) A. Eskin and M. Mirzakhani, *Invariant and stationary measures for the $SL_2()$-action on Moduli space*, Publication mathématique de l'IHES **127** (2018), 95-324.

(d) D. Simmons and B. Weiss., *Random walks on homogeneous spaces and Diophantine approximation on fractals*, Invent. math. **216**, (2019), 337-394.

Additional support

Lecture notes will be provided.

Geometric group theory (L16)

Henry Wilton

The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts.

This reading course will provide an introduction to the basic ideas of the subject.

The course will be based on Löh's book *Geometric group theory*, and will cover the following chapters:

- Generating groups: free groups, finitely presented groups, amalgamated free products.
- Cayley graphs.
- Group actions
- Quasi-isometry
- Hyperbolic groups

The other books listed below could be used to provide alternative perspectives on some of this material.

Prerequisites

Part IB Geometry and Part II Algebraic Topology (or equivalents) are required.

Literature

(a) M. R. Bridson and A. Haefliger *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999. xxii+643 pp.

(b) C. Druţu and M. Kapovich, *Geometric group theory.* With an appendix by Bogdan Nica. American Mathematical Society Colloquium Publications, 63. American Mathematical Society, Providence, RI, 2018. xx+819 pp.

(c) C. Löh, *Geometric group theory – an introduction.* Universitext. Springer, Cham, 2017. xi+389 pp.

(d) J. Meier, *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups.* London Mathematical Society Student Texts, 73. Cambridge University Press, Cambridge, 2008. xii+231 pp.

Additional support

Three examples sheets will be provided.

Characteristic classes and K-theory (L24)

Oscar Randal-Williams

This is an advanced course on the algebraic topology of vector bundles. In the first part of the course we will describe the basic constructions on vector bundles, and prove the classification of vector bundles by Grassmannians. We will then describe cohomological invariants of vector bundles, the Chern, Stiefel–Whitney, and Pontrjagin characteristic classes, establish their basic properties and give applications.

In the second part of the course we will define complex K-theory, a topological invariant analogous to cohomology but defined in terms of vector bundles rather than cochains. We will prove the fundamental Bott Periodicity Theorem, establish the basic properties of K-theory such as long exact sequences, representability, and the Thom isomorphism, then move on to operations on K-theory, applications (such as the celebrated Hopf Invariant One Theorem), and relations between K-theory and cohomology.

Prerequisites

Part III Algebraic Topology is essential.

Literature

(a) M. F. Atiyah, *K-theory.* Lecture notes by D. W. Anderson, W. A. Benjamin, Inc., New York-Amsterdam 1967.

(b) M. Karoubi, *K-theory. An introduction.* Grundlehren der Mathematischen Wissenschaften, Band 226. Springer-Verlag, Berlin-New York, 1978.

(c) A. Hatcher, *Vector Bundles and K-Theory.* Available at https://www.math.cornell.edu/ hatcher/VBKT/VB.pdf

(d) D. Husemoller, *Fibre bundles.* Third edition. Graduate Texts in Mathematics, 20. Springer-Verlag, New York, 1994.

(e) J. W. Milnor and J. D. Stasheff, *Characteristic classes.* Annals of Mathematics Studies, No. 76. Princeton University Press, Princeton, N. J., 1974.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Complex Manifolds (L24)

Alexei Kovalev

This course introduces the basic theory of complex manifolds. The subject relates to both Differential Geometry (a complex manifold is a differentiable manifold endowed with certain additional structure) and Algebraic Geometry (a non-singular algebraic variety over $\mathbf{C}$ gives an instance of a complex manifold). I expect to cover much of the following:

- *Local analysis and complex structures.* Introduction to several complex variables. Definition and examples of complex manifolds. Holomorphic tangent bundle, (p, q)-forms. Almost complex structures and integrability. Dolbeault cohomology and $\bar{\partial}$-Poincaré lemma.
- *Holomorphic geometry.* Holomorphic line bundles, the Picard group, meromorphic sections. Subvarieties. Divisors and line bundles. Chern connection and the first Chern class. The adjunction formula. Blowing up.
- *Kähler geometry.* Hermitian metrics. Hodge theorem, Serre duality. Kähler manifolds (from several points of view). Kähler identities, Hodge and Lefschetz decompositions. Calculation of basic invariants for hypersurfaces in $\mathbf{CP}^n$. Ricci form and the Calabi–Yau manifolds.

Pre-requisites

Smooth manifolds, tangent bundles, differential forms. Basic knowledge of Riemannian metrics and curvature (including the Ricci curvature). I suppose that all of these will be covered in the course *Differential Geometry* offered in Michaelmas Term which is an ideal pre-requisite.
Basic theory of holomorphic functions in one complex variable will be assumed. Some familiarity with Riemann surfaces will be useful, but not essential.

Literature

[1] D. Huybrechts. *Complex geometry. An introduction.* Springer 2005.

[2] P. Griffiths and J. Harris. *Principles of algebraic geometry*, Wiley 1978.

[3] S. Kobyashi and K. Nomizu. *Foundations of differential geometry*, vol. 2. Wiley 1996.

[4] R.O. Wells. *Differential analysis on complex manifolds.* Springer 1980.

You might like to browse in the first chapter of [1] in advance, the book also has a generous collection of examples. A possible alternative is [2], Ch. 0 and 1 (further chapters may be of interest if you are taking algebraic geometry courses). On the other hand, [4] elaborates on the analysis side of things. The two volumes of [3] are regarded as a canonical reference on differential geometry topics; volume 2 has a chapter on the foundations of complex manifolds.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

10 Foundations

Category Theory (M24)

Peter Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

- **Categories, functors and natural transformations**. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]
- **Locally small categories**. The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]

- **Adjunctions**. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]
- **Limits**. Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]
- **Monads**. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

- **Filtered colimits**. Finitary functors, finitely-presentable objects. Applications to universal algebra.
- **Regular categories**. Embedding theorems. Categories of relations, introduction to allegories.
- **Abelian categories**. Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.
- **Monoidal categories**. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.
- **Fibrations**. Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Prerequisites

There are no specific prerequisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

(a) S. Mac Lane *Categories for the Working Mathematician.* Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.

(b) S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.

(c) T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.

(d) E. Riehl *Category Theory in Context.* Dover Publications 2016. A recent account of the subject by someone who first encountered it as a Part III student fifteen years ago.

(e) F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

A full set of written notes for the course will be provided; these contain exercises for the reader, and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Large Cardinals (L16)

Professor Benedikt Löwe

The following definitions and facts should be familiar from an introductory course on set theory: a cardinal κ is called *regular* if every unbounded subset of κ has cardinality κ; successor cardinals, i.e., cardinals of the form $\aleph_{\alpha+1}$, are always regular; the usual limit cardinals, e.g., $\aleph_\omega$, $\aleph_{\omega+\omega}$, or $\aleph_{\omega_1}$, are not.

Thus, the following is a natural question:

"Are there any uncountable regular limit cardinals?".

If they exist, they must be very large, in particular, much larger than any of the mentioned limit cardinals.

It turns out that this question is intricately connected with the incompleteness phenomenon in set theory: if there is an uncountable regular limit cardinal, then there is a model of ZFC; therefore, ZFC is consistent, and hence (by Gödel's Second Incompleteness Theorem) ZFC cannot prove the existence of these cardinals (unless, of course, it is inconsistent).

Regular limit cardinals (a.k.a. *weakly inaccessible cardinals*) are the smallest examples of set-theoretic notions called *large cardinals*: cardinals with properties that imply that they must be very big and whose existence cannot be proved in ZFC.

In this course, we shall get to know a number of these large cardinals, study their behaviour, observe consequences of their existence for set theory, and develop techniques to determine the logical strength of large cardinals

(the so-called *consistency strength hierarchy*). In modern set theory, large cardinals are used as the standard way to calibrate logical strength of extensions of ZFC.

Prerequisites

The Part II course *Logic & Set Theory* or an equivalent course is essential.

The Part III courses *Logic & Computability* (Michaelmas 2022) and *Model Theory* (Lent 2023) are not directly related to the material of this course, but provide useful supplementary insights into various metamathematical phenomena that will be useful to understand the material in this course.

Literature

(a) Thomas Jech, *Set Theory*, The Third Millenium Edition, revised and expanded, Springer 2003 [Springer Monographs in Mathematics]

(b) Akihiro Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings.* Springer 2003 [Springer Monographs in Mathematics]

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Logic and Computability (M24)

José V. Paiva Miranda de Siqueira

This course builds upon the foundations laid out by the Part II courses *Logic and Set Theory* and *Automata and Formal Languages* by exploring the relationship between Logic and Computability. We will reflect upon the meaning of *truth* in mathematics and how logical considerations mirror computability concerns and vice-versa. The intent is to cover the following topics:

- Natural deduction and intuitionistic logic;
- The simply typed λ-calculus and the Curry-Howard correspondence;
- Heyting and Kripke semantics for intuitionistic logic;
- Negative translation;

- The untyped λ-calculus, recursive functions, and λ-definability;
- Computability in Logic: encodings, incompleteness, Craig's Theorem, nonstandard models of arithmetic and Tennenbaum's Theorem;
- Oracles, Turing degrees, and the Friedberg-Muchnik Theorem;

The course will primarily be of interest to students concerned with logic, programming languages, computability theory, and topos theory. If time permits, we will also discuss duality in algebraic logic and/or realisability.

Prerequisites

Familiarity with the contents of the Part II courses *Logic and Set Theory* and *Automata and Formal Languages* (or equivalent) is essential, particularly acquaintance with propositional and predicate logic and associated results (completeness, compactness, Löwenheim-Skolem theorems), Zorn's Lemma, cardinals, and basic computability theory (Church's Thesis, recursive and recursively enumerable sets, the Halting Problem, the s-m-n theorem and the recursion theorem). Some of the relevant facts and definitions pertaining to computability, however, will be revised at a fast pace.

Literature

A primary source for intuitionistic logic and its relation to the λ-calculus is '*Proofs and Types*' [1]. It doesn't address the Heyting and Kripke semantics, however, and thus has to be complemented in other ways. A categorically minded student will enjoy Ghilardi's '*A First Introduction to the Algebra of Sentences*' [2], but everyone can appreciate Palmgren's notes '*Semantics of intuitionistic propositional logic*' [3] — they were both made available online. This is also the case for the very good and straightforward notes on the untyped λ-calculus by Barendregt and Barendsen '[4]'. Kaye's '*Models of Peano Arithmetic*' [5] is the go-to reference on anything related to the formal theory of arithmetic, while Soare's book [6] has the best exposition of Turing degrees and finite priority injury methods I am aware of. Discussions on duality will benefit from Priestley's own book [7], while Streicher's notes [8] include a treatment of realizability and a valuable companion to the course as a whole.

References

[1] J. Girard, Y. Lafont, P. Taylor. *Proofs and Types*, Cambridge University Press, 1989.

[2] S. Ghilardi. *A First Introduction to the Algebra of Sentences*, Lecture Notes, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, 2000, http://users.mat.unimi.it/users/ghilardi/allegati/dispcesena.pdf.

[3] E. Palmgren. *Semantics of intuitionistic propositional logic*, Lecture Notes, Department of Mathematics, Uppsala University, 2009, http://www2.math.uu.se/ palmgren/tillog/heyting3.pdf.

[4] H. Barendregt, E. Barendsen. *Introduction to Lambda Calculus*, Revised edition, 2000, https://www.cse.chalmers.se/research/group/logic/TypesSS05/Extra/geuvers.pdf.

[5] R. Kaye. *Models Of Peano Arithmetic*, Oxford Logic Guides **15**, Oxford University Press, 1991.

[6] R. I. Soare. *Turing Computability: Theory and Applications*, Theory and Applications of Computability Series, Springer, 2016.

[7] B. A. Davey, H. A. Priestley. *Introduction to Lattices and Order*, 2nd Edition, Cambridge University Press, 2002.

[8] T. Streicher. *Introduction to Constructive Logic and Mathematics*, https://www2.mathematik.tu-darmstadt.de/ streicher/CLM/clm.pdf.

[9] A. S. Troelstra, D. van Dalen. *Constructivism in Mathematics: an introduction*, Vol. 1, Elsevier, 1988.

[10] H. Rogers Jr. *Theory of Recursive Functions and Effective Computability*, Higher Mathematics Series, McGraw-Hill, 1967.

Additional support

Four Example Sheets (and accompanying Example Classes) are planned for this course. There will also be a one-hour revision class in Easter Term.

Model Theory (L16)

Silvia Barbina

Model theory is a branch of mathematical logic. Initially, the focus was on how far a set of sentences in a first-order language determines the class of structures it describes. Later, the subject evolved in several directions. For example, part of the attention shifted to contexts where the subsets of a structure that are *definable* are also mathematically meaningful – for instance, the definable sets in an algebraically closed field are the constructible sets. This is one of several ways in which model theory interacts with other areas of mathematics. Connections have emerged to

real and algebraic geometry, number theory and, more recently, combinatorics. This course introduces some basic model-theoretic tools and ideas up to initial notions in stability theory.

We aim to cover a selection of the following topics:

- preliminaries: structures, theories, elementarity (including elementary substructures, Tarski-Vaught test, downward Löwenheim-Skolem theorem)
- examples of relational structures (dense linear orders, the random graph)
- universal homogeneous models
- the compactness theorem for theories and for types
- saturation and the monster model
- omitting types theorem, ω-categorical theories and small theories
- preservation theorems and quantifier elimination
- strongly minimal structures
- imaginaries and the eq-expansion
- formulas without the order property (externally definable sets, stability and the number of types, stationary types and canonical bases).

Prerequisites

The Part II course *Set Theory and Logic*, or an equivalent course, is essential.

Literature

References and notes will be provided during the course. A relevant model theory textbook is

(a) Katrin Tent and Martin Ziegler, *A Course in Model Theory* (Lecture Notes in Logic 40). Cambridge University Press, 2012.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

11 Information and Finance

Information Theory (M16)

Ioannis Kontoyiannis

Information theory is the mathematical foundation of the ideas and tools required for the quantitative description and analysis of the notion of information. The origin of the theory in Claude Shannon's landmark 1948 paper is motivated by questions in communications engineering, but since then information theory has forged deep connections with many areas of mathematics, most notably with probability and statistics. This will be an introductory course to the main information-theoretic ideas, results, and techniques. We will discuss entropy as a measure of information, relative entropy as a natural distance between probability distributions, and mutual information as a universal dependence measure between random variables. Their properties will be established (monotonicity, chain rules, data processing inequalities, asymptotic equipartition) and different versions of Shannon's source coding theorems will be proved, including their extensions to universal data compression by Shtarkov and Rissanen. Connections with large deviations theory and with additive combinatorics will also be developed.

Prerequisites

The only pre-requisite is knowledge of basic probability, although a certain level of maturity and familiarity with the use of probabilistic techniques will be helpful. Knowledge of advanced (including measure-theoretic) probability is not necessary.

Literature

(a) Mainstream introduction to information theory:
T.M. Cover and J. Thomas. *Elements of Information Theory.* 2nd edition. Wiley-Interscience, 2006.

(b) A more applied perspective:
D. MacKay. *Information Theory, Inference, and Learning Algorithms.* Cambridge University Press, 2003. Available free online at: https://www.inference.org.uk/itprnn/book.pdf.

(c) Theoretical connections with ergodic theory and probability...
P. Billingsley. *Ergodic Theory and Information.* J. Wiley, 1965.

(d) ... and: P. Shields. *The Ergodic Theory of Discrete Sample Paths.* American Mathematical Society, 1996.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Advanced Financial Models (L24)

M.R. Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability and Stochastic Calculus & Applications.

The course will cover a selection of topics including:

- *Discrete-time models.* Arbitrage, martingale deflators, the fundamental theorem of asset pricing. Numéraires, equivalent martingale measures. Forwards, options, futures, bonds, interest rates. Attainable claims, market completeness. The Breeden–Litzenberger formula. Fourier pricing. American claims.
- *Continuous-time models.* Admissible strategies. Absolute and relative arbitrage. Existence of replicating strategies. Pricing and hedging via partial differential equations. The implied volatility surface. Dupire's formula. Stochastic volatility models. The HJM approach to term structure. Merton's problem.

Prerequisites

Familiarity with measure-theoretic probability will be assumed.

Literature

(a) M. Baxter & A. Rennie. *Financial calculus: an introduction to derivative pricing.* Cambridge University Press, 1996

(b) M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling.* Springer, 2006

(c) D. Kennedy. *Stochasic Financial Models.* Chapman & Hall, 2010

(d) D. Lamberton & B. Lapeyre.*Introduction to stochastic calculus applied to finance.* Chapman & Hall, 1996

(e) S. Shreve. *Stochastic Calculus for Finance: Vol. 1 and 2.* Springer-Finance, 2005

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Concentration inequalities (L16)

Varun Jog

The topic of this course is the study of concentration inequalities and their applications. Concentration inequalities quantify the random fluctuations of functions of independent random variables and find applications in many domains including statistics, learning theory, discrete mathematics, random matrix theory, information theory, and high-dimensional geometry.

The course will cover a selection of topics including:

- Markov's inequality and Chebyshev's inequality; inequalities for sub-Gaussian and sub-Gamma random variables; Hoeffding's inequality and related inequalities
- Efron-Stein inequality and Poincaré inequality; basic information-theoretic inequalities; log-Sobolev inequalities; the entropy method
- Marton's transport cost inequalities; the transport method; Talagrand's convex distance inequality

Applications of the above inequalities to dimensionality reduction, random matrices, Boolean analysis, combinatorics, and learning theory will be considered throughout the course.

Prerequisites

The only pre-requisite is knowledge of basic probability, although a certain level of maturity and familiarity with the use of probabilistic techniques will be helpful. Knowledge of advanced (including measure-theoretic) probability is not necessary.

Literature

(a) S. Boucheron, G. Lugosi and P. Massart, *Concentration Inequalities: A Non-Asymptotic Theory of Independence*, Oxford University Press, 2013.

(b) M. Raginsky and I. Sason, *Concentration of Measure Inequalities in Information Theory, Communications and Coding*, Foundations and Trends in Communications and Information Theory, 2013.

(c) M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a 90 minute revision class in the Easter Term.

12 Number Theory

Local Fields (M24)

Rong Zhou

The p-adic numbers Q_p were introduced by Hensel at the end of the 19th century and are now a ubiquitous tool in modern number theory as well as many other fields including algebraic topology, representation theory and algebraic geometry. The idea is to consider the completion of Q with respect to the absolute value defined by $|x|_p = p^{-n}$ for non-zero $x \in Q$ where $x = p^n \frac{a}{b}$ with $a, b, n \in Z$ and a, b coprime to p. The resulting field Q_p gives a neat way of packaging the information of congruences modulo p^n for all n and is the basic example of a local field. From this point of view, local fields are objects lying on the interface between algebra and analysis and the techniques used to study them involve an interesting mix of the two subjects.

This course will cover the basic theory of local fields, as well as some more advanced topics such as local class field theory, and is likely to be useful for students interested in studying other Part III courses pn number theory. Topics likely to be covered include:

Absolute values on fields; valuations; Hensel's Lemma; structure of local fields;

Extensions of complete fields; Dedekind domains; Decomposition groups;

Ramification theory; higher ramification groups; Weil group;

Statements of local class field theory; formal groups; Lubin–Tate theory.

Prerequisites

Basic algebra up to and including Part II Galois theory as well as knowledge of concepts in point set topology and metric spaces are essential prerequisites. It will be assumed that students have attended a first course in algebraic number fields.

Literature

(a) J.W.S. Cassels, *Local fields*, Cambridge University Press, 1986. .

(b) J.W.S. Cassels; A. Fröhlich, *Algebraic Number Theory*, Academic Press, 1967

(c) J. Neukirch, *Algebraic Number Theory*, Springer–Verlag, 1999.

(d) J. P. Serre, *Local fields*, Graduate Texts in Mathematics, 67. Springer–Verlag, 1979.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Modular Forms (L24)

Jack Thorne

Modular forms are holomorphic functions on the complex upper half plane which are invariant under an action of the group $SL_2(\mathbf{Z})$ (or a finite index subgroup). They feature in many different parts of mathematics, but are most famous in number theory for their role in the proof of Fermat's Last Theorem, itself based on a proof of the Shimura–Taniyama conjecture concerning the modularity of elliptic curves over $\mathbf{Q}$.

In this course we will introduce the theory of modular forms from a number-theoretic point of view, including the theory of modular curves, Hecke operators, Eisenstein series, and the connection with L-functions.

Prerequisites

Essential: IB Complex Analysis (or equivalent), IB Groups, Rings & Modules

Useful: Part II Riemann Surfaces (or equivalent)

Literature

(a) J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Maths. 7, Springer, New York, 1973

(b) F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Maths. 228, Springer, New York, 2005

(c) J. Milne, *Modular Functions and Modular Forms*, Lecture notes available at
https://www.jmilne.org/math/CourseNotes/MF.pdf

(d) M. Ram Murty, Applications of symmetric power L-functions. In *Lectures on automorphic L-functions*, pp. 203–283, Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Algebraic Number Theory (L24)

Hanneke Wiersema

This course constitutes a second course in algebraic number theory and as such will cover various topics in the subject. We will study objects central to current research in number theory, such as ζ-functions and L-functions, and discuss applications of these. Moreover, we will talk about global class field theory, which aims to describe abelian extensions of number fields. We will consider both the ideal and idele theoretic versions, and we will see how these are related. Further we will discuss the Chebotarev density theorem and some applications of this result.

There is plenty of literature in this area. We will mostly focus on the books [2], [3] and [6].

Topics likely to be included (not in order):

- Dedekind ζ-functions, analytic class number formula.
- L-functions.
- Ideles and adeles.
- Statements of global class field theory.
- Dirichlet density, Chebotarev density theorem.

Prerequisites

Galois Theory, Number Fields, Local Fields.

Literature

(a) J.W.S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Academic Press, 1967.

(b) D.A. Cox, *Primes of the Form $p = x^2 + ny^2$: Fermat, Class Field Theory, and Complex Multiplication*, John Wiley & Sons, 1989.

(c) G.J. Janusz, *Algebraic Number Fields*, Academic Press, 1973.

(d) S. Lang, *Algebraic Number Theory*, Springer GTM 110, 1970.

(e) D. Marcus, *Number Fields*, Springer-Verlag, 1977.

(f) J. Neukirch, *Algebraic Number Theory*, Springer–Verlag, 1999.

(g) H.P.F. Swinnerton-Dyer, *A Brief Guide to Algebraic Number Theory* (London Mathematical Society Student Texts), Cambridge University Press, 2001.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Analytic Number Theory (M16)

Non-Examinable (Part III Level)

Mansur Boase

Analytic number theory is the application of analytic tools to derive results in number theory. The subject has seen an explosion of new results in the past couple of decades. Green & Tao showed in 2004 that there exist arbitrarily long arithmetic progressions in the primes. Helfgott showed in 2013 that every odd number > 5 is a sum of three primes. Maynard received a Fields Medal for his 2016 simplification and improvement on Zhang's proof of bounded gaps between primes infinitely often. There has also been a revived interest in mean values of multiplicative functions, which are related to possibly the most famous unsolved problem in all of mathematics: the Riemann Hypothesis!

The purpose of the course is to cover the classical bases. Although the focus will be on prime numbers, it should have broad appeal for anyone

interested in analytic, combinatorial and probabilistic tools for counting things. The course will cover the following topics:

- Generating functions, Dirichlet series & Euler products, Euler-Maclaurin Summation.
- Dirichlet's theorem on primes in arithmetic progressions, Contour integration, Gauss sums.
- Arithmetical functions, Chebyshev estimates, Selberg's Formula.
- Riemann's memoir, The Riemann Hypothesis, Littlewood's Criterion.
- Hadamard Products, The Prime Number Theorem, Explicit Formulae.
- Sieve methods.
- Waring's problem, the circle method.
- Mean values of multiplicative functions.

The Modular Forms course lectured by Jack Thorne in Michaelmas will provide an alternative and complementary viewpoint to some elements of this course, including in particular D. J. Newman's analytic proof of the prime number theorem.

Prerequisites

A first course in complex analysis & number theory will be assumed and reminded of.

Literature

(a) Apostol, T., *Introduction to Analytic Number Theory,* Springer, 2013.

(b) Baker, A., *A comprehensive course in number theory,* CUP 2014.

(c) Cojocaru & Murty, *An introduction to sieve methods and their applications,* CUP, 2012.

(d) Koukoulopoulos, D., *The distribution of prime numbers,* AMS, 2019.

(e) Murty, M., *Problems in analytic number theory,* Springer, 2010.

(f) Newman, D. J., *Analytic number theory,* Springer, 2000.

(g) Tenenbaum, G., *Introduction to analytic and probabilistic number theory,* AMS, 2015.

Elliptic Curves (L24)

Tom Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points.

The following topics will be covered, and possibly others if time is available.

- **Weierstrass equations and the group law.** Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process.
- **Isogenies.** The degree of an isogeny is a quadratic form. The invariant differential and separability. The torsion subgroup over an algebraically closed field.
- **Elliptic curves over finite fields.** Hasse's theorem and zeta functions.
- **Elliptic curves over local fields.** Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.
- **Elliptic curves over number fields.** The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Prerequisites

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. The first few lectures will include a review of the necessary geometric background, but some previous knowledge of algebraic curves (at the level of the Part II course *Algebraic Geometry* or the first two chapters of [3]) would be very helpful. Later in the course, some basic knowledge of the Part III course *Local Fields* will be assumed.

Preliminary Reading

(a) J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Literature

(a) J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.

(b) J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

Additional support

Four example sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

13 Particle Physics and Quantum Fields

Quantum Field Theory (M24)

Nick Dorey

Quantum Field Theory is a marriage of quantum mechanics and special relativity which provides the mathematical framework for describing the interactions of elementary particles. This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalars, (Dirac) spinors, and gauge fields. The relativistic covariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem. The quantisation of free fields is developed using Hamiltonian methods in terms of operators which create and annihilate particles and anti-particles. The Fock space of quantum physical states and the associated realisation of particle statistics is described. Interacting field theory is developed next using Dyson's formula for the scattering amplitude. Perturbation theory is formulated by applying Wick's theorem and Feynman diagrams are introduced as an efficient way of organising the resulting calculations. Physically measurable quantities such as decay rates and cross-sections are defined. Spinors and the Dirac equation are explored in detail, along with parity and chirality. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions. We review the relativistic formulation of Maxwell's equations formulating a corresponding action principle. The significance of gauge invariance is discussed. Lorentz gauge quantisation of the electromagnetic field is accomplished by imposing the Gupta-Bleuler condition. Finally, quantum electrodynamics (QED) is developed. Interactions between photons and charged matter are introduced via the principal of minimal coupling. We discuss the calculation of tree-level scattering amplitudes in QED.

Prerequisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics including Dirac's "bra/ket" notation. A basic knowledge of group theory is also very useful for this course.

Literature

(a) D. Tong, *Lectures on Quantum Field Theory* http://www.damtp.cam.ac.uk/user/tong/qft.html. These printed lecture notes have a large overlap with the current course.

(b) T. Lancaster and S.J. Blundell, *Quantum field theory for the gifted amateur* Oxford University Press, 2015 is an introductory text that Part III students have found useful.

(c) M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory* AddisonWesley, 1996 is a classic, and also covers aspects of the Standard Model.

(d) A. Zee, *Quantum Field Theory in a Nutshell* Princeton University Press, 2010 gives a modern take with a lot of physical intuition, possibly taking the subject into topics more theory-specialised and advanced than the references above.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a weekly office hour in Michaelmas term. One revision class will be given in Easter term.

Symmetries, Particles and Fields (M24)

Matthew B Wingate

Lie groups and Lie algebras are important in the construction of quantum field theories which describe interactions between known particles. Particle states and quantum fields are described in terms of irreducible representations of the Poincaré group, a Lie group. Gauge theories, which describe many of the interactions in the Standard Model of particle physics, rely on Lie groups.

After some basic properties and preliminaries, we introduce Lie matrix groups which rely on continuous parameters. Differentially, these act as

a Lie algebra. The exponential map connects the Lie algebra to the Lie group. We then introduce representations in terms of square matrices, describing how to construct various new representations in terms of old ones.

The group of rotations in three-dimensional space $SO(3)$ is examined, along with $SU(2)$ and the connection to angular momentum states in quantum theory. Representations of each are covered.

The relativistic symmetries (Lorentz group and Poincaré group in four dimensions) are studied from the point of view of their group elements and Lie algebras.

Analysis of compact simple Lie algebras and their finite representations comes from mapping them to a geometrical picture involving roots and weights via the Cartan matrix. An overview of the results of the Cartan classification of simple Lie algebras is included.

An application in terms of representations of a global $SU(3)_F$ flavour symmetry explains some features of the spectrum of hadronic particles. Further properties of the spectrum lead one to introduce an additional local $SU(3)_c$ colour symmetry leading a particular gauge theory called quantum chromodynamics.

We shall close by covering gauge theory, first $U(1)$, then general Lie groups and different representations of matter.

Prerequisites

Linear algebra, including direct sums and tensor products of vector spaces. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices.

Literature

(a) B. C. Allanach, *Symmetries, Particles and Fields.* Kindle Direct Publishing, 2021 will form basis of the course.

(b) P. Ramond, *Group Theory: A Physicist's Survey.* Cambridge University Press, 2010.

(c) G. Costa and G. Fogli, *Symmetries and Group Theory in Particle Physics.* Springer, 2012 has nice geometrical aspects.

(d) For a more basic introduction to the use of groups in physics: Zee, *Group Theory in a Nutshell for Physicists.* Princeton University Press, 2016.

(e) H.F. Jones, Taylor and Francis, *Groups, Representations, and Physics.* CRC Press, 1998.

Additional support

Four examples sheets will be provided and four associated examples classes will be given (the final one in Lent term). There will be a one-hour revision class in the Easter Term.

Statistical Field Theory (M16)

Christopher Thomas

This course introduces the renormalization group, focusing on statistical systems such as spin models with further connections to quantum field theory.

After presenting the Ising Model, Landau's mean field theory is introduced and used to describe phase transitions. The extension to the Landau-Ginzburg theory reveals broader aspects of fluctuations whilst consolidating connections to quantum field theory. At second order phase transitions, also known as 'critical points', renormalisation group methods play a starring role. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems, including those with continuous symmetries.

Prerequisites

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Literature

(a) N Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Addison-Wesley (1992).

(b) M Kardar, *Statistical Physics of Fields*, Cambridge University Press (2007).

(c) J Cardy, *Scaling and Renormalisation in Statistical Physics*, Cambridge University Press (1996).

(d) J M Yeomans, *Statistical Mechanics of Phase Transitions*, Clarendon Press (1992).

(e) M Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press (1991).

(f) J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992).

(g) D Amit and V Martin-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005).

(h) L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press (1996).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in Easter Term.

Non-Equilibrium Statistical Field Theory (M8)

Non-Examinable (Part III Level)

Rosalba Garcia-Millan

This course introduces Master Equations – an ODE for time dependent probability distributions – as a model for reaction systems on lattices. After briefly exploring its basic properties and models such as random walkers as well as branching and coagulation processes, they are transformed into a second-quantized form using bosonic ladder operators and coherent states. The meaning and properties associated with shifts of the creation and annihilation operators are explored as well as the interpretation of observables. Then, the second quantized equation is formally solved in a path integral. At this point, Doi-Peliti field theory starts and a few example models are presented: diffusion, branching, coagulation as well as coupled reactions. Feynman diagrams, loop corrections, n-point correlation functions are all concepts which are introduced and Part III students taking *SFT* will hopefully enjoy a non-equilibrium point of view on them.

Prerequisites

It will be very beneficial to take the Part III course *Statistical Field Theory* (*SFT*) alongside this course. Although both courses talk about different statistical field theories, they explore many of the same concepts.

Literature

(a) N.G. van Kampen, *Stochastic Processes in Physics and Chemistry* Elsevier, 1992

(b) Uwe Täuber *Critical Dynamics* 1st edition. Cambridge University Press, 2014.

(c) John Cardy, *Lecture Notes on Field Theory and Nonequilibrium Statistical Mechanics*, Available at
https://www-thphys.physics.ox.ac.uk/people/JohnCardy/

(d) Gunnar Pruessner *Lecture Notes on Non-equilibrium Statistical Mechanics* Available at
http://wwwf.imperial.ac.uk/ pruess/publications/Gunnar$_Pruessner_field_theory_notes.pdf$

Additional support

Lecture notes and voluntary example sheets will be provided. However, since this is a non-examinable course, there will be no example classes.

Advanced Quantum Field Theory (L24)

Ron A Reid-Edwards

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess and is the basic theoretical framework for describing elementary particles and their interactions (excluding gravity). QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course introduces techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalization of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie group at every point in space-time and yet still describe the same physics. Quantizing a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry.

A further major component of the course is to study renormalization. Wilson's picture of renormalization is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the renormalization group (RG) flow. The course explores renormalization systematically using dimensional regularization in perturbative loop integrals. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a nonabelian gauge theory can effectively become small

at high energies. Known as *asymptotic freedom*, this phenomenon revolutionized our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrize possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

Prerequisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be helpful for the section on nonabelian gauge theory. The section on the renormalization group has some overlap with Statistical Field Theory but no knowledge of this course will be assumed.

Literature

(a) Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory.* Perseus Books, 1995.

(b) Srednicki, M., *Quantum Field Theory.* CUP, 2007. A pre-publication draft is available at http://web.physics.ucsb.edu/ mark/qft.html.

(c) Schwartz, M., *Quantum Field Theory and the Standard Model.* CUP, 2014.

(d) Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2. CUP, 1996.

Additional support

Four examples sheets will be provided and four associated examples classes will be given (the fourth will be scheduled for early Easter term). There will be a one-hour revision class later in Easter Term.

The Standard Model (L24)

Fernando Quevedo

The Standard Model of particle physics is, by far, the most successful application of quantum field theory (QFT). At the time of writing, it ac-

curately describes all experimental measurements involving strong, weak, and electromagnetic interactions. The course aims to demonstrate how this model, a QFT with gauge group $SU(3) \times SU(2) \times U(1)$ and fermion fields for the leptons and quarks, is realised in nature. It is intended to complement the QFT and Advanced QFT courses and should be essentially understood as a third course on QFT since it covers some key QFT subjects such as the Goldstone theorem, the Higgs mechanism and anomalies which are not covered in the two QFT courses.

We begin by describing the important role of symmetries in relativistic quantum physics and quantum field theory. We start with spacetime symmetries including representations of the Poincaré group and discrete symmetries (C,P,T). Then move to non-abelian gauge symmetries. Ideas of spontaneous symmetry breaking are applied to discuss Goldstone's theorem and the Higgs mechanism. We later apply these concepts to describe the weak interactions and their unification with electromagnetic interactions and also Quantum Chromodynamics (QCD) that describes strong interactions in terms of an $SU(3)$ gauge theory.

We put all this together to define the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content (spin-half leptons and quarks, spin-one gauge bosons and spin-zero Higgs boson).

Throughout the lectures, general consistency and fundamental properties such as the structure of gauge anomalies and asymptotic freedom will be highlighted. Also, some key phenomenological properties of the Standard Model will be introduced such as the Cabibbo-Kobayashi-Maskawa (CKM) mixing, the Glashow-Iliopoulos-Maiani (GIM) mechanism, neutrino oscillations, etc.

Both very high-energy experiments and very precise experiments are currently striving to observe effects that cannot be described by the Standard Model alone. We comment on how the Standard Model is treated as an effective field theory (EFT) to accommodate (so far hypothetical) effects beyond the Standard Model. General open questions and potential answers for the standard model will be discussed at the end of the course.

General Outline

(a) Introduction and History

(b) Spacetime Symmetries

(c) Internal Symmetries

(d) Broken Symmetries

(e) Weak Interactions: Electroweak unification

(f) Strong Interactions: QCD

(g) The Standard Model and Effective Field Theories.

Prerequisites

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or to be familiar with the material covered in them. It would be advantageous to attend the Advanced QFT course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

Literature

(a) M.D. Schwartz, *Quantum Field Theory and the Standard Model.* Cambridge University Press, 2014.

(b) S. Weinberg, *The Quantum Theory of Fields, Volume 1,2.* Cambridge University Pres, 1995.

(c) C.P. Burgess and G. Moore, *The Standard Model: A Primer.* Cambridge University Press, 2007.

(d) S. Raby, *Introduction to the Standard Model and Beyond.* Cambridge University Press, 2021.

(e) C.P. Burgess *Effective Field Theories.* Cambridge University Press, 2020.

(f) M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory.* Addison-Wesley, 1995.

(g) F. Halzen and A.D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics.* Wiley, 1984.

(h) I.J.R. Aitchison and A.J.G. Hey, *Gauge Theories in Particle Physics.* CRC Press (two volumes or earlier 1989 edition in one volume).

(i) J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model.* Cambridge University Press, 2014.

(j) H. Georgi, *Weak Interactions and Modern Particle Theory.* Benjamin/Cummings, 1984.

(k) T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics.* Oxford University Press, 1984.

(l) M. Thomson, *Modern Particle Physics.* Cambridge University Press, 2013.

Additional support

Four example sheets will be provided and four associated examples classes will be given by Andreas Schachner. There will also be a revision class in

Easter Term. Template of solutions will be distributed at the end of the course.

String Theory (L24)

David Skinner

String theory is the quantum theory of interacting one-dimensional extended objects (strings). What makes the theory so appealing is that it is a quantum theory that contains gravitational interactions and therefore provides the first tentative steps towards a full quantum theory of gravity. It has become clear that string theory is also much more than this. It has become a framework in which to understand problems in quantum field theory, to ask meaningful questions about what we expect from a quantum theory of gravity, and as a crucible for new ideas in mathematics.

This course provides an introduction to String Theory. We begin by generalising the worldline of a particle to the two-dimensional surface swept out by a string. The quantum theory of the embedding of these surfaces in space-time is governed by a two-dimensional quantum field theory and we shall study the simplest example – the bosonic string – in detail.

An introduction to relevant ideas in Conformal Field Theory (CFT) will be given. The quantisation of the string will be studied, its spectrum obtained, and the relationship between states on the two dimensional CFT and fields in space-time will be discussed. We will see the necessity of the critical dimension of space-time.

The path integral approach to the theory will be discussed. Fadeev-Popov and BRST methods will be introduced to deal with the redundancies that appear in the theory. Vertex operators will be introduced and scattering amplitudes will be computed at tree level. Perturbation theory at higher loops and the role played by moduli space of Riemann surfaces will be sketched.

The course will focus on closed strings, but time permitting, open strings and the role of D-branes may be discussed. There may also be some discussion of more stringy phenomena such as symmetry enhancement and duality.

Prerequisites

Knowledge of the Quantum Field Theory course in Michaelmas term is assumed. Advanced Quantum Field Theory will complement this course but will not be assumed.

Literature

(a) J. Polchinski, *String Theory, Vol. 1: An Introduction to the Bosonic String*, CUP 1998.

(b) M. Green, J. Schwarz and E. Witten, *Superstring Theory, Vol. 1: Introduction*, CUP 1987.

(c) R. Blumenhagen, D. Lüst and S. Theisen, *Basic Concepts of String Theory*, Springer 2013.

(d) D. Tong, *Lectures on String Theory*, https://arxiv.org/abs/0908.0333

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Supersymmetry and Duality (L24)

David Tong

Supersymmetry is a peculiar symmetry that relates fermions and bosons. This course will start with an introduction to supersymmetric quantum field theories, explaining how one can realise supersymmetry with a carefully curated collection of fields and interactions. We do this by introducing the concept of superspace, a generalisation of Minkowski space that includes extra dimensions that are anti-commuting.

The magic of supersymmetry is that it allows us to make exact statements about quantum field theories even when the interactions are strong and usual perturbative, Feynman diagram methods fail to work. This means that supersymmetric theories provide a collection of solvable toy models that give is insight into what kinds of things can happen in a quantum field theory. In the second part of this course, we will study the supersymmetric version of QCD. This will give us a handle on a number of interesting phenomena, such as confinement and chiral symmetry breaking, that are also known to arise in non-supersymmetric theories. We will end the course by describing a surprising phenomenon known as "Seiberg duality", in which two different quantum field theories give rise to the same low-energy physics.

Prerequisites

You should be comfortable with Quantum Field Theory from Michaelmas term and parts AQFT from Lent, including the basics of path integrals, non-Abelian gauge theories and beta functions.

In addition, there are connections to a number of other courses. We will start by studying representations of the supersymmetry algebra and some familiarity with Lie algebras will be helpful. The latter part of the course involves understanding how quantum field theories look at different length scales and an appreciation of Wilsonian RG, whether from Statistical Field Theory or AQFT, will be useful. Finally, the subjects of anomalies and symmetry breaking will also be covered in the Standard Model.

Literature

Full lecture notes for this course can be downloaded from the course webpage

http://www.damtp.cam.ac.uk/user/tong/susy.htmlhttp://www.damtp.cam.ac.uk/user/tong/susy.html.

You can also find a number of other lecture notes and review articles collected there. Two books that might be worth consulting are:

(a) Wess and Bagger, *Supersymmetry and Supergravity*, Princeton University Press

(b) John Terning, "*Modern Supersymmetry: Dynamics and Duality*, Oxford University Press

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Applications of Quantum Field Theory (E16)

Sean Hartnoll

This course will explore quantum field theories that emerge as the long distance, continuum, description of quantum many-body systems. These quantum field theories share important conceptual tools with those encountered in particle physics, such as renormalisation group flow, spontaneous symmetry breaking, diagrammatic perturbation theory and gauge invariance. However, due to the immense diversity of many-body systems, which can have 1, 2 or 3 spatial dimensions, are not Lorentz invariant, and admit a variety of different collective modes, the quantum field theories to be considered in this course exhibit a range of rich and novel

behaviours. Furthermore, while collider physics occurs in the vacuum, many-body physics takes place in states with a large number of particles and often at nonzero temperature. This course will describe the interplay between quantum and thermal fluctuations.

Some subset of the following topics will be covered, as time allows:

- Spontaneous symmetry breaking in quantum many-body physics. Effective field theories of superfluids and magnets. Nonlinear sigma models. Coherent state path integral and Berry phases.
- Dimerized and valence-bond solid phases. Spin liquids. Spinons and emergent gauge fields.
- Quantum Criticality. Scaling exponents. Quantum fluctuations. Vector large N expansion and epsilon expansion. Deconfined quantum criticality and monopoles. Spin chains, topological terms and large S expansion.
- Nonzero temperature. Periodicity in imaginary time. Matsubara frequencies. Spectral functions. Fluctuation-dissipation theorem. Symmetry restoration.
- Effective field theory of Fermi surfaces. Renormalisation group picture of Fermi liquids. Superconducting and other instabilities of Fermi surfaces. Non-Fermi liquids.
- Quantum Hall liquids. Fractional statistics. Chern-Simons theory. Ground state degeneracy and topological order.

Prerequisites

Understanding of quantum field theory at the level of the Advanced Quantum Field Theory course is required. In particular, familiarity with path integrals and Wilsonian renormalisation will be assumed. Exposure to material in the Statistical Field Theory course (in particular, classical Landau-Ginzburg theory) and The Standard Model course (in particular, spontaneous symmetry breaking in quantum field theory) may be useful but will not be assumed. No prior exposure to solid state physics will be assumed.

Literature

(a) S. Sachdev, *Quantum Phase Transitions*, 2nd Edition, CUP 2011.

(b) S. Sachdev, "Quantum Magnetism and Criticality", Nature Physics **4**, 173 (2008). Available at https://www.nature.com/articles/nphys894.

(c) X-G. Wen, *Quantum Field Theory of Many-Body Systems*, OUP 2004.

(d) A. Zee, *Quantum Field Theory in a Nutshell*, PUP, 2003.

(e) A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd Edition, CUP 2010.

(f) E. Fradkin, *Field Theories of Condensed Matter Physics*, 2nd Edition, CUP 2013.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Physics Beyond the Standard Model (E12)

Non-Examinable (Part III Level)

Maria Ubiali

This graduate course gives a brief overview on the successes and theoretical problems of the Standard Model (SM). It then discusses the guiding ideas and a number of useful techniques to explore physics Beyond the Standard Model (BSM).

The course will cover a selection of topics including:

- Recap on the Standard Model and open questions
- Basic concepts about Effective Field Theories
- The concept of naturalness in BSM physics
- Standard Model Effective Field Theory, focussing on dimension 5 and 6 operators
- Supersymmetry, with emphasis on particle physics phenomenology
- Chiral symmetry breaking in QCD
- Composite Higgs models
- Phenomenology landscape of BSM physics searches at the LHC and beyond

Prerequisites

Background knownledge of Standard Model and Quantum Field Theories is essential.

Literature

(a) A. A. Petrov, A. Blechman, *Effective field theories.* World Scientific https://www.worldscientific.com/doi/10.1142/8619

(b) B. Gripaios, *Lectures on Physics Beyond the Standard Model.* https://arxiv.org/abs/1503.02636 [hep-ph]

(c) S. Willenbrock and C. Zhang, *Effective Field Theory Beyond the Standard Model.* https://arxiv.org/abs/1401.0470 [hep-ph]

(d) A. Wulzer, *Behind the Standard Model*, https://arxiv.org/abs/1901.01017 [hep-ph]

(e) I. Brivio, M. Trott, *The Standard Model as an Effective Field Theory.* https://arxiv.org/abs/1706.08945 [hep-ph]

(f) S. P. Martin, *A Supersymmetry Primer.* https://arxiv.org/abs/hep-ph/9709356 [hep-ph]

(g) R. Contino, *The Higgs as a Composite Nambu-Goldstone Boson.* World Scientific https://www.worldscientific.com/doi/abs/10.1142/9789814327183$_0$005; $https://arxiv.org/abs/1005.4269[hep-ph]$

14 Probability

Advanced Probability (M24)

Perla Sousi

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

- **Review of measure and integration:** sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini's theorem.
- **Conditional expectation:** Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.
- **Martingales:** Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.
- **Stochastic processes in continuous time:** Kolmogorov's criterion, regularization of paths; martingales in continuous time.
- **Weak convergence:** Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.

- **Sums of independent random variables:** Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.
- **Brownian motion:** Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.
- **Poisson random measures:** Construction and properties; integrals.
- **Lévy processes:** Lévy-Khinchin theorem.

Prerequisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Literature

(a) Lecture notes available online

(b) D. Applebaum, *Lévy processes* 2nd ed. Cambridge University Press, 2009.

(c) R. Durrett, *Probability: Theory and Examples* 4th ed. CUP, 2010.

(d) O. Kallenberg, *Foundations of Modern Probability.* Springer-Verlag, 1997.

(e) D. Williams, *Probability with martingales.* CUP, 1991.

Additional support

Four example sheets will be provided along with associated supervisions or examples classes. There will be a revision class in Easter term.

Lattice Models (M24)

Roland Bauerschmidt

Lattice models are probabilistic models defined on $\mathbf{Z}^d$ (or other lattices). Examples include the Ising model, percolation, the Potts and clock models,

the XY and Heisenberg models, dimer models, the self-avoiding walk, spanning trees and forests, lattice gauge theory, and others. Most of these models are motivated from statistical physics and/or quantum field theory. Part of the appeal of this active research subject within probability and statistical physics is that the models are simple to define, yet they often have very rich behaviour. A variety of different mathematical methods exist for the analysis of many important aspects, while other questions remain out of reach of present mathematics (one of them is at the heart of one of the $1,000,000 problems).

This course will focus on probabilistic aspects and techniques and cover a sample of models and results about these, starting from the Ising model and its phase transition. The methods discussed will include correlation inequalities, random geometric representations, differential inequalities, expansions, among others.

Prerequisites

This course assumes familiarity with probability, measure theory, and analysis at Part II level. Parallel participation in Advanced Probability is encourged. Previous familiarity with statistical physics will not be assumed.

Literature

The course will use material from a number of sources. Some helpful references are:

(a) S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems*, Cambridge University Press
Available here: https://www.unige.ch/math/folks/velenik/smbook/

(b) G. Grimmett, *Probability on Graphs*, Cambridge University Press
Available here: http://www.statslab.cam.ac.uk/ grg/books/pgs.html

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Stochastic Calculus (L24)

Jason Miller

This course will be an introduction to Itô calculus and will aim to cover the following topics:

- **Brownian motion** Existence and sample path properties.
- **Stochastic calculus for continuous processes** Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita-Watanabe theorem, and Itô's formula.
- **Applications to Brownian motion and martingales** Lévy characterization of Brownian motion, Dubins-Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- **Stochastic differential equations** Strong and weak solutions, notions of existence and uniqueness, Yamada-Watanabe theorem, strong Markov property, and relation to second order partial differential equations.
- **Stroock–Varadhan theory** Diffusions, martingale problems, equivalence with SDEs, approximations of diffusions by Markov chains.

Prerequisites

We will assume knowledge of measure theoretic probability as taught in Part III Advanced Probability. In particular we assume familiarity with discrete-time martingales and Brownian motion.

Literature

(a) R. Durrett, *Probability: theory and examples.* Cambridge, 2010.

(b) I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus.* Springer, 1998.

(c) P. Morters and Y. Peres, *Brownian Motion.* Cambridge, 2010.

(d) D. Revuz and M. Yor, *Continuous martingales and Brownian motion.* Springer, 1999.

(e) L.C. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales.* Cambridge, 2000.

Schramm-Loewner Evolutions (L16)

Konstantinos Kavvadias

In this course, we will introduce and study the Schramm-Loewner Evolution (SLE), a one-parameter family of non-crossing random fractal curves on planar domains. SLE was introduced in 1999 by Schramm as a way to describe the scaling limit of two-dimensional statistical mechanics models (such as loop-erased random walk, percolation, the Ising model and the Gaussian free field) at criticality and has been very fruitful in this regard. However, SLE is not only a useful tool for deriving results on statistical mechanics models. SLE is highly interesting in its own right, being defined by combining ideas from complex analysis and probability. Thus, it ended up as a topic filled with exceptionally beautiful mathematics.

The main goal of the course is to define and study SLE. The course will start with a review of complex analysis before going into the specifics of conformal maps and their relations to planar Brownian motion; distortion estimates and the Loewner differential equation. We then quickly review stochastic calculus before moving on to defining and studying the geometry of SLE, as well as its relation to the Gaussian free field.

Prerequisites

The student should know measure-theoretic probability (see the course Advanced Probability). Moreover, the student is assumed to be familiar with conformal maps and the Riemann mapping theorem. Stochastic calculus is needed to understand the second half of the course.

Literature

The course will follow the lecture notes by Jason Miller: http://www.statslab.cam.ac.uk// jpm205/teachin

For extra material on SLE the student can consult:

(a) Gregory F. Lawler, *Conformally Invariant Processes in the Plane,* volume 114 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.

(b) Wendelin Werner, *Random planar curves and Schramm-Loewner evolutions :* https://arxiv.org/pdf/math/0303354.pdf

For refreshing your stochastic calculus knowledge, the following is recommended:

(a) Jason Miller, *Stochastic Calculus :* http://www.statslab.cam.ac.uk// jpm205/teaching/lent2016/lecture$_n otes.p$

For extra material on conformal maps, see the following: (Note that the references below are beyond the prerequisites of the course. We will cover what is needed but the interested student will enjoy the books.)

(a) Dimitry Beliaev, *Conformal maps and geometry, Advanced Textbook in Mathematics.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.

(b) John B. Garnett and Donald Marshall, *Harmonic Measure,* New Mathematical Monographs: 2. CUP, 2005.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

15 Quantum Computation, Information and Foundations

Quantum Computation (M24)

Sergii Strelchuk

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

This course will be a 'second' course in the subject, following the Part II course Quantum Information and Computation (see below in prerequisites) that was introduced in 2017-2018.

In this course we will aim to cover the following topics:

- The hidden subgroup problem and quantum Fourier transform on a group;
- The quantum phase estimation algorithm and applications;
- Amplitude amplification and applications;

- Quantum simulation for local hamiltonians, QMA-completeness of the local Hamiltonian problem
- Variational Quantum Algorithms;
- The Harrow-Hassidim-Lloyd quantum algorithm for systems of linear equations;
- Introduction to Clifford operations; Classical simulation properties of Clifford circuits (Gottesman-Knill theorem);
- The Stabilizer formalism;
- The Pauli based model of quantum computing (Bravyi, Smith and Smolin 2016).

Prerequisites

This course will assume a prior basic acquaintance with quantum computing, to the extent presented in the course notes for the Cambridge Part II course Quantum Information and Computation available at http://www.qi.damtp.cam.ac.uk/part-iii-quantum-computation

In particular you should be familiar with Dirac notation and principles of quantum mechanics, as presented in the course notes sections 2.1, 2.2 and 2.3. You should also have a basic acquaintance with quantum computation to the extent of the second half of the course notes, pages 47 to 86 (Chapters 6-11). It would be desirable for you to look through this material before the start of the course.

Literature

Further useful literature includes the following.

(a) Nielsen, M. and Chuang, I., *Quantum Computation and Quantum Information*. CUP, 2000.

(b) John Preskill *Lecture Notes on Quantum Information Theory* (especially Chapter 6) available at http://theory.caltech.edu/ preskill/ph229/

(c) Ronald de Wolf, *Quantum Computing: Lecture Notes* https://arxiv.org/pdf/1907.09415.pdf

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in Quantum Foundations (L16)

Boris Groisman

This course introduces a number of key foundational questions in quantum theory and quantum information, with a focus on the relationship between quantum measurement, nonlocality and relativistic causality.

The course will cover a selection of topics including:

- Overview of basic concepts in quantum theory and quantum information: Hilbert space, pure and mixed states, density matrix, the Schmidt decomposition, entanglement, etc.
- Relationship between quantum theory and relativity: the no-signalling principle.
- The EPR argument, Bell's theorem and Bell-nonlocality, local hidden variable model, CHSH inequality; relationship between Bell-nonlocality and entanglement; Quantum nonlocality without entanglement.
- Measurement in Quantum Mechanics - the quantum measurement paradigm: definition and formalism (pointers, Born-rule, interaction Hamiltonian, etc.)
- Quantum measurement and Relativity: constraints on measurements of joint properties of several quantum systems separated in space (instantaneous measurements of nonlocal variables).
- Two-vector formalism and weak measurements; Interaction-free measurements.
- Interpretation of measurement in different theories of Quantum Mechanics: Everettian (Many-Worlds) quantum theory and collapse-based versions of quantum theory.

Prerequisites

Familiarity with undergraduate level quantum mechanics is essential. An advanced course on quantum mechanics including Dirac's "bra/ket" notation, Heisenberg picture and Heisenberg equations of motion will be very useful for this course. Familiarity with a first course in quantum information theory, such as the Cambridge Part II Quantum Information and Computation course (specifically familiarity with Dirac notation, qubits, entanglement, Pauli operations, no-signalling theorem and quantum teleportation), would be advantageous.

Literature

(a) Y. Aharonov and D. Rohrlich, *Quantum paradoxes: quantum theory for the perplexed.* WILEY-VCH Verlag, 2005: Chapters 3,7,8,10,14 and 16.

(b) A. Einstein, B. Podolsky, and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?* Phys. Rev. **47,** 777 (1935).

(c) John Bell, *Speakable and Unspeakable in Quantum Mechanics.* CUP, 2nd edition: Chapters 1,2 and 22.

(d) C.H. Bennett, D.P. DiVincenzo, C.A. Fuchs, T. Mor, E. Rains, P.W. Shor, J.A. Smolin, and W.K. Wootters, *Quantum nonlocality without entanglement.* Phys. Rev. **A 59,** 1070 (1999);
B. Groisman and L. Vaidman, *Nonlocal variables with product-state eigenstates.* J. Phys. A: Math. Gen. **34,** 6881 (2001).

(e) Y. Aharonov, D.Z. Albert and L. Vaidman, *Measurement process in relativistic quantum theory.* Phys. Rev. **D 34,** 1805 (1986);
S. Popescu and L. Vaidman, *Causality constraints on nonlocal quantum measurements.* Phys. Rev. **A 49,** 4331 (1994).

(f) B. Groisman, and B. Reznik, *Measurements of semi-local and non-maximally entangled states.* Phys. Rev. **A 66,** 022110 (2002);
L. Vaidman, *Instantaneous Measurement of Nonlocal Variables,* Phys. Rev. Lett. **90,** 010402 (2003).

(g) S. R. Clark, A. J. Connor, D. Jaksch and S. Popescu, *Entanglement consumption of instantaneous nonlocal quantum measurements,* New J. Phys. **12,** 083034 (2010);
B. Groisman and S. Strelchuk, *Optimal amount of entanglement to distinguish quantum states instantaneously.* Phys. Rev. **A 92,** 052337 (2015).

(h) S. Saunders, J. Barrett, A. Kent, and D. Wallace (eds.), *Many Worlds?: Everett, Quantum Theory, and Reality.* OUP (2010);
L. Vaidman, *Many-Worlds Interpretation of Quantum Mechanics.* https://plato.stanford.edu/entries/qm-manyworlds/

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topological Quantum Matter (L16)

Benjamin Béri

Topologically ordered systems display quantum mechanical features that are insensitive to the local details of the system but are robustly set by its topology. Such features include "anyonic" particles beyond the boson/fermion dichotomy, a "holographic" correspondence between the system's bulk and its boundary, or a topology-dependent ground-state degeneracy. These remarkable properties, fundamentally linked to certain gauge theories, have intriguing potential applications, including the use of topological ground-state degeneracy in quantum error correction or employing so-called non-Abelian anyons in robust schemes for quantum computation.

This course is an introduction to topological order and its quantum computing applications. We begin with some basics of conventional (i.e., symmetry-breaking) order, illustrated using the quantum Ising chain. This will serve as a reference with which to compare topological order.

We then embark on our study of topological order. Concepts we shall aim to discuss include:

- ground-state degeneracy
- anyons
- the robustness of topological order
- bulk-boundary correspondence
- Majorana zero modes
- non-Abelian anyons.

We shall use the toric code and the Kitaev chain – a boson-fermion dual of the quantum Ising chain – as illustrative examples.

Of the quantum computing applications of topological order, we shall aim to discuss quantum error correction with the toric code, and topological quantum computation using non-Abelian anyons, in particular those based on Majorana zero modes.

Some of our discussions will use second quantisation, the basics of path integrals, and of quantum error correction. These shall be reviewed or introduced at the appropriate points of the course.

Prerequisites

It will be assumed that you have taken an advanced quantum mechanics course, similar to Part II Principles of Quantum Mechanics. Familiarity with the basics of band theory (e.g., from Part II Applications of Quantum Mechanics) and of

quantum computation (e.g., from Part II Quantum Information and Computation) will also be useful. While not a prerequisite, having attended Part III Quantum Computation in Michaelmas term can be beneficial.

Literature

(a) S. H. Simon, *Topological Quantum: Lecture Notes and Proto-Book.* Available at Steve Simon's http://www-thphys.physics.ox.ac.uk/people/SteveSimon/website. Chapter 3 and Parts II, V are especially useful for this course.

(b) S. Sachdev, *Quantum Phase Transitions.* CUP, 2011. Parts of chapters 1, 5, 10.

(c) B. A. Bernevig with T. L. Hughes, *Topological Insulators and Topological Superconductors.* Princeton University Press, 2013. Chapter 16.

(d) S. Das Sarma, M. Freedman, C. Nayak, *Majorana zero modes and topological quantum computation.* https://www.nature.com/articles/npjqi20151npj Quantum Inf. **1**, 15001 (2015).

(e) A. Altland and B. Simons, *Condensed Matter Field Theory.* CUP, 2010. Chapters 2, 3 for background on second quantisation and path integrals.

Additional support

Three example sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Tensor Networks (M8)

Non-Examinable (Part III Level)

Thorsten Wahl

Understanding quantum many body systems poses some of the most outstanding challenges in quantum physics. The computational effort required to describe these systems generally grows exponentially with the number of particles. It turns out that for the types of interactions found in nature, in many relevant cases, explored in this course, such a description has only polynomial complexity in the particle number. For local gapped Hamiltonians defined on a lattice, this reduction of complexity is manifest in the language of tensor network states, which have not only revolutionised our ability to simulate quantum many-body systems, but also the way we think about quantum matter itself. As a result, the utility of tensor network states has by now far exceeded their traditional realm of condensed matter physics. They are now being applied in areas as

distant as lattice gauge theories, quantum computing, quantum gravity and, most recently, machine learning.

This course will introduce the main one- and two-dimensional tensor network states and indicate how they can be used numerically to overcome the quantum many-body problem. We will see how symmetries and topological properties are reflected locally in the constituting tensors and discuss important extensions of the standard tensor network states.

The course will cover a selection of topics, including:

- The quantum many-body problem and the area law of entanglement
- Matrix Product States (MPS), Projected Entangled Pair States (PEPS)
- The graphical representation of tensor networks
- Efficient contractions of tensor network states
- Gauge transformations and canoncial forms
- Parent Hamiltonians
- Symmetries of tensor network states
- Topological features of tensor network states, classification of topological phases
- Extensions of tensor network states: Multiscale Entanglement Renormalisation Ansatz (MERA), fermionic PEPS, continuous MPS

Prerequisites

Basic quantum mechanics.

Literature

(a) D. Pérez-García, F. Verstraete, M. M. Wolf, and J. I. Cirac, *Matrix Product State Representations*, Quantum Information and Computation **7**, 401 (2007). https://arxiv.org/abs/quant-ph/0608197 (hard one)

(b) J. I. Cirac and F. Verstraete, *Renormalization and tensor product states in spin chains and lattices*, Journal of Physics A: Mathematical and Theoretical **42**, 504004 (2009). https://iopscience.iop.org/article/10.1088/1751-8113/42/50/504004

(c) R. Orús, *A practical introduction to tensor networks: Matrix product states and projected entangled pair states*, Annals of Physics **349**, 117 (2014). https://www.sciencedirect.com/science/article/pii/S0003491614001596?via%3Dihub

(d) J. Eisert, *Entanglement and tensor network states*, Modeling and Simulation **3**, 520 (2013). https://www.cond-mat.de/events/correl13/manuscripts/eisert.pdf

(e) R. Orús, *Tensor networks for complex quantum systems*, Nature Reviews Physics **1**, 538 (2019). https://www.nature.com/articles/s42254-019-0086-7

(f) S.-J. Ran, E. Tirrito, C. Peng, X. Chen, L. Tagliacozzo, G. Su, and M. Lewenstein, *Tensor Network Contractions*, Saint Philip Street Press (2020). https://link.springer.com/book/10.1007/978-3-030-34489-4

(g) J. I. Cirac, D. Pérez-García, N. Schuch, and F. Verstraete, *Matrix product states and projected entangled pair states: Concepts, symmetries, theorems*, Reviews of Modern Physics **93**, 045003 (2021). https://journals.aps.org/rmp/abstract/10.1103/RevModPhys.93.045003

Additional support

For scientific or organisational questions, feel free to get in touch via tw344@cam.ac.uk.

16 Relativity and Cosmology

Cosmology (M24)

Blake Sherwin

This course discusses what we know (and don't know) about the evolution of our universe, from inflationary quantum fluctuations in the first fraction of a second to the formation of galaxies and structures today. It also seeks to illustrate how cosmology can serve as a uniquely powerful laboratory for understanding fundamental physics.

In detail, the course will cover the following topics:

- Geometry and dynamics of our Universe
- Inflation
- Thermal history
- Cosmological perturbation theory
- Structure formation
- Cosmic microwave background basics
- Initial conditions from inflation

Pre-requisites

Although the course is intended to be as self contained as possible, knowledge of relativity, quantum mechanics and statistical mechanics will be very useful. General relativity and quantum field theory courses may allow a deeper understanding of some of the material covered.

Literature

(a) S. Dodelson, *Modern Cosmology*
(b) E. Kolb and M. Turner, *The Early Universe*
(c) S. Weinberg, *Cosmology*

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

General Relativity (M24)

Claude Warnick

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

This is a second course on General Relativity, albeit one that could just about be followed without prior exposure to the subject. The first half of the course will give an introduction to differential geometry, the mathematics that underlies curved spacetime. The second half of the course will discuss the physics of gravity.

Prerequisites

Familiarity with Newtonian gravity, special relativity, finite-dimensional vector spaces and the Euler-Lagrange equations is essential. Knowledge of the relativistic formulation of MaxwellÕs equations is highly desirable.

Most students attending this course have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before then you should read an introductory book (e.g. Hartle or Rindler) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Literature

(a) J.B. Hartle, *Gravity: An introduction to Einstein's General Relativity,* Addison-Wesley, 2003.

(b) W. Rindler, *Relativity: Special, General, and Cosmological*, 2nd ed., OUP, 2006.

(c) R.M. Wald, *General Relativity,* Chicago UP, 1984.

(d) C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation,* W.H. Freeman, 1973.

(e) S.M. Carroll, *Spacetime and geometry: an introduction to General Relativity,* Addison- Wesley, 2004.

(f) J.M. Stewart, *Advanced General Relativity,* CUP, 1993.

(g) S. Weinberg, *Gravitation and Cosmology,* Wiley, 1972.

The course will follow a similar approach to Wald and Misner, Thorne and Wheeler. Carroll's book is a very readable introduction at about the same level. Stewart's book is based on a previous version of this course. Weinberg's book gives a good discussion of the Equivalence Principle.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Black Holes (L24)

Jorge E. Santos

A black hole is a region of spacetime that is causally disconnected from the rest of the Universe. These objects appear to be pervasive in Nature, and their properties have direct implications for the recent advances in gravitational wave astronomy. Besides being astrophysically relevant, black holes also play a fundamental role in quantum theory and are a natural arena to study and test any consistent quantum theory of gravity.

The following topics will be discussed:

- Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
- The initial value problem, strong cosmic censorship.
- Causal structure, null geodesic congruences, Penrose singularity theorem.

- Penrose diagrams, asymptotic flatness, weak cosmic censorship.
- Reissner-Nordström and Kerr black holes.
- Energy, angular momentum and charge in curved spacetime.
- The laws of black hole mechanics. The analogy with laws of thermodynamics.
- Quantum field theory in curved spacetime. The Hawking effect and its implications.

Prerequisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

(a) H. S. Reall, *Part 3 Black Holes: lecture notes* available at www.damtp.cam.ac.uk/user/hsr1000.

(b) R. M. Wald, *General relativity*, University of Chicago Press, 1984.

(c) S. W. Hawking and G.F.R. Ellis, *The large scale structure of spacetime*, Cambridge University Press, 1973.

(d) V. P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.

(e) N. D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.

(f) R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term

Field Theory in Cosmology (L24)

Enrico Pajer

This course discusses applications of classical, statistical and quantum field theory to cosmology. The course comprises three interconnected topics:

- Cosmological inflation and primordial quantum perturbations (QFT in curved spacetime)

- The matter and galaxy distribution in the Large Scale Structure of the Universe (statistical field theory)
- The physics of the Cosmic Microwave Background (classical and statistical field theory)

The goals of the course are: to discuss open problems in cosmology and describe their intimate relation to fundamental high energy physics; to provide the basic knowledge to understand modern research literature in cosmology; to explore how field theory provides a unifying formalism to describe disparate physical processes from the birth of the Universe to the highly non-linear cosmic web.

More specifically, after a general introduction to open current research and open problems in cosmology, we review inflation and introduce the Effective Field Theory of cosmological perturbations and its connection to field theories with non-canonical interactions. Then we present the so-called "in-in" or Schwinger-Keldysh formalism to compute cosmological correlators and discuss some simple examples, focusing on the leading non-Gaussian statistic, the bispectrum. After showing that cosmological perturbations become classical, we review some basic properties of stochastic fields and correlation functions. The equations determining the dynamics of Large Scale Structure are then introduced together with the concept of renormalization. As an application, we derive a prediction for the matter and galaxy power spectrum at next-to-leading order. Finally, we introduce the Boltzmann equations for the coupled photon-baryon "fluid" and use them to compute the observed temperature anisotropies in the Cosmic Microwave Background.

Prerequisites

Some familiarity with introductory Quantum Field Theory and General Relativity, as for example provided by the respective Michaelmas courses, is highly recommended. Basic knowledge of introductory Cosmology is essential. Students who did not attend the Michaelmas course on Cosmology may still follow this course after reviewing the relevant course notes.

Literature

Lecture notes including references will be provided by the lecturers.

Additional support

Four example sheets will be provided and four associated example classes will be given. There will be a one-hour revision class in Easter Term.

Solitons, Instantons, and Geometry (L16)

Maciej Dunajski

Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation.

Instantons are non–singular solutions of classical field equations in Euclidean space whose action is finite. Instantons describe quantum-mechanical tunnelling between the classical minima of the potential in Minkowski space.

Solitons and instantons are globally characterised by some topological charge, related to the behaviour of solutions at spatial infinity. Its understanding requires ideas from topology, and geometry.

The course will cover a selection of topics including:

- Geometry of Lie Groups
- Solitons (kinks in 1D, vortices in 2D)
- Yang Mills instantons
- Fibre bundles and instantons: connection, curvature and characteristic classes.

Prerequisites

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

Literature

(a) Dunajski, M. *Solitons, Instantons, and Twistors, Oxford Graduate Texts in Mathematics,* Oxford University Press, 2009.

(b) Eguchi, T., Gilkey, P. and Hanson. A. J. *Physics Reports,* **66** (1980) 213-393

(c) Manton, N. and Sutcliffe, P. *Topological Solitons,* Cambridge University Press, , 2004.

Additional support

Three examples sheets will be provided and three examples classes will be given. There will be a one-hour revision class in the Easter Term.

Gauge-Gravity Duality (E16)

Aron Wall

Gauge/Gravity duality (also known as AdS/CFT) is an amazing duality that relates theories of quantum gravity (with a negative cosmological constant) to certain quantum field theories living in a smaller dimensional spacetime. This is the most precise known realization of the holographic principle, the idea that all information in the universe is encoded somehow at the boundary of the universe. These lectures will describe in detail the "dictionary" used to relate observables on the bulk side to observables on the boundary side.

Topics covered:

- Anti-de Sitter spacetime
- Conformal field theory
- Wave equations in AdS, and their relationship to CFT operators and sources
- The duality between black holes and thermal states
- holographic entanglement entropy
- If time permits, recent developments concerning bulk reconstruction and the black hole information puzzle.

Prerequisites

Required: General Relativity, Black Holes, Advanced Quantum Field Theory

Helpful: Some basic aspects of quantum information theory and conformal symmetry will play an important role in this course, but the relevant aspects will be reviewed in a self-contained manner.

Not Required: String Theory, Supersymmetry. Although most of the specific known examples of AdS/CFT come from superstring theories, these aspects will not be emphasized in these lectures. (There will be one or two lectures on Maldacena's original derivation of black hole entropy in string theory, but this topic will not be examinable.)

Literature

Lecture notes for the course will be provided in pdf format. Additional information about the course, including a list of relevant review articles, is available here:

http://www.damtp.cam.ac.uk/user/aw846/AdSCFT.html

Additional support

Two examples sheets will be provided, and two associated examples classes will be given. There will be a one-hour revision class in the Easter Term, subsequent to the lectures.

Gravitational Waves and Numerical Relativity (E16)

Non-Examinable (Part III Level)

Ulrich Sperhake

The direct detection of a gravitational-wave signal from the black-hole binary merger GW150914 detected by LIGO in September 2015 has promoted gravitational-wave physics to a rich field combining theory with observation. With about 100 further events observed over the ensuing years, a vast range of physical and astrophysical phenomena are now accessible to a qualitatively new channel of observations. New gravitational-wave observatories on Earth and in Space will greatly enhance the sensitivity and frequency range in future decades. Gravitational-wave observations heavily rely on accurate theoretical predictions that require source modelling in the framework of Einstein's theory of relativity. Due to the complexity of Einstein's field equations and the highly dynamical character of the physical systems under consideration, the calculation of such templates is a formidable challenge that requires the combination of analytic and numerical techniques. The main goal of these lectures is to discuss the theoretical foundation of gravitational waves and discuss techniques used to compute the waveforms generated by the most common astrophysical sources, in particular the inspiral and merger of compact binaries.

For this purpose, the course will cover a selection of topics including:

- Characteristic surfaces and the classification of partial differential equations.
- The structure of the Einstein field equations.
- Gravitational waves in the characteristic (aka *Bondi-Sachs*) formalism of general relativity.
- The space-time split of the Einstein equations and well-posed formulations of the field equations.
- The main ingredients of (successful) numerical relativity.
- Diagnostics and their relation to gravitational-wave observables.

Prerequisites

This course assumes knowledge of Part III General Relativity and some knowledge of Part III Black Holes, most notably the Kerr solution.

Literature

(a) T. W. Baumgarte and S. L. Shapiro *Numerical Relativity.* Cambridge University Press, 2010.

(b) T. W. Baumgarte and S. L. Shapiro *Numerical Relativity: Starting from Scratch.* Cambridge University Press, 2021.

(c) M. Alcubierre, *Introduction to 3+1 Numerical Relativity*, Oxford University Press, 2008.

(d) E. Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, Springer, New York, 2012. See also https://arxiv.org/abs/gr-qc/0703035 .

(e) M. Maggiore, Gravitational Waves. Vols. 1 (Theory and Experiments) and 2 (Astrophysics and Cosmology), Oxford University Press, 2018.

(f) R. d'Inverno, *Introducing Einstein's Relativity*, Oxford Clarendon Press, 1992.

Additional support

Three examples sheets will be provided and accompanying office hours will be offered.

17 Statistics

Modern Statistical Methods (M24)

Rajen Shah

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data where we have huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the

number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to study such datasets. We aim to cover some of the following topics.

- Kernel machines: Ridge regression, the kernel trick, reproducing kernel Hilbert spaces, kernel ridge regression, the support vector machine, the hashing trick.
- Penalised regression: Model selection, the Lasso, variants of the Lasso.
- Conditional independence and high-dimensional inference: Conditional independence graphs, Neighbourhood selection, the graphical Lasso, conditional independence testing, the debiased Lasso.
- Multiple testing: the closed testing procedure and the Benjamini–Hochberg procedure.

Prerequisites

Basic knowledge of statistics, probability, linear algebra and real analysis. Some background in optimisation would be helpful but is not essential.

Literature

(a) T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning.* 2nd edition. Springer, 2001.

(b) P. Bühlmann, S. van de Geer, *Statistics for High-Dimensional Data.* Springer, 2011.

(c) T. Hastie, R. Tibshirani and M. Wainwright, *Statistical learning with sparsity: the lasso and generalizations.* CRC Press, 2015.

(d) C. Giraud, *Introduction to High-Dimensional Statistics.* CRC Press, 2014.

(e) M. Wainwright, *High-Dimensional Statistics: A Non-Asymptotic Viewpoint.* Cambridge University Press, 2019.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topics in Statistical Theory (M16)

Richard J. Samworth

This course will provide an introduction to the theory behind a selection of statistical problems that play a key role in modern statistics. Most undergraduate statistics courses are restricted to the study of parametric models; here we will no longer assume that our distributions belong to finite dimensional classes and will instead study fundamental nonparametric problems such the estimation of a distribution function, a density function or a regression function. We will also study minimax lower bounds, which characterise the intrinsic difficulty of a statistical problem, and provide benchmarks against which statistical procedures can be compared.

An outline of the course is as follows:

- **An introduction to nonparametric statistics** the basics of empirical process theory, Glivenko–Cantelli theorem, Dvoretzky–Kiefer–Wolfowitz inequality, order statistics, quantile estimation and associated asymptotic distribution theory. Concentation inequalities, including Hoeffding, Bennett and Bernstein inequalities.
- **Kernel density estimation** Definition, bounds on bias and variance, uniform nonasymptotic bounds on MSE and MISE. Bandwidth selection via least squares cross validation and Lepski's method, choice of kernel, multivariate density estimation.
- **Nonparametric regression** Local poynomial estimation, bounds on weights, bias, variance and MSE. Cubic splines, natural cubic smoothing splines, choice of smoothing parameter.
- **Minimax lower bounds** Reduction to testing, f-divergences, Le Cam's two point lemma, Assoaud's lemma, the data processing inequality, Fano's lemma; examples.

Prerequisites

A good background in undergraduate probability theory, elements of linear algebra and real analysis. Measure theory is not necessary but may be helpful; similarly for a preliminary course in mathematical statistics. Though the material in the Modern Statistical Methods course will not be needed here, the two courses complement each other well.

Literature

The lecturer is currently writing a book based on the course, and this should be available (if not published) in time for the course. Some of the material is

covered in:

(a) S. Boucheron, Lugosi, G. and Massart, P. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.

(b) A. Tsybakov, *Introduction to Nonparametric Estimation*. Springer, 2009.

(c) M. J. Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge University Press, 2019.

Additional support

Three examples sheets will be provided and associated examples classes will be given. There will be weekly office hours during Michaelmas and a revision class in the Easter Term.

Causal Inference (M16)

Qingyuan Zhao

From its onset, modern statistics engages in the problem of inferring causality from data. A common mindset is that causal inference is only possible using randomised experiments, but developments in statistics and related fields have shown that this view is oversimplified and restrictive. We now have a much better understanding of the assumptions and methodologies that enable causal inference from observational, non-experimental data. This course aims to cover some of the most fundamental ideas in causal inference, a vibrant research area where statistical theory meets scientific practice.

(a) **Motivations:**
- Principles of causal inference: motivations; historical perspectives; basic concepts.
- Randomised experiments: randomisation tests, regression adjustment and its asymptotic inference.
- Path analysis and linear structural equation models (SEMs);

(b) **Languages for causality:**
- Probabilistic directed acyclic graphical (DAG) models: Markov properties, d-separation, structure discovery.
- Counterfactual causal models: nonparametric SEMs; single-world intervention graphs; g-computation formula.
- Causal identification: back-door criterion, front-door criterion; counterfactual calculus; other examples.

(c) **Design and statistical methods:**

- Observed confounders: matching, randomisation inference, Rosenbaum's sensitivity analysis; influence functions and semiparametric inference.
- Instrumental variables (IV): core IV assumptions; generalised method of moments; principal stratification.
- Other selected topics: regression discontinuity design; difference in differences and negative control methods; mediation analysis; longitudinal data and time-varying treatments; meta-analysis and evidence synthesis.

Pre-requisites

This course assumes familiarity with undergraduate-level probability and statistics.

Literature

(a) Hernán M. A. and Robins, J. M. *Causal Inference: What If.* Chapman & Hall, 2020.

(b) Imbens, G. W. and Rubin, D. B. *Causal Inference in Statistics, Social, and Biomedical Sciences.* Cambridge University Press, 2015.

(c) Lauritzen, S. L. *Graphical Models.* Clarendon Press, 1996.

(d) Angrist, J. D. and Pischke, J. S. *Mostly Harmless Econometrics: An Empiricist's Companion.* Princeton University Press, 2008.

Additional support

Lecture notes will be provided. Three examples sheets will be provided and three associated examples classes will be given. A fourth example class will ask the student to read and present an applied research article. There will also be a revision class in the Easter Term.

Statistics in Medicine (3 units)

Statistics in Medical Practice (M12)

Lecturers from the MRC Biostatistics Unit

This part of the course includes three modules covering a range of statistical methods and their application in three areas of biostatistics.

A. Stochastic Models for Chronic and Infectious Diseases [4 Lectures] **(C. Jackson, D. De Angelis, P. Birrell)**

Continuous-time multi-state and Markov models: properties and quantities of interest, and fitting models to individual disease history data. Applications to modelling the onset and progression of chronic diseases. Multi-state modelling to estimate incidence of infectious diseases from population-level prevalence data. Backcalculation methods for the estimation of incidence of disease with long incubation periods. Dynamic modelling of infectious disease transmission.

B. Causal Inference [4 Lectures] **(S. Burgess and others)**

It is well known that "correlation is not causation". But how then do you assess causal claims? Is it possible to show that X is a cause of Y? What does it even mean to say that X is a cause of Y? In this module, we introduce definitions of causal concepts, starting with the work of Rubin, Pearl, and Robins, and discuss practical approaches for assessing causal claims from observational data.

C. Design and Analysis of Randomised Trials [4 Lectures] **(S. Villar, D. Robertson & H. Zheng)**

Sample size estimation for clinical trials; Adaptive and multi-stage designs (including group-sequential designs). Treatment effect estimation following a group-sequential trial. Types of randomisation procedures. Optimal response-adaptive procedures. Precision medicine trials with multiple strata: estimation of stratum-specific parameter(s), Bayesian methods for borrowing information, and mixture priors.

Statistics in Medicine (3 units)

Analysis of Survival Data (L12)

P. Treasure

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

D. Time-to-Event Analysis [4 Lectures]

'Survival analysis' is generalised to *time-to-event* analysis. The implications of event times which are unknown or in the future (*censored* data) are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (*Kaplan-Meier*) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (*log-rank* test).

E. Modelling Hazard [4 Lectures]

The *hazard* function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size...) on the time-to-event distribution (*proportional hazards* modelling). Model checking procedures are introduced with an emphasis on excess event (*Martingale*) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (*competing risk* analysis) and (ii) the cause of death is unknown (*net survival*). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar year (e.g. 2019) are addressed: *period* survival analysis.

Statistics in Medicine (3 units)

Additional Information

Prerequisites

Undergraduate-level statistics and probability: including analysis and interpretation of data, maximum likelihood estimation, hypothesis testing, basic stochastic processes.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. A few books to complement the course material are listed below.

(a) Armitage P, Berry G, Matthews JNS, *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001. [A good introductory companion to the whole course]

(b) van den Hout, A, *Multi-State Survival Models for Interval-Censored Data.* Chapman and Hall, 2016 [Module A]

(c) Keeling, M. J., & Rohani, P. *Modeling Infectious Diseases in Humans and Animals.* Princeton University Press, 2008 [Module A]

(d) Burgess S, Thompson SG, *Mendelian Randomization: Methods for Using Genetic Variants in Causal Estimation* Chapman and Hall, 2015 [Module B]

(e) Senn, S. *Statistical Issues in Drug Development.* Wiley, 2007. [Module C]

(f) Jennison C, Turnbull B, *Group Sequential Methods with Applications to Clinical Trials.* Chapman and Hall, 2000. [Module C]

(g) Rosenberger, William F., and John M. Lachin. *Randomization in clinical trials: theory and practice.* John Wiley & Sons, 2015. [Module C]

(h) Cox DR, Oakes D, *Analysis of Survival Data.* Chapman and Hall, 1984 [Modules D, E, F: the classic text]

(i) Collett D, *Modelling Survival Data in Medical Research.* CRC Press, 2015 [Modules D, E, F: modern, applied, supports and extends lectures.]

(j) Aalen OO, Borgen Ø, Gjessing HK, *Survival and Event History Analysis: A Process Point of View.* Springer, 2008 [Modules D, E, F: excellent modern approach]

Additional support

[Modules A, B, C] A two-hour example class, supported by question sheets and solutions, will be given at the start of Lent term.

[Modules D, E, F] A 90-minute example class, supported by question sheets and solutions, will be given in each of the Lent and Easter Terms. A 90-minute revision class will be held just before the examination. There will be regular office hours by videoconference.

Bayesian Modelling and Computation (L24)

Sergio Bacallado

The course deals with Bayesian inference in hierarchical models using Monte Carlo methods. We shall emphasize practical aspects of model design and evaluation, and the construction of efficient inference algorithms. The example sheets will include practical problems.

- Exponential families and conjugate priors. Explicit inference with Gaussian posteriors. Hierarchical models. Bayesian and frequentist mixed-effects models. Posterior predictive checks and model checking.

- Monte Carlo integration. Markov Chain Monte Carlo. Metropolis–Hastings algorithm, Gibbs sampling, and Metropolis-within-Gibbs. Convergence diagnostics.
- Gibbs sampling for hierarchical location models. Centred and non-centred parametrisations. Data augmentation and mixture models.
- Geometric ergodicity and asymptotic variance. Fundamentals of spectral theory, functional inequalities, Lyapunov drift/minorisation techniques.
- Metropolis–Hastings and dimensionality, optimal scaling analysis. Pre-conditioned Crank–Nicolson method. Bayesian inverse problems.
- Diffusion-based samplers, canonical characterisation. The Langevin algorithm, Barker proposal, and Hamiltonian Monte Carlo. Probabilistic programming languages.
- State-space models and Feynman–Kac formulation. Forward-backward algorithm and Kalman filtering. Convergence and stability of particle filters.

Prerequisites

This course assumes familiarity with probability, basic Markov chain theory, and statistical modelling. Prior knowledge of Bayesian analysis is helpful.

Reference textbooks

(a) Gelman, A., Carlin, J. B., Stern, H. S., Dunson, D. B., Vehtari, A., & Rubin, D. B., *Bayesian data analysis (3rd Edition)*. Chapman and Hall/CRC, 2020.

(b) Douc, R., Moulines, E., Priouret, P., & Soulier, P., *Markov chains*. Cham, Switzerland: Springer International Publishing, 2018.

(c) Chopin, N., & Papaspiliopoulos, O., *An introduction to sequential Monte Carlo*. New York: Springer International Publishing, 2020.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be one revision class in the Easter Term.

Functional Data Analysis (L16)

John Aston

Functional Data Analysis (FDA) is the study of the statistics of curves and surfaces. In most statistical analyses, the object of study is either a number (one dimensional univariate statistics) or a vector (finite dimensional multivariate statistics). In FDA the object of study is usually considered to be infinite dimensional, often but not always, with some inherent smoothness characteristics. In this course, we will examine both the theory behind such data analysis and the myriad of different places that these data arise.

The course will likely cover the following topics:

- Definition of Functional Data
- Functional Principal Component Analysis
- Covariance Operators
- Functional Linear Models

Prerequisites

Basic knowledge of statistics, probability, linear algebra and real analysis.

Literature

(a) Ramsay, J.O. and Silverman B.W. *Functional Data Analysis* 2nd edition, Springer, 2005.

(b) Horvath, L. and Kokoszka, P. *Inference for Functional Data with Applications*, Springer, 2012

(c) Kokoszka, P. and Reimherr, M. *Introduction to Functional Data Analysis*, Chapman and Hall, 2017

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistical Learning in Practice (L24)

Randolf Altmeyer

Statistical learning aims to build statistical models for learning relationships and structures from data, mainly with the goal of predicting future outcomes.

It blends classical statistics with techniques from modern machine learning and tries to provide concrete algorithms, for example for regression or classification tasks. This course consists of 12 lectures and 12 practical classes. We will get to know some of the most successful and widely used statistical methodologies in modern applications, with a focus on practical aspects and concrete problems. The practical classes will cover an introduction to the programming language R, exploratory data analysis and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models for regression and classification
- Model selection and regularisation
- Bayesian regression
- Mixed effects models
- Linear discriminant analysis and Support Vector Machines
- Introduction to Deep learning and Random forests
- Principal component analysis
- Introduction to time series

Prerequisites

Elementary probability theory, maximum likelihood estimation, hypothesis tests and confidence intervals, linear models. Previous experience with R is not essential and can be obtained during the course (only a good working knowledge of R will be necessary, as provided, for instance, in the lab sessions of the first reference below).

Literature

(a) G. James, D. Witten, T. Hastie and R. Tibshirani. *An Introduction to Statistical Learning (with Applications in R).* Springer, 2013. Also available at https://www.statlearning.com.

(b) T. Hastie, R. Tibshirani and J. Friedman. *The Elements of Statistical Learning.* Springer, Second Edition.

(c) A. Agresti. *Foundations of linear and generalized linear models.* John Wiley & Sons, 2015.

(d) C. Robert and G. Casella. *Introducing Monte Carlo Methods with R.* Springer, 2009.

(e) P.J. Brockwell and R.A. Davis. *Introduction to time series and forecasting.* Springer, 1996.

(f) A.J. Dobson and A. Barnett. *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC, 2008.

(g) L. Wasserman. *All of Statistics.* Springer, 2003.

Additional support

This course includes practical classes, in which statistical methods are introduced in a practical context and students carry out analysis of datasets using R. In the practical classes, the students have the opportunity to discuss statistical questions with the lecturer. Four example sheets will be provided and there will be four associated example classes (the last one probably to be held in Easter term). There will be a revision class in the Easter Term.

Bayesian Nonlinear Statistical Inverse Problems (L16)

Non-Examinable (Part III Level)

Richard Nickl

Common examples for nonlinear inverse problems range from parameter identification in PDEs to tomography and data assimilation problems.

They naturally involve high or infinite dimensional parameter spaces and appropriate statistical noise models lead to a class of non convex inference problems that present substantial challenges in contemporary data science. In influential work, A. Stuart [4.] has proposed a unified Bayesian approach to solve such problems. It is computationally feasible via Gaussian process priors and high dimensional MCMC algorithms and provides important uncertainty quantification methodology ('error bars' or confidence regions) based on posterior distributions. Despite evident empirical success, the theoretical understanding of the performance of such methods has been limited until recently. Specifically in non linear settings Bayesian methods are distinct from optimisation based algorithms and their analysis requires a very different set of mathematical ideas.

In these lectures we will summarise recent developments that allow to give rigorous statistical and computational guarantees for these algorithms in high dimensional and non convex settings. The general theory will be illustrated in two non linear model examples arising with elliptic partial differential equations.

Prerequisites

Knowledge of probability and measure theory and functional analysis will be useful to follow this course, e.g., see [1.]. Background on nonparametric statistical inference can be found in [2.], and a comprehensive set of lecture notes [3.] is available on my website.

Literature

(a) R.M. Dudley, *Real analysis and probability.* Cambridge University Press, 2002 (paperback 2018).

(b) E. Giné, R. Nickl *Mathematical foundations of infinite dimensional statistical models.* Cambridge University Press, 2016 (paperback 2021).

(c) R. Nickl, *Bayesian non linear statistical inverse problems.* lecture notes, 2022.

(d) A. Stuart, *Inverse problems: A Bayesian perspective*, Acta Numer. 19 (2010), 451–559. **56**.

Additional support

Three examples sheets will be provided and accompanying office hours will be offered.