Mathematical Tripos
Part III Lecture Courses in 2020-2021

Department of Pure Mathematics & Mathematical Statistics

Department of Applied Mathematics & Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*, and are indicated as such after the title. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2019-20. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures and associated course materials as offered in this academic year define the syllabus. Each course lecturer has discretion to vary the material covered.
- Some courses have no writeup available at this time, in which case you will see "No description available" in place of a description. Course descriptions will be added to the online version of the Guide to Courses as soon as they are provided by the lecturer. Until then, the descriptions for the previous year (available at http://www.maths.cam.ac.uk/postgrad/mathiii/courseguide.html) may be helpful in giving a rough idea of course content, but beware of the comments in the preceding item on what defines the syllabus.

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Algebra

Finite dimensional Lie and associative algebras (M24)

Christopher Brookes

The main aim of this course is to study Lie algebras that are finite dimensional as complex vector spaces. Lie algebras (and Lie groups) appear in many branches of mathematics and mathematical physics, the Lie algebra arising as the tangent space to the identity element in the associated Lie group. They consist of infinitesimal symmetries, a linearised approximation to the groups. One reason for their importance is that the finite dimensional complex representations of the simple Lie algebras correspond exactly to those of the groups. So instead of needing to study the topology and geometry of the simple Lie groups, or the algebraic geometry of the simple algebraic groups, we can concentrate on the Lie algebras using methods from linear algebra and still completely describe these representations.

The classification of the finite dimensional simple complex Lie algebras boils down to a study of root systems labelled by Dynkin diagrams. These root systems are fundamental combinatorial structures consisting of a collection of data accompanied by an associated Weyl group of symmetries.

Root systems and Dynkin diagrams also appear in classification problems when studying the representation theory of associative algebras. For example they arise in Gabriel's classification of the quivers (directed graphs, with multiple edges and loops allowed), and their associated path algebras, with only finitely many indecomposable complex representations. Thus root systems form the central theme of the course.

The Lie algebra material is entirely standard and can be found in many places. Serre's little book provides an excellent summary but with quite a bit of detail suppressed. I shall briefly introduce the definitions and basic structure theory. The core topics are:

Cartan subalgebras, root systems, Weyl groups, Dynkin diagrams, the finite simple Lie algebras. Classification of finite dimensional representations, Verma modules, Weyl character formula.

Similarly I shall run through the basic structure of finite dimensional associative algebras, before concentrating on the representation theory of quivers. A good source that covers all of this associative material is the book by Pierce. Core topics are:

Jacobson radical, Artin-Wedderburn theorem (classification of finite dimensional semisimple associative algebras, not necessarily complex). Quivers, path algebras and their representations. Gabriel's theorem.

Pre-requisites

Linear algebra. A first course on rings and modules (e.g. IB groups, rings and modules). Experience of some representation theory (e.g. the II course or equivalent) will be very useful but not essential.

Literature

1. K. Erdmann and M.J. Wildon, *Ingtroduction to Lie algebras.* Springer, 2006.
2. J.E. Humphries, *Introduction to Lie algebras and representation theory.* Springer, 1972.
3. N. Jacobson, *Lie algebras.* Dover, 1979.
4. R.S. Pierce, *Associative algebras.* Springer, 1982.
5. J.-P. Serre, *Complex semisimple Lie algebras* Springer, 1987.

Additional support

There will be four examples sheets, with associated examples classes, and a revision class in the Easter Term.

Commutative Algebra (24 lectures, MT20)

Stuart Martin

This course provides an introduction to the theory of commutative noetherian rings and modules over such rings. It forms a theory which is an essential ingredient in algebraic geometry, algebraic number theory and both modular and integral representation theory.

I hope to cover most of the following

- ideals for noetherian and artinian rings; Hilbert Basis Theorem and the Nullstellensatz;
- localisations and completions;
- tensor products;
- dimension theory; polynomial nature of Hilbert functions;
- projective and injective modules, resolutions and the Koszul complex;
- assorted methods from (co)homology,; Hilbert's syzygy theorem.

Four sheets of examples will be provided.

Desirable Previous Knowledge

You will have attended a first course in ring theory, such as the IB course Groups, Rings and Modules. Experience of more advanced material such as Part II courses Galois Theory, Representation Theory, Algebraic Geometry and Number Fields is desirable but not essential.

Books

There is no shortage of books on commutative ring theory. Notable amongst them is the 1969 classic [1], which is a clear, concise and efficient textbook aimed at beginners and with a good spread of topics. So it has remained popular. However its age and flaws are apparent. In particular many details are sketchy and important results relegated to the exercises. Sharp [6] and Kaplansky [3] are decent accounts which fill in some of the gaps. Matsumura [4] is good for the homology theory but is hard going as an introduction. Reid's book [5] is a companion to his other book on algebraic geometry, so the topics and examples can be overly specific. As in the rest of the series, [2] is encyclopedic, but it's a bit like reading a car manual. [7] is a bit dense.

1. M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra, Addison–Wesley, 1969.
2. N. Bourbaki, Commutative algebra, Elements of Mathematics, Springer, 1989.
3. I. Kaplansky, Commutative rings, Allyn and Bacon 1970.
4. H. Matsumura, Commutative ring theory, Cambridge Studies 8, CUP, 1989.
5. M. Reid, Undergraduate commutative algebra, LMS Student Texts 29, CUP 1995.
6. R.Y. Sharp, Steps in commutative algebra, LMS Student Texts 19, CUP 1990.
7. O. Zariski and P. Samuel, Commutative algebra (2 volumes), Springer GTM series.

Representation Theory of Symmetric Groups (L24)

Stacey Law

The representation theory of symmetric groups is a classical subject with many connections across mathematics, computer science and physics. The rich interplay between algebra and combinatorics in the representation theory of symmetric groups has led to the development of a deep yet versatile theory, which has kept the topic at the forefront of modern research.

In this course, we will develop some of the techniques underpinning both the proof of the McKay Conjecture for symmetric groups, as well as other key results in the area from the last few years. Time permitting, we will cover:

- Combinatorics of partitions I: Young diagrams and tableaux.
- Specht modules and simple modules for symmetric groups.
- Standard basis for Specht modules.
- Hook length formula for dimensions.
- Combinatorics of partitions II: Olsson's approach.
- McKay numbers for symmetric groups.
- Representation theory of the Sylow subgroups of the symmetric groups.

Pre-requisites

Part IB Groups, Rings and Modules and Part II Representation Theory (or equivalent).

Literature

1. G. D. James, *The Representation Theory of the Symmetric Groups.* Lecture Notes in Mathematics **682**, Springer, Berlin, 1978.
2. G. D. James and A. Kerber, *The Representation Theory of the Symmetric Group.* Encyclopedia of Mathematics and its applications vol. 16, Addison-Wesley, 1981.
3. J. Olsson, *Combinatorics and Representations of Finite Groups*, Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen, Heft 20, 1994.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Profinite Groups and Group Cohomology (L24)

Gareth Wilkes

Profinite groups are groups which are the limit of a family of finite groups in a certain sense, and arise naturally in algebraic geometry, geometric group theory and Galois theory. As such they link together several regions of mathematics, being at the same time infinite groups and complete metric spaces, and exhibit properties of the theory of both finite and infinite groups. This course will introduce profinite groups and their basic properties. We will go on to explore their relations with classical groups and certain algorithmic questions: for instance, the question of deciding whether two given infinite groups are isomorphic or not.

We will also introduce the immensely powerful and versatile theory of group cohomology. As well as exhibiting further links between group theory and topology, cohomology provides computable invariants which bring out the finer structure of a group. We introduce this theory both for classical groups and profinite groups, and discuss the similarities and differences of the two theories.

Course synopsis:

- Inverse and direct limits of groups. Definitions and basic properties of profinite groups. Gaschutz's Lemma.
- Residual finiteness properties of groups. Profinite completions. Constructing finite quotients of free groups. Distinguishing discrete groups through their finite quotients ('profinite rigidity' questions).
- Pro-p groups and the p-adic integers. Frattini subgroups. Hensel's lemma. Serre's theorem on uniqueness of topology.
- Elementary group cohomology: definitions for both discrete groups and profinite groups. Comparison of theory for discrete groups vs pro-p groups. Extensions and semi-direct products. Groups of cohomological dimension one.

Pre-requisites

Part IB Groups, Rings and Modules and Part IB Metric and Topological Spaces are essential. Concepts from Part II Algebraic Topology will be referenced, but detailed knowledge of proofs is not necessary.

Literature

Notes and example sheets will be made available on the lecturer's webpage `https://www.dpmms.cam.ac.uk/~grw46/partiiiprofinite.html`.

1. K. Brown, *Cohomology of Groups.* Springer 1982.
2. L. Ribes and P. Zalesskii, *Profinite Groups.* Springer 2000.
3. J-P. Serre, *Galois Cohomology.* Springer 2013.
4. J.S. Wilson, *Profinite Groups.* Clarendon Press 1998.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Lie algebras, Vertex algebras, Shtuka (L24)

Non-Examinable (Graduate Level)

Ian Grojnowski

This is a course about semisimple Lie algebras and their generalisations.

I'll assume you are familiar with the basic structure and representation theory of the simple Lie algebras, and start again.

The point of view will be a strange mix of homotopical nonsense and down to earth calculation.

Hopefully some of the results and constructions will be new.

Pre-requisites

This course will be self contained, and proceed from the definitions. However the basics of Lie algebras are probably necessary. It would also help if you aren't scared by homological algebra—Weibel's book is a good basic reference, as is the lovely paper of Goerss-Schemmerhorn on model categories.

When (if?) we get to the chiral algebras, you will be asked to take on faith certain properties of algebraic curves, their moduli, and the stack of G-bundles on them. I'll give references in class.

Likewise for the characteristic p and spectral examples.

You are *not* supposed to have already looked at the papers and books listed below. Borcherd's paper is the definition of vertex algebras; Beilinson and Drinfeld's monograph is the definition of chiral homology. The rest are beautiful expository texts.

The last few years have seen some remarkable work on vertex algebras and quantum field theory, on D=4 N=2 SCFT, and D=3 N=4 gauged mirror symmetry. This work is related, but somewhat different to the intended topics.

Literature

Goerss, P. G., Schemmerhorn, K., Model Categories and Simplicial Methods, arXiv:math/0609537 (2006).

Weibel, C, An introduction to homological algebra, Cambridge Studies in Advanced Mathematics (38), 1994.

Borcherds, R. E., Vertex algebras, Kac-Moody algebras, and the Monster, Proc Natl Acad Sci USA 83, 3068–3071 (1986).

Beilinson, A, Drinfeld V, Chiral Algebras AMS Colloquium Publications, vol. 51, 2004

Frenkel, E., Ben-Zvi, D. Vertex Algebras and Algebraic Curves. Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, 2004.

Sofic Groups (L16)

Non-Examinable (Graduate Level)

Henry Bradford

A group is *sofic* if it can be "approximated" by finite groups, in a rather weak sense. In fact, the definition of soficity is so permissive that there is no group which has been proven not to satisfy it. The question of whether all groups are sofic is one of the major unsolved problems in group theory, as a positive answer to this question would also resolve famous open problems in dynamical systems; operator algebras, and ring theory. The first goal of this course is to explain what sofic groups are, and why soficity is an interesting and powerful property for a group to possess. The second goal, which will take up most of our time, is to indicate why finding a non-sofic group is so difficult. This will be achieved by showing that a wide range of natural group-theoretic conditions on a group imply soficity, which will in turn give us an opportunity to explore some of the awesome diversity of finitely generated infinite groups.

1. Definition of soficity and first properties. An application: either Gottschalk's Surjunctivity Conjecture or Kaplansky's Directed Finiteness Conjecture.
2. Amenable groups: definition via Følner sets and characterization in terms of invariant means; examples and closure properties; the Hausdorff-Banach-Tarski paradox and Tarski's Theorem; sofic-by-amenable groups are sofic; elementary amenability, word growth and Grigorchuk's first group.
3. Residual finiteness: examples including nilpotent and free groups; Abels' group as a non-example; Mal'cev's Theorem. Local embeddability: wreath products and symmetric enrichments; the space of marked groups; Cornulier's construction of a non-amenable isolated sofic group.
4. Construction of a candidate for a non-sofic group.

Pre-requisites

This course should be understandable to anyone with an advanced undergraduate-level background in group theory, and some familiarity with rudiments of measure theory and functional analysis.

Literature

1. K. Juschenko, *Sofic groups,*

   `https://web.ma.utexas.edu/users/juschenko/files/soficgroups.pdf`

2. V.G. Pestov, *Hyperlinear and Sofic Groups: A Brief Guide.* The Bulletin of Symbolic Logic, Vol. 14, No. 4 (Dec., 2008), pp. 449–480

Algebraic Geometry

Algebraic Geometry (M24)

Mark Gross

This will be a basic course introducing the tools of modern algebraic geometry. The most relevant reference for the course is the book of Hartshorne and the notes of Vakil.

The course will begin with a quick review of the theory of varieties as presented in the Part II algebraic geometry course (see e.g., the book of Reid for this background) and then proceeding to sheaves and the notion of an abstract variety. We then turn to an introduction to scheme theory, explaining why we want schemes and what they will do for us. We define schemes and introduce projective schemes. From there, we will pass to coherent sheaves, and introduce a number of tools, such as sheaf cohomology, necessary for any practicing algebraic geometer, with applications to problems in projective geometry.

Pre-requisites

Basic theory on rings and modules will be assumed. It is strongly recommended that students either have had a previous course on Commutative Algebra or had a quick read of the book on Commutative Algebra by Atiyah and MacDonald, and/or the elementary text by Reid on Algebraic Geometry.

Literature

Introductory Reading

1. M. Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press (1988) (preliminary reading).
2. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison–Wesley (1969) (basic text also for the commutative algebra we'll need).

Reading to complement course material

1. U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner, 2010.
2. R. Hartshorne, *Algebraic Geometry*, Springer (1977) (more advanced text).
3. R. Vakil, *The rising sea. Foundations of Algebraic Geometry*, available at http://math.stanford.edu/∼vakil/216blog/index.html

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Algebraic surfaces (L24)

Dhruv Ranganathan

This course will provide an introduction to the theory of complex algebraic surfaces. The geometry of algebraic surfaces is intricate but also accessible, and provides concrete ground in which to use the basic techniques of modern algebraic geometry, including bundles, sheaves, and birational methods.

The course will begin with a rapid review of the geometry of curves and their classification, followed by a study the intersection theory of curves on surfaces, Riemann–Roch for surfaces, rational and birational maps, and the geometry of blowups and contractions. The rest of the course will proceed by analyzing several examples carefully and explicitly. We will study del Pezzo, ruled, K3, and abelian surfaces in detail, and aim to sketch the classification of surfaces of general type. Each of these examples is an entry point into active modern areas of research, spanning the minimal model program, mirror symmetry, and hyperkähler geometry.

The course will stress examples and explicit calculations and is an appropriate follow up to a first course in schemes.

Pre-requisites

The basic pre-requisite is familiarity with the basics of algebraic varieties, line bundles, and cohomology, though there will be frequent reminders to much of this material. The Part II and Part III courses in algebraic geometry, or the equivalent, will be more than sufficient background. The course may also be suitable for students who have background in complex geometry but are less familiar with scheme theory.

Literature

1. A. Beauville, *Complex Algebraic Surfaces.* 2nd edition. London Mathematical Society Student texts 34,Cambridge University Press, 1996.
2. M . Reid, *Chapters on algebraic surfaces.* Available at `https://arxiv.org/abs/alg-geom/9602006`, 1996.

Additional support

Detailed handwritten lecture notes will be made available online.

Four example sheets will be provided with associated examples classes. There will be a revision class during the Easter term.

Geometric aspects of p-adic Hodge theory (M16)

Non-Examinable (Graduate Level)

Dr. Tamas Csige

p-adic Hodge theory is a p-adic counterpart of classical Hodge theory: it studies the natural structures found on the cohomology of algebraic varieties over a p-adic field. Some of these structures (such as a Galois action) arise from the arithmetic of the base field, while others (such as a Frobenius action) arise from the geometry of integral models. The aim of this course is twofold. On the one hand, we will discuss some of these structures, in particular prove the Hodge-Tate decomposition theorem. On the other hand, we aim to do so while touching some of the recent important theories and tools in this field, developed by B. Bhatt, M. Morrow and P. Scholze. In particular we will use the language of perfectoid spaces and we will give a short introduction to pro-étale cohomology.

Pre-requisites

This course assumes that you have some knowledge about étale cohomology, rigid analytic varieties, formal schemes and perfectoid spaces.

Literature

1. B. Bhatt, M. Morrow, P. Scholze, Integral p-adic Hodge theory *Publ. Math. Inst. Hautes Etudes Sci.* 128 (2018)

2. G. Faltings, p-adic Hodge theory *J. Amer. Math. Soc.* 1. (1988)
3. P. Scholze, Perfectoid spaces *Publl. Math. Inst. Hautes Etudes Sci.* 116 (2012)
4. P. Scholze, p-adic Hodge theory for rigid analytic varieties *Forum of Mathematics, Pi* Vol. 1 (2013)

Additional support

If there is interest, example sheets can be provided for this course.

Analysis and PDEs

An Introduction to Non Linear Analysis (M24)

Pierre Raphaël

This class in an introduction to the basic analytic tools needed for the mathematical study of nonlinear problems arising from mathematical physics (non linear wave equations, fluid mechanics). A particular emphasis will be made on the classical description of fundamental non linear waves discovered in the 19th century: solitons or solitary waves. The exact role of these bubbles of energy in many systems is still mysterious and the subject of an intense research activity. The last twenty years have seen spectacular progress in the understanding of the stability properties of these objects, and their role in the description of *singularity formation mechanisms.*

We will review the main classical methods at hand for the mathematical description of these objects, and build bridges with very recent developments on singularity formation problems both for nonlinear waves and fluid mechanics problems. Lecture notes will be available online.

Syllabus

1. Basic tool box of modern analysis

- L^p spaces, Hölder, complex interpolation.
- Convolution, Young and Hardy-Littlewood-Sobolev inequality.
- Hilbert spaces: weak convergence and compact operators.
- Sobolev spaces $H^s(\mathbb{R}^d)$: definition, Hilbertian structure.
- The Sobolev embedding and compactness.

2. A canonical model: the Non Linear Schrödinger equation

- The linear Schrödinger semi group in $\mathbb{R}^d$.
- Dispersion and Strichartz estimates.
- Virial identity, scattering and blow up.
- Open problems and recent progress.

3. Solitons: an introduction to variational methods

- Phase portrait in dimension 1.
- Variational approach: existence of a ground state.
- Introduction to spectral theory: the case of the harmonic oscillator.
- Bifurcation and the Lyapounouv Schmidt theorem.
- The stability problem: the concentration compactness lemma.
- Solitons in fluid mechanics: the example of vortices.

4. Singularity formation in non linear evolution equations

- The minimal mass blow up solution for (NLS).
- Merle's classification theorem (1992).
- Self similar blow up: recent progress and perspectives.
- The blow up problem in fluid mechanics: recent progress and perspectives.

Pre-requisites

Basic notions of functional analysis (Hilbert and Banach spaces). Basis notion of distributions theory ($\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$). Basic notion of continuous Fourier transform (Plancherel).

Literature

1. T. Cazenave : *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, **10**, NYU, CIMS, AMS 2003.
2. P. Raphaël: Concentration compacité à la Kenig-Merle, Séminaire Bourbaki, Exp. No. 1046, *Astérisque*, **352** (2013).
3. T. Tao: Nonlinear dispersive equations. Local and global analysis. CBMS Regional Conference Series in Mathematics, **106**, American Mathematical Society, Providence, RI, 2006.

Additional support

Examples sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Fractal Geometry and Additive Combinatorics(L16)

Han Yu

Fractal Geometry, roughly speaking, studies sets with complicated but interesting infinitesimal structures. In this course, we will mainly focus on fractals with self-similar structures. We shall see that many GMT(geometric measure theory) properties of such fractals boil down to subtle additive combinatorial and Diophantine properties of numbers. In the other direction, there are several number theoretic problems that can be represented as certain geometric properties of fractals. This allows us to study those number theoretic/additive combinatorial problems by understanding the geometry of fractal sets.

The course is divided into three parts. For the first part (around 3-4 lectures), I plan to cover some basics in fractal geometry including various notions of dimensions, self-similar sets, and basic geometric properties of fractal sets, for example, projections and slicings. For the second part (4-6 lectures), I will introduce several challenging open problems which are of combinatorial nature, and some basic techniques to approach them. For the third part (all the rest), I will explain a result of Bourgain on sum-product estimates of fractal sets. If time permits, we will continue going along Bourgain's route and obtain a recent result of Hochman on the Hausdorff dimension of self-similar sets.

Part of the following topics will be covered (depends on the schedule):

- Part 1 ($\leq$ 1 lecture per topic)

1 Self-similar sets, Hausdorff dimension, box counting dimensions

2 Marstrand's projection theorem and related results

3 Slicing properties of fractals

- Part 2 ($\leq$ 2 lectures per topic)

1 Kakeya problem, Falconer's distance problem

2 Furstenberg's slicing problem

3 Bernoulli convolutions

- Part 3

1 Bourgain's sum-product theorem for fractal sets.

2 (If we have enough time left) Hochman's inverse entropy theorem and its application to the study of self-similar sets

Pre-requisites

Basic mathematical analysis, basic geometric measure theory, basic ergodic theory. Actually, I will recall most of the definitions.

Literature

Lecture notes, once finished, will be avaiable on the lecture's webpage. The following two books cover most of the basic theory in part 1 as well as some materials in part 2.

1. K. Falconer *Fractal Geometry: Mathematical Foundations and Applications.* 2nd edition. John Wiley & Sons, 2003.
2. P. Mattila *Fourier Analysis and Hausdorff Dimension*, CUP, 2015.

For part 3, we will focus on the following two articles.

1. J. Bourgain *The discretized sum-product and projection theorems* Journal d'Analyse Mathématique (112), 193-236 (2010).
2. M. Hochman *On self-similar sets with overlaps and inverse theorems for entropy*, Annals of Mathematics (180), issue 2, 773-822, (2014).

Metric Embeddings (L24)

András Zsák

In the area of metric embeddings, one is mostly concerned with the following problem. Given metric spaces X and Y, is there a bi-Lipschitz embedding of X into Y, and what is the best distortion of such embeddings? In most situations of interest, X is a finite metric space and Y is a Banach space, particularly L_1, L_2 or a more general L_p space. Other types of embeddings, uniform and coarse embeddings, are also important. The aim of this course is to demonstrate the richness of this theory and the variety of the techniques (analytic, combinatorial and probabilistic) through a number of topics and major results in the field. We will also indicate the connections to other areas of mathematics (optimization, graph theory, computer science, geometry, Banach space theory). We aim to cover as many of the following topics as time permits.

Basic definitions and examples

Fréchet embeddings, Aharoni's theorem

L_1-embeddings and combinatorial optimization

Euclidean distortion, Dvoretzky's theorem, Bourgain's embedding theorem

Obstructions to embeddability: Poincaré inequalities, expander graphs

Dimension reduction in L_2, the Johnson–Lindenstrauss lemma

Impossibility of dimension reduction in L_1, Diamonds and Laakso graphs

Local theory of Banach spaces, Ribe's rigidity theorem, the Ribe programme, metric characterization of some Banach space properties

Metric theory of type and cotype, non-linear Dvoretzky theorem

Coarse embeddings into c_0 and ℓ_2; coarse embeddings of ℓ_2 into Banach spaces

Pre-requisites

Undergraduate level analysis, general topology, probability and functional analysis. In terms of the Cambridge Tripos, the Part IA Probability, the Part IB Analysis and Topology and the Part II Linear Analysis courses suffice for much of the course. In addition, knowledge of the basics of the weak and weak-star topologies in Banach spaces would be helpful.

Literature

1. Mikhail I. Ostrovskii *Metric Embeddings.* de Gruyter, 2013.

2. Jiří Matoušek *Lecture notes on metric embeddings.* online notes, 2013.

   `http://kam.mff.cuni.cz/~matousek/ba-a4.pdf`

Additional support

Up to four examples sheets will be provided and associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Combinatorics

Topics in Combinatorics (M16)

W. T. Gowers

This course will present a number of different results in order to showcase the wide variety of techniques, problems and results to be found in combinatorics, where "combinatorics" will be interpreted somewhat broadly. There will be an emphasis on results with proofs that surprised experts with their brevity or simplicity: often this means that the proof presented in the course will not be the first proof discovered. The content of the course will overlap substantially with the following topics.

1. Sperner's theorem and the Erdős-Ko-Rado theorem.
2. Well-separated familes of sets.
3. Székely's proof of the Szemerédi-Trotter theorem.
4. Solymosi's sum-product theorem.
5. Greene's short proof of the Kneser conjecture.
6. The Marcus-Tardos theorem.
7. Entropy and the number of P3s in a bipartite graph.
8. Dvir's solution to the finite-field Kakeya problem.
9. Croot, Lev, Pach, Ellenberg, Gijswijt, and the cap-set problem.
10. Levy's isoperimetric inequality.
11. An approximate Ramsey theorem for colourings of high-dimensional spheres.
12. Szarek's proof of the Kashin decomposition.
13. Huang's proof of the sensitivity conjecture.
14. Gurvits's simpler proof of the van der Waerden conjecture on permanents of doubly stochastic matrices.

Pre-requisites

There are few prerequisites beyond some knowledge of linear algebra and a familiarity with basic definitions connected with graphs.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Ramsey Theory (L16)

Prof. I. B. Leader

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. A typical example is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions.

The flavour of the course is combinatorial. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. We shall cover a number of 'classical' Ramsey theorems, such as Gallai's theorem and the Hales-Jewett theorem, as well as some more recent developments. There will also be several indications of open problems. We hope to cover the following material.

Monochromatic Systems

Ramsey's theorem (finite and infinite). Canonical Ramsey theorems. Colourings of the natural numbers; focusing and van der Waerden's theorem. Combinatorial lines and the Hales-Jewett theorem. Applications, including Gallai's theorem.

Partition Regular Equations

Definitions and examples. The columns property; Rado's theorem. Applications. (m, p, c)-sets and Deuber's theorem. Ultrafilters; the Stone-Čech compactification. Idempotent ultrafilters and Hindman's theorem.

Infinite Ramsey Theory

Basic definitions. Not all sets are Ramsey. Open sets and the Galvin-Prikry lemma. Borel sets are Ramsey. Applications.

Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey's theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness.

Literature

B. Bollobás, *Combinatorics*, C.U.P. 1986

R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, John Wiley 1990

Additive Combinatorics (L24)

Thomas Bloom

Additive combinatorics lies in the intersection of number theory and combinatorics, and studies the generic behaviour of simple arithmetic operations, such as addition and multiplication, on finite sets. For example, under what conditions can we guarantee that a set contains a three-term arithmetic progression? How does the size of the sumset $A + A$ differ from the size of A? How do addition and multiplication interact in sets of integers? It differs from other branches of combinatorics in that it introduces basic algebraic operations, and differs from other branches of number theory in that it rarely assumes much about the sets we are working in, other than very basic information such as the size of the set.

Despite the basic nature of the objects it studies, additive combinatorics as a field is relatively recent, and there have been many exciting recent developments. In this course our goal is to build up the basic theory of the subject from scratch, including some recent new techniques, so that we can discuss and prove some of the great theorems of the area, and engage with the latest research.

Topics that will be covered include

- Basic sumset estimates (such as the Balog-Szemerédi-Gowers lemma and the Plünnecke-Petridis bound),
- Fourier analysis in finite abelian groups, and how this can be used to prove Roth's theorem on three-term arithmetic progressions,
- Almost-periodicity, a probabilistic sampling technique,
- Long arithmetic progressions in sumsets, and
- The Freiman-Ruzsa inverse theorem and its quantitative improvements by Sanders.

Anyone who enjoys either combinatorics and/or number theory should find something of interest in this course.

Pre-requisites

There are no pre-requisites for this course.

Literature

Parts of this course are quite recent, and not yet present in any textbook. Complete PDF lecture notes will be provided by the lecturer. Some of the topics covered are discussed in the following books.

1. A. Geroldinger and I. Ruzsa, *Combinatorial Number Theory and Additive Group Theory.* Advanced Courses in Mathematics, 2009.
2. A. Granville, M. Nathanson, and J. Solymosi, *Additive Combinatorics.* CRM Proceedings and Lecture Notes, Volume 43, 2007.
3. T. Tao and V. Vu, *Additive Combinatorics.* Cambridge University Press, 2010.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Higher-order uniformity and applications (E12)

Non-Examinable (Graduate Level)

Julia Wolf

The discrete Fourier transform is a useful, classical tool that allows us to measure the extent to which a set of integers is uniformly distributed. Higher-order generalisations of the Fourier transform provide a more sensitive measure of uniform distribution, and first appeared in work of Gowers on Szemerédi's theorem in the late 1990s. They were further developed by Green and Tao (amongst others), culminating in their celebrated proof that the primes contain arbitrarily long arithmetic progressions.

In this course we shall examine the foundations of higher-order Fourier decompositions, the analogous regularity results for hypergraphs (generalising Szemerédi's famous graph regularity lemma from the 1970s), and some of their applications. In addition to the Green-Tao theorem, we shall cover recent progress of Peluse on polynomial patterns in dense subsets of the integers.

If time permits, we shall give an outline of the more abstract framework for higher-order Fourier analysis developed over the past decade by Szegedy and others.

Pre-requisites

This course assumes familiarity with the discrete Fourier transform and Szemerédi's regularity lemma for graphs. The Part III course *Additive Combinatorics* offered in the Lent term will be useful.

Literature

1. B.J. Green, *Montréal lecture notes on quadratic Fourier analysis.* Additive Combinatorics (Montréal 2006, ed. Granville et al.), CRM Proceedings vol. 43, AMS (2007), 69-102. Available at `https://arxiv.org/abs/math/0604089`.

2. W.T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi theorem.* Annals of Mathematics, 166 (2007), 897–946. Available at `https://arxiv.org/abs/0710.3032`.

3. J. Fox, D. Conlon, Y. Zhao, *The Green-Tao theorem: an exposition.* EMS Surv. Math. Sci. 1 (2014), 249-282. Available at `https://arxiv.org/abs/1403.2957`.

4. S. Peluse, *On the polynomial Szemerédi theorem in finite fields*, Duke Math. J. 168, no. 5 (2019), 749-774. Available at `https://arxiv.org/abs/1802.02200`.

Additional support

Two examples sheets will be provided and associated office hours will be offered.

Differential Geometry and Topology

Algebraic Topology (M24)

Ivan Smith

Algebraic Topology permeates modern pure mathematics and theoretical physics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover singular and cellular (co)homology; degrees of maps and cup-products; cohomology with compact supports and Poincaré duality; and Thom isomorphism and the Euler class. The course will not specifically assume any knowledge of algebraic topology, but will go quite fast in order to reach more interesting material, so some previous exposure to chain complexes (e.g. simplicial homology) would certainly be helpful.

Pre-requisite Mathematics

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. Some knowledge of the fundamental group would be helpful though not a requirement. Hatcher's book and Bott and Tu's book are especially recommended for accompanying the course, but there are many other suitable texts.

Level: Basic

Literature

1. Bott, R. and Tu, L. *Differential forms in algebraic topology.* Springer, 1982.
2. Hatcher, A. *Algebraic Topology.* Cambridge Univ. Press, 2002.
3. May, P. *A concise course in algebraic topology.* Univ. of Chicago Press, 1999.
4. Sutherland, W. *Introduction to metric and topological spaces.* Oxford Univ. Press, 1999.

Additional support

The course will be accompanied by four questions sheets, which will involve applying the general theory to do explicit calculations and solve geometric problems.

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Differential Geometry (M24)

Jack Smith

Differential geometry is the study of manifolds—spaces built from smoothly gluing together open sets in Euclidean space—and structures that live on or in them. The goal of this course is to introduce the main ideas on both the abstract conceptual ('coordinate-free') level and the concrete computational ('in coordinates') level, and to develop fluency in passing between them. This will lay the foundation for future study in geometry and topology, and provide the language for modern theoretical physics. Throughout the emphasis will be on building up geometric intuition. Topics will include:

- Manifolds, tangent and cotangent spaces, smooth maps and their derivatives. Tangent and cotangent bundles, tensors. Vector fields, flows, the Lie derivative.

- Differential forms, the exterior derivative, de Rham cohomology. Orientability. Integration and Stokes's theorem. Frobenius integrability.
- Lie groups and algebras. Principal bundles, connections (from multiple perspectives), curvature. Associated bundles, reduction of the structure group, vector bundles.
- Riemannian metrics, the Levi-Civita connection, geodesics and the exponential map. The Riemann tensor and its symmetries and contractions. The Hodge star, the Laplacian, statement of the Hodge decomposition.

Pre-requisites

Familiarity with point set topology (including compactness), multi-variable calculus (including the inverse function theorem), and linear algebra (including dual spaces and bilinear forms) is essential. No previous exposure to geometry will be assumed.

Literature

1. Liviu I. Nicolaescu, *Lectures on the geometry of manifolds.* 2nd edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.
2. John M. Lee, *Introduction to smooth manifolds.* 2nd edition. Graduate Texts in Mathematics, 218. Springer, New York, 2013.
3. (For a physics perspective) Mikio Nakahara, *Geometry, topology and physics.* 2nd edition. Graduate Student Series in Physics. Institute of Physics, Bristol, 2003.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Mapping class groups (L16)

Henry Wilton

Let S be a compact, orientable surface, with finitely many punctures and boundary components. The group of orientation-preserving self-homeomorphisms of S, $\mathrm{Homeo}^+(S)$, is too large to conveniently study, so we instead pass to the quotient

$$\mathrm{Mod}(S) = \mathrm{Homeo}^+(S)/\mathrm{Homeo}_0(S)$$

by factoring out the path component of the identity element. The resulting group – the *mapping class group* of S – is both tractable to study, and encodes a great deal of information about the topology and geometry of S. Mapping class groups are ubiquitous, appearing in subjects as diverse as algebraic geometry, combinatorial group theory, symplectic geometry, dynamics and 3-manifold topology.

This course introduces some of the basic techniques used to study mapping class groups, and applies them to compute examples and to prove some fundamental results. We will certainly cover the following topics.

1. The bigon criterion, which makes it possible to determine if two homeomorphisms are isotopic.
2. The simplest examples of mapping class groups, including the important case when S is a torus.
3. Twists, the most fundamental mapping classes of infinite order;
4. The complex of curves, which will enable us to prove that $\mathrm{Mod}(S)$ is finitely generated.

Pre-requisites

Part Ib Geometry is essential. Part II Algebraic Topology is essential. Part II Riemann Surfaces is useful. Part III Algebraic Topology, taken concurrently, is useful.

Literature

1. B. Farb and D. Margalit *A primer on mapping class groups.* Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012. xiv+472 pp.

Additional support

Printed notes are available online. Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Complex Dynamics (M16)

Holly Krieger

This course will introduce the study of iteration of rational functions of one complex variable. We will cover the local theory and the global theory, and introduce key modern ideas in the field to form a basis for further study.

Pre-requisites

Essential: IB Complex Analysis, IB Analysis and Topology.

Helpful: IB Geometry, II Algebraic Topology, II Dynamical Systems.

Literature

Milnor is the primary resource, though there will be some departure in presentation and material, particularly towards the end of the course.

1. J. Milnor *Dynamics in One Complex Variable.* Any version is fine, including the early online version available on the arXiv:

   `https://arxiv.org/pdf/math/9201272.pdf`

   Additional resources for students seeking more detailed or more accessible presentations of mostly the same material:

2. A. Beardon, *Iteration of Rational Functions.* Springer, 1991.
3. L. Carleson, T. W. Gamelin, *Complex Dynamics.* Springer, 1993.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term. The lecturer may be contacted by email at

`hkrieger@dpmms.cam.ac.uk`.

Please note that this course will differ in substantive ways from the Lent 2020 Complex Dynamics course. In particular, this course does *not* require Riemann surfaces as a pre-requisite, and will not include the study of quasiconformal maps.

Symplectic topology (L16)

Ailsa Keating

The study of symplectic manifolds originated as an extension of classical mechanics; it has since developed into a field of its own right, with connections to e.g. low-dimensional topology, algebraic geometry, and theoretical physics. The course will focus on the core foundations of symplectic topology, with an emphasis on explicit geometric techniques and examples.

Time allowing, topics are expected to include:

- Symplectic linear algebra. Hamilton's equations, cotangent bundles. Lagrangian submanifolds. Symplectic submanifolds. Moser's trick, Darboux and Weinstein neighbourhood theorems.
- Surgery constructions: blow ups, symplectic fibre sums. Lefschetz pencils. Gompf's theorem on fundamental groups of symplectic 4-manifolds.
- Almost complex structures and compatible triples. Some properties of Kaehler manifolds.

Pre-requisites

Essential: Algebraic Topology and Differential Geometry, at the level of the Part III Michaelmas courses. Basic concepts from Algebraic Geometry (at the level of the Part II course) will be useful.

Literature

1. A. Cannas da Silva, *Lectures on symplectic geometry*, Springer-Verlag, 2001.
2. D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd edition. Oxford University Press, 2017.
3. D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, 2nd edition. American Mathematical Society, 2012.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Homotopy Theory (L24)

Oscar Randal-Williams

This will be an advanced course in Algebraic Topology, first describing higher homotopy groups and their associated theory and then introducing two of the most important computational tools in this subject: spectral sequences and cohomology operations. After developing the basic theory we will focus on calculations and applications of these tools.

Topics that might be covered are

1. Higher homotopy groups; CW complexes and Whitehead's theorem; homotopy excision; the Hurewicz theorem; fibre bundles and fibrations; Eilenberg–MacLane spaces;
2. the spectral sequence of a filtered space; the Serre spectral sequence; example calculations; applications to computing homotopy groups;
3. Steenrod squares and the Steenrod algebra; example calculations; applications to Stiefel–Whitney classes, and vector fields on spheres.

Pre-requisites

Part III Algebraic Topology.

Literature

1. A. Hatcher *Algebraic Topology.* Cambridge University Press, 2002. Available at
   `http://pi.math.cornell.edu/~hatcher/AT/AT.pdf`
   as well as an online-only Chapter 5 at
   `http://pi.math.cornell.edu/~hatcher/AT/ATch5.pdf`
2. J. Strom *Modern Classical Homotopy Theory.* Graduate Studies in Mathematics, 127.
3. R. E. Mosher and M. C. Tangora *Cohomology operations and applications in homotopy theory.* Dover.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Foundations

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Category Theory (M24)

Prof. P.T. Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]

Locally small categories. The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]

Limits. Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

Filtered colimits. Finitary functors, finitely-presentable objects. Applications to universal algebra.

Regular categories. Embedding theorems. Categories of relations, introduction to allegories.

Abelian categories. Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.

Monoidal categories. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

Fibrations. Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Pre-requisites

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. S. Mac Lane *Categories for the Working Mathematician.* Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.
2. S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.
3. T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.
4. E. Riehl *Category Theory in Context.* Dover Publications 2016. A new account of the subject by someone who first encountered it as a Part III student a dozen years ago.
5. F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Model Theory (M24)

Gabriel Conant

Model theory is a branch of mathematical logic in which one studies classes of models of first-order theories. In the early 20th century, a driving question was the extent to which a mathematical structure can be characterized by its first-order properties. The discovery and application of the Compactness Theorem marked a fundamental failure in this pursuit, since it showed that, no matter how rich a language one chooses, any infinite first-order structure admits elementarily equivalent structures of arbitrarily large cardinality. This led to the study of categoricity and classification of models of first-order theories, which came to fruition in the 1960s and 70s. More recent applications of model theory to other areas of mathematics have shown that this consequence of the Compactness Theorem, which may have initially been viewed as a setback, is in fact an extremely powerful tool. Indeed, one can gain deep insights into a mathematical structure by working instead with an elementarily equivalent structure satisfying various desirable properties, which provide leverage over what may have initially appeared to be intricate or mysterious behavior in the original structure.

This course will cover a selection of topics from model theory, such as:

- complete theories and axiomatizations of structures,
- quantifier elimination and applications,
- type spaces and saturated models,
- the Omitting Types Theorem and applications,
- categoricity and the number of countable models,
- an introduction to stability theory.

Pre-requisites

Logic and Set Theory (part II); or equivalent. I assume familiarity with first-order languages and structures, the Compactness Theorem, and the Lowenheim-Skolem Theorems.

Literature

1. D. Marker *Model Theory: An Introduction.* Springer, 2002.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Infinite Games (L24)

Professor Benedikt Löwe

Infinite two-player perfect information games are connected to many topics in the foundations of mathematics: central concepts from analysis and topology can be reformulated in game-theoretic terms using infinite two-player perfect information games. Examples are the concepts of Lebesgue measurability, the property of Baire, as well as the perfect set property.

The central game-theoretic notion is the concept of *determinacy*: the full axiom of determinacy AD ("all infinite two-player perfect information games with natural number moves are determined") contradicts the axiom of choice AC, but definable fragments of AD can be proved in ZFC or extensions of ZFC. The axiom of determinacy itself yields an interesting alternative foundations of mathematics.

We shall treat several of the following topics:

Basic theory of determinacy. Applications in topology and measure theory. Incompatibility of AC and AD. Basics of descriptive set theory: the Borel hierarchy and the projective hierarchy.

Proving determinacy. Open determinacy. Low-level Borel games. Borel determinacy.

Proving determinacy from large cardinals. Introduction to large cardinals: inaccessible cardinals and measurable cardinals. Proving Π^1_1-determinacy from a measurable cardinal.

The axiom of determinacy. Combinatorial consequences: $\aleph_1$ and $\aleph_2$ are measurable. Infinite exponent partition relations.

Stronger axioms of determinacy. The axiom of real determinacy. Inconsistent extensions of the axiom of determinacy. Long games.

The Wadge hierarchy. Definition and structure theory of the Wadge hierarchy under AD.

A course webpage will be available at

`https://www.math.uni-hamburg.de/home/loewe/Lent2021/`.

Pre-requisites

The Part II course *Logic and Set Theory* or an equivalent course is essential. The Part Ib course *Metric and Topological Spaces* is useful.

Literature

1. Akihiro Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings.* Springer 2003 [Springer Monographs in Mathematics]

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Number Theory

Local Fields (M24)

Rong Zhou

The p-adic numbers $\mathbb{Q}_p$ were introduced by Hensel at the end of the 19th century and are now a ubiquitous tool in modern number theory as well as many other fields including algebraic topology, representation theory and algebraic geometry. The idea is to consider the completion of $\mathbb{Q}$ with respect to the absolute value defined by $|x|_p = p^{-n}$ for non-zero $x \in \mathbb{Q}$ where $x = p^n \frac{a}{b}$ with $a, b, n \in \mathbb{Z}$ and a, b coprime to p. The resulting field $\mathbb{Q}_p$ gives a neat way of packaging the information of congruences modulo n for all n and is the basic example of a local field. From this point of view, local fields are objects lying on the interface between algebra and analysis and the techniques used to study them involve an interesting mix of the two subjects.

This course covers the basic theory of local fields and is likely to be useful for students interested in studying other Part III courses on number theory such as Elliptic Curves. Topics to be covered will include:

Absolute values on fields; valuations; structure of local fields;

Extensions of complete fields; Galois theory; the different and discriminants;

Decomposition groups; Hensel's lemma; ramification theory;

Local class field theory (statements only).

Pre-requisites

Basic algebra up to and including Part II Galois theory as well as knowledge of concepts in point set topology and metric spaces are essential pre-requisites. It will be assumed that students have attended a first course in algebraic number fields.

Literature

1. J.W.S. Cassels, *Local fields*, Cambridge University Press, 1986.
2. J. Neukirch, *Algebraic number theory*, Springer–Verlag, 1999.
3. J. P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, 7. Springer–Verlag, 1973.
4. J. P. Serre, *Local fields*, Graduate Texts in Mathematics, 67. Springer–Verlag, 1979.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Algebraic Number Theory (L24)

Professor A J Scholl

In recent decades one of the most growing areas of research in number theory has been Arithmetic Algebraic Geometry, in which the techniques of algebraic number theory and abstract algebraic geometry are applied to solve a wide range of deep number-theoretic problems. These include the celebrated proof of Fermat's Last Theorem, work on the Birch-Swinnerton-Dyer conjectures, the Langlands Programme and the study of special values of L-functions. In this course we will study one half of the picture: Algebraic

Number Theory. I will assume some familiarity with the basic ideas of number fields, although these will be reviewed briefly at the beginning of the course.

Topics likely to be covered (not in order):

- Decomposition of primes in extensions, decomposition and inertia groups. Discriminant and different.
- Completion, adeles and ideles, the idele class group. Application to class group and units.
- Dedekind zeta function, analytic class number formula.
- Class field theory (statements and applications). L-functions.

Pre-requisites

Basic algebra up to and including Galois theory is essential. Familiarity with the Michaelmas term *Local Fields* course (or equivalent) will be assumed.

Exposure to number fields (at the level of the Part II course) is highly desirable.

Literature

1. J.W.S. Cassels and A. Frohlich: *Algebraic Number Theory.* London Mathematical Society 2010 (2nd ed.)
2. A. Frohlich, M.J. Taylor: *Algebraic Number Theory.* Cambridge, 1993.
3. J. Neukirch, *Algebraic number theory.* Springer, 1999.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Elliptic Curves (M24)

Tom Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points. The following topics will be covered, and possibly others if time is available.

Weierstrass equations and the group law. Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process.

Isogenies. The degree of an isogeny is a quadratic form. The invariant differential and separability. The torsion subgroup over an algebraically closed field.

Elliptic curves over finite fields. Hasse's theorem and zeta functions.

Elliptic curves over local fields. Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.

Elliptic curves over number fields. The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Pre-requisites

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. The first few lectures will include a review of the necessary geometric background, but some previous knowledge of algebraic curves (at the level of the Part II course *Algebraic Geometry* or the first two chapters of [3]) would be very helpful. Later in the course, some basic knowledge of the field of p-adic numbers will be assumed.

Preliminary Reading

1. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Literature

2. J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.
3. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Modular Forms (L24)

Professor J A Thorne

Modular forms are holomorphic functions on the complex upper half plane which are invariant under an action of the group $SL_2(\mathbf{Z})$ (or a finite index subgroup). They feature in many different parts of mathematics, but are most famous in number theory for their role in the proof of Fermat's Last Theorem, itself based on a proof of the Shimura–Taniyama conjecture concerning the modularity of elliptic curves over $\mathbf{Q}$.

In this course we will introduce the theory of modular forms from a number-theoretic point of view, including the theory of modular curves, Hecke operators, Eisenstein series, and the connection with L-functions.

Pre-requisites

Essential: IB Complex Analysis (or equivalent)

Useful: IB Groups, Rings & Modules, Part II Riemann Surfaces (or equivalent)

Literature

1. J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Maths. 7, Springer, New York, 1973
2. F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Maths. 228, Springer, New York, 2005
3. J. Milne, Modular Functions and Modular Forms, Lecture notes available at

   `https://www.jmilne.org/math/CourseNotes/MF.pdf`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Multiplicative functions (M16)

Aled Walker

The classical proof of the prime number theorem, using Cauchy's residue theorem from complex analysis, is a beautiful piece of mathematics. It is also a little troubling. How can we really claim to understand the properties of the integers under multiplication if we have to resort to such witchcraft as the residue theorem?

In recent years there has been a movement, spearheaded by Andrew Granville and Kannan Soundararajan but building on earlier work of many people, to consider an alternative approach to the subject, one which avoids the use of the residue theorem. This has involved turning the focus away from the primes themselves and focussing instead on multiplicative functions. The programme has had numerous successes, not just in reinterpreting pre-existing theorems but in proving extremely surprising new results on multiplicative functions themselves.

In this course we will try to cover the following topics from the modern theory of multiplicative functions (in greater or lesser detail, depending on time constraints):

- pretentious multiplicative functions and Halász's theorem;
- Granville–Soundararajan's improvement on the Pólya–Vinogradov inequality;
- the Matomäki–Radziwiłł theorem;
- Tao's proof of the logarithmically-averaged Chowla conjecture.

Pre-requisites

I will assume familiarity with basic undergraduate real and complex analysis, including a little harmonic analysis (essentially just the Fourier inversion formula). It is *not* a pre-requisite to have previously attended a first course in analytic number theory, but it will certainly be helpful to have done so, not least for putting the results of this course into their full context.

Literature

1. Granville, A. and Soundararajan, K., *Large character sums: pretentious characters and the Pólya–Vinogradov theorem.* Journal of the American Mathematical Society, **20**(2), 357-384 (2007).
2. Granville, A., Harper, A. and Soundararajan, K., *A more intuitive proof of a sharp version of Halász's theorem.* Proceedings of the American Mathematical Society, 146(10), pp.4099-4104 (2018).
3. Matomäki, K. and Radziwiłł, M., *Multiplicative functions in short intervals.* Annals of Mathematics 1015-1056 (2016).
4. Soundararajan, K., *The Liouville function in short intervals [after Matomäki and Radziwiłł].* Séminaire Bourbaki (2016), p.68ème.
5. Tao, T., *The logarithmically averaged Chowla and Elliott conjectures for two-point correlations.* Forum of Mathematics, Pi (Vol. 4). Cambridge University Press 2016.
6. Walker, A. *Lecture notes*, to appear, 2020

Diophantine Analysis (L24)

Peter Varju

This course will discuss two classical methods in Diophantine Analysis. The first one can be traced back to Thue, who proved the following result in Diophantine Approximation. Let α be an algebraic number of degree d. Then for any

$$\kappa > \frac{d}{2} + 1,$$

there is a constant c such that

$$\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^\kappa}$$

for all $r, s \in \mathbf{Z}$. This result is a significant improvement of Liouville's bound, in which the exponent is $\kappa = d$. Thue used his result to bound the number of solutions of certain Diophantine equations. The method has been subsequently improved by Siegel, Dyson and finally Roth, who achieved the optimal exponent $\kappa = 2 + \varepsilon$. Schmidt generalized Roth's result to the setting of a system of inequalities for linear forms, which is known as the Subspace Theorem.

The second method that will be discussed in the course goes back to the work of Baker. Gelfond and Schneider independently proved that α^β is transcendental whenever $\alpha \neq \{0, 1\}$ and β is an irrational algebraic number, which was Hilbert's seventh problem. This can be reformulated as

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0$$

for any non-zero algebraic numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$, provided $\log \alpha_1$ and $\log \alpha_2$ are linearly independent over the rationals. Baker generalized this result to linear forms in arbitrarily many logarithms, and, moreover, he gave lower bounds for the absolute value of such a form. These estimates have been revisited and improved on by many authors.

Both methods have been utilized by many authors for a wide range of applications in number theory and beyond. In the course, we will discuss Roth's theorem and some estimates for linear forms in logarithms, and we will sample from their applications.

Pre-requisites

Some knowledge of calculus, complex analysis, linear algebra and Galois theory will be assumed.

Literature

The course will not follow any particular source, but there are many excellent books that discuss some of the course material including the following.

1. A. Baker, *Transcendental number theory.* Cambridge University Press, Cambridge, 1990.
2. E. Bombieri and W. Gubler, *Heights in Diophantine geometry.* Cambridge University Press, Cambridge, 2006.
3. Y. Bugeaud, *Linear forms in logarithms and applications.* European Mathematical Society (EMS), Zürich, 2018.
4. J. W. S. Cassels, *An introduction to Diophantine approximation.* Hafner Publishing Co., New York, 1972.
5. D. Masser, *Auxiliary polynomials in number theory.* Cambridge University Press, Cambridge, 2016.

Probability

Advanced Probability (M24)

James Norris and Wei Qian

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

Review of measure and integration: sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini's theorem.
Conditional expectation: Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.
Martingales: Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.
Stochastic processes in continuous time: Kolmogorov's criterion, regularization of paths; martingales in continuous time.
Weak convergence: Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.
Sums of independent random variables: Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.
Brownian motion: Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.
Poisson random measures: Construction and properties; integrals.
Lévy processes: Lévy-Khinchin theorem.

Pre-requisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Literature

- Lecture notes online: `www.statslab.cam.ac.uk/~james/Lectures/ap.pdf`
- D. Applebaum, Lévy processes (2nd ed.), Cambridge University Press 2009.
- R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
- O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.
- D. Williams, Probability with martingales, CUP 1991.

Additional support

Four example sheets will be provided along with supervisions or examples classes. There will be a revision class in Easter term.

Mixing times of Markov chains (M16)

Perla Sousi

An ergodic Markov chain converges to its equilibrium distribution as time goes to infinity. But how long should one wait until the distribution is “close” to the invariant one? How many times should one shuffle a deck of cards until the order becomes uniform? This question lies at the heart of the modern theory of mixing times for Markov chains. The classical theory of Markov chains studied fixed chains and the focus was on large time asymptotics of their distribution. Recently the need to analyse large spaces has increased and the focus has shifted on studying asymptotics of the mixing time as the size of the state space tends to infinity. The area of mixing times is at the interface of mathematics, statistical physics and theoretical computer science.

In this course we will develop the basic theory and some of the main techniques and tools from probability and spectral theory used to estimate mixing times. We will apply them to study the mixing time of several chains of interest. We shall also discuss the *cutoff* phenomenon which was first discovered by Diaconis in the context of card shuffling and it says that a Markov chain converges to equilibrium abruptly. This phenomenon seems to be widespread but it remains a challenging question to obtain criteria for cutoff for general classes of chains.

Pre-requisites

This course assumes almost no background, except for prior exposure to Markov chains at an elementary level.

Literature

1. D. Levin and Y. Peres and E. Wilmer *Markov chains and Mixing Times.* American Mathematical Society, 2008.
2. D. Aldous and J. Fill, *Reversible Markov Chains and Random Walks on Graphs.* book in preparation available online at https://www.stat.berkeley.edu/∼aldous/RWG/book.html
3. R. Montenegro and P. Tetali, *Mathematical aspects of mixing times in Markov chains.* Foundations and Trends in Theoretical Computer Science: Vol. 1: No. 3, pp 237-354, 2006.

Percolation and Related Topics (M16)

Geoffrey Grimmett

This introductory course in discrete random geometry is centred around a number of processes of topical significance.

The percolation process is the simplest probabilistic model for a random medium in finite-dimensional space. It has a central role in the theory of mathematical disordered systems, with strong links to combinatorics and statistical mechanics. Amongst its connections of current importance are those to Schramm–Loewner evolutions (SLE), to the combinatorics of self-avoiding walks, and to the theory of phase transitions in physics.

The related topics may include self-avoiding walks, and further models from interacting particle systems including the contact model for the spread of infection, and (if time permits) certain physical models for the ferromagnet such as the Ising model.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

The following text will cover the majority of the course, and is available online.

1. Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2nd edn, 2018; see http://www.statslab.cam.ac.uk/∼grg/books/pgs.html

Additional support

Three examples sheets will be provided with three associated examples classes.

Random planar geometry (L16)

Jason Miller

This course will be an introduction to two-dimensional random geometric structures, both discrete and continuous.

The first part of the course will be on *random planar graphs.* Recall that a *tree* is a connected graph without cycles. A *plane tree* is a tree together with an embedding into the plane. Since there are only a finite number of plane trees with a fixed number of edges, one can pick one uniformly at random. We will discuss how random plane trees are related to random walks on $\mathbf{Z}$. We will also describe their continuous counterpart, the so-called *continuum random tree*, which is a random tree defined using Brownian motion. Plane trees turn out to serve as the basic building block for more elaborate geometric structures. One important example is a *planar map*, which is a graph together with an embedding into the plane so that no two edges cross. Since there are only a finite number of planar maps with a fixed number of faces, one can also talk about picking one uniformly at random. This is an example of a *random planar map.* The study of random planar maps goes back to work of Tutte in the 1960s in his attempt to prove the four color theorem. In recent years, random planar maps have been the subject of intense study, in part due to their deep connection with understanding different models in statistical mechanics (e.g., the percolation model and loop-erased random walk).

The second part of the course will be on the Schramm-Loewner evolution (SLE), which is a family of non-crossing curves in the plane indexed by a parameter $\kappa \geq 0$. SLE was introduced by Schramm in 1999 to describe the scaling limit of many different models in statistical mechanics (e.g., the percolation model and loop-erased random walk) in the same way that Brownian motion describes the scaling limit of simple random walk. SLE is defined in a very interesting way, combining ideas from complex analysis and probability. In this part of the course, we will introduce SLE and derive some of its basic properties.

Time permitting, we will discuss more advanced topics, such as the Gaussian free field and its connection with random planar maps and SLE.

Pre-requisites

Advanced probability. Stochastic calculus is a co-requisite.

Literature

1. G. Miermont *Aspects of random maps.* 2014 St Flour lecture notes.

   http://perso.ens-lyon.fr/gregory.miermont/coursSaint-Flour.pdf

2. J. Miller *Schramm-Loewner evolutions.*

   http://statslab.cam.ac.uk/~jpm205/teaching/lent2019/sle_notes.pdf

3. N. Berestycki and J.R. Norris, *Lectures on Schramm-Loewner evolutions.*

   http://www.statslab.cam.ac.uk/~james/Lectures/sle.pdf

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Stochastic Calculus and Applications (L24)

M. Tehranchi

This course is an introduction to the theory of continuous-time stochastic processes, with an emphasis on the central role played by Brownian motion. It complements the material in Advanced Probability.

- *Stochastic integration.* Martingales, local martingales and semi-martingales. Quadratic variation and co-variation. Itô's isometry and definition of stochastic integral. Kunita–Watanabe's theorem. Itô's formula.
- *Stochastic calculus for Brownian motion.* Lévy's characterization of Brownian motion. Dambis–Dubins–Schwartz theorem. Girsanov's theorem. Martingale representation theorems.
- *Stochastic differential equations.* Strong and weak solutions. Notions of existence and uniqueness. Yamada–Watanabe theorem. Strong Markov property. Kolmogorov, Fokker–Planck and Feynmann–Kac partial differential equations.
- *Applications.* Replication of contingent claims in finance. Elements of stochastic control.

Pre-requisites

Knowledge of measure theoretic probability at the level of Part III Advanced Probability will be assumed, especially familiarity with discrete-time martingales and basic properties of Brownian motion.

Literature

1. I. Karatzas and S. Shreve. *Brownian Motion and Stochastic Calculus.* Springer. 1998
2. D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer. 2001
3. L.C. Rogers and D. Williams.*Diffusions, Markov Processes and Martingales. Vol.1 and 2.* Cambridge University Press. 2002

Additional support

Four sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Information Theory (L16)

Ioannis Kontoyiannis

Information theory is the mathematical foundation of the ideas and tools required for the quantitative description and analysis of the notion of information. The origin of the theory in Claude Shannon's landmark 1948 paper is motivated by questions in communications engineering, but since then information theory has forged deep connections with many areas of mathematics, most notably with probability and statistics. This will be an introductory course to the main information-theoretic ideas, results, and techniques. We will discuss entropy as a measure of information, relative entropy as a natural distance between probability distributions, and mutual information as a universal dependence measure between random variables. Their properties will be established (monotonicity, chain rules, data processing inequalities, asymptotic equipartition) and Shannon's two main theorems will be proved: The source coding theorem that establishes the entropy as the fundamental limit for data compression, and the channel coding theorem which identifies the capacity as the fastest possible rate of reliable data transmission through a noisy channel.

Pre-requisites

The only pre-requisite is knowledge of basic probability, although a certain level of maturity and familiarity with the use of probabilistic techniques will be helpful. Knowledge of advanced (including measure-theoretic) probability is not necessary.

Literature

1. Mainstream introduction to information theory:
   T.M. Cover and J. Thomas. *Elements of Information Theory.* 2nd edition. Wiley-Interscience, 2006.

2. A more applied perspective:
   D. MacKay. *Information Theory, Inference, and Learning Algorithms.* Cambridge University Press, 2003. Available free online at: `https://www.inference.org.uk/itprnn/book.pdf`.

3. Theoretical connections with ergodic theory and probability...
   P. Billingsley. *Ergodic Theory and Information.* J. Wiley, 1965.

4. ... and: P. Shields. *The Ergodic Theory of Discrete Sample Paths.* American Mathematical Society, 1996.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistics

The courses in statistics form a coherent Masters-level course in statistics, covering statistical methodology, theory and applications. You may take all of them, or a subset of them. Core courses are Modern Statistical Methods and Applied Statistics in the Michaelmas Term.

All statistics courses for examination in Part III assume that you have taken an introductory course in statistics and one in probability, with syllabuses that cover the topics in the Cambridge undergraduate courses Probability in the first year and Statistics in the second year. It is helpful if you have taken more advanced courses, although not essential. For Applied Statistics and other applications courses, it is helpful, but not essential, if you have already had experience of using a software package, such as R or Matlab, to analyse data. The statistics courses assume some mathematical maturity in terms of knowledge of basic linear algebra and analysis. However, they are designed to be taken without a background in measure theory, although some knowledge of measure theory is helpful for Topics in Statistical Theory.

The desirable previous knowledge for tackling the statistics courses in Part III is covered by the following Cambridge undergraduate courses. The syllabuses are available online at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
First	*Essential*	Probability
Second	*Essential*	Statistics
	Helpful for some courses	Markov Chains
Third	*Helpful*	Principles of Statistics
	Helpful for applied statistics courses	Statistical Modelling
	For additional background	Probability and Measure

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation. If you have more time, then it would be helpful to review other courses as indicated.

Causal Inference (M16)

Qingyuan Zhao

From its onset, modern statistics engages in the problem of inferring causality from data. A common mindset is that causal inference is only possible using randomised experiments, but developments in statistics and related fields have shown that this view is oversimplified and restrictive. We now have a much better understanding of the assumptions and methodologies that enable causal inference from observational, non-experimental data. This course aims to cover some of the most fundamental ideas in causal inference, a vibrant research area where statistical theory meets scientific practice.

1. **Motivations:**
    - Principles of causal inference: motivations; historical perspectives; basic concepts.
    - Randomised experiments: randomisation tests, regression adjustment and its asymptotic inference.
    - Path analysis and linear structural equation models (SEMs);
2. **Languages for causality:**
    - Probabilistic directed acyclic graphical (DAG) models: Markov properties, d-separation, structure discovery.

- Counterfactual causal models: nonparametric SEMs; single-world intervention graphs; g-computation formula.
- Causal identification: back-door criterion, front-door criterion; counterfactual calculus; other examples.

3. **Design and statistical methods:**
   - Observed confounders: matching, randomisation inference, Rosenbaum's sensitivity analysis; semiparametric inference.
   - Instrumental variables (IV): core IV assumptions; generalised method of moments; principal stratification.
   - Other selected topics: regression discontinuity design; difference in differences and negative control methods; mediation analysis; longitudinal data and time-varying treatments; meta-analysis and evidence synthesis.

Pre-requisites

This course assumes familiarity with undergraduate-level probability and statistics.

Literature

1. Imbens, G. W. and Rubin, D. B. (2015) *Causal Inference in Statistics, Social, and Biomedical Sciences.* Cambridge University Press.
2. Hernán M. A. and Robins, J. M. (2020) *Causal Inference: What If.* Chapman & Hall.
3. Lauritzen, S. L. (1996). *Graphical Models.* Clarendon Press.
4. Angrist, J. D. and Pischke, J. S. (2008) *Mostly Harmless Econometrics: An Empiricist's Companion.* Princeton University Press.

Additional support

Lecture notes will be provided. Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

Robust Statistics (L16)

Po-Ling Loh

This is a topics course on the use of robustness in estimation and inference. We will begin with an overview of robustness theory from classical statistics, including notions such as the influence function and breakdown point, and the asymptotic theory of M-estimators. We will then survey a variety of more recent directions in machine learning and theoretical computer science where ideas in classical robustness theory have been leveraged and expanded. We will also briefly touch upon the topic of robust optimization and compare/contrast techniques from that field with statistical notions of robustness.

Below is a tentative list of topics:

- M-estimation: quantitative robustness and asymptotic theory; scale estimates; regression estimates
- Robust covariance estimation
- Robust hypothesis testing
- Robust high-dimensional estimation
- Estimation under adversarial contamination
- Robust optimization; distributional robustness

Pre-requisites

This course is appropriate for students with a general background in statistics. We will also assume proficiency in linear algebra and basic optimization. Some familiarity with machine learning will be helpful.

Literature

1. Huber and Ronchetti, *Robust Statistics*, 2011.
2. Hampel, Ronchetti, and Rousseeuw, *Robust Statistics: The Approach Based on Influence Functions*, 2011.
3. Maronna, Martin, and Yohai, *Robust Statistics: Theory and Methods*, 2006.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

Topics in Statistical Theory (M16)

Richard J. Samworth

This course will provide an introduction to the theory behind a selection of statistical problems that play a key role in modern statistics. Most undergraduate statistics courses are restricted to the study of parametric models; here we will no longer assume that our distributions belong to finite dimensional classes and will instead study fundamental nonparametric problems such the estimation of a distribution function, a density function or a regression function. We will consider the canonical machine learning problem of classification, and may also cover some extreme value theory including anologues of the Central Limit Theorem for the maxima and minima of a sample. Minimax lower bounds are studied as a way of quantifying the intrinsic difficulty of a statistical problem, and provide limits on how well any estimator can perform in a given situation.

A tentative outline of the course is as follows:

- An introduction to nonparametric statistics: the basics of empirical process theory, Glivenko–Cantelli theorem, Dvoretzky–Kiefer–Wolfowitz theorem, order statistics, quantile estimation and associated asymptotic distribution theory.

- Kernel density estimation: histograms, bias and variance expansions, aymptotically optimal bandwidth, canonical kernels, higher order kernels, bandwidth selection, multivariate density estimation.

- Nonparametric regression: kernel nonparametric regression, bias and variance expansions. Cubic splines, natural cubic smoothing splines, choice of smoothing parameter, other splines, equivalent kernel. Classification problems, the Bayes classifier, nearest neighbour classifiers.

- Minimax theory: notion of information-theoretic lower bounds, distance and divergence between distributions, optimal rates, Le Cam's two points lemma.

- Extreme value theory: the extremal types theorem, domains of attraction, max-stability.

Prerequisites

A good background in undergraduate probability theory, elements of linear algebra and real analysis. Measure theory is not necessary but may be helpful; similarly for a preliminary course in mathematical statistics. Though the material in the Modern Statistical Methods course will not be needed here, the two courses complement each other well.

Literature

No book will be explicitly followed, but some of the material is covered in

L. Devroye, L. Györfi, G. Lugosi, *A Probabilistic Theory of Pattern Recognition*, Springer 1996.

A. Tsybakov, *Introduction to Nonparametric Estimation*, Springer 2009.

M. J. Wainwright, *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*, Cambridge University Press, 2019.

Additional support

Three example sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Statistics in Medicine (3 units)

Statistics in Medical Practice (M12)

Statistics in Medicine (3 units)

Lecturers from the MRC Biostatistics Unit

This part of the course includes three modules covering a range of statistical methods and their application in three areas of biostatistics.

A. Stochastic Models for Chronic and Infectious Diseases [4 Lectures] **(C. Jackson, A. Presanis & D. De Angelis)**

Continuous-time multi-state and Markov models: properties and quantities of interest, and fitting models to individual disease history data. Applications to modelling the onset and progression of chronic diseases. Multi-state modelling to estimate incidence of infectious diseases from population-level prevalence data. Backcalculation methods for the estimation of incidence of disease with long incubation periods. Dynamic modelling of infectious disease transmission.

B. Causal Inference [4 Lectures] **(S. Burgess)**

It is well known that "correlation is not causation". But how then do you assess causal claims? Is it possible to show that X is a cause of Y? What does it even mean to say that X is a cause of Y? In this module, we introduce definitions of causal concepts, starting with the work of Rubin, Pearl, and Robins, and discuss practical approaches for assessing causal claims from observational data.

C. Design and Analysis of Randomised Trials [4 Lectures] **(M. Pilling, D. Robertson & S. Seaman)**

Sample size estimation for clinical trials; group-sequential designs and treatment effect estimation following a group-sequential trial. Adaptive and multi-stage designs. Types of randomisation procedures. Non-parametric and parametric response-adaptive procedures. Handling missing data: classification of missingness mechanisms, maximum likelihood, and multiple imputation.

Analysis of Survival Data (L12)

Statistics in Medicine (3 units)

P. Treasure

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

D. Time-to-Event Analysis [4 Lectures]

'Survival analysis' is generalised to *time-to-event* analysis. The implications of event times which are unknown or in the future (*censored* data) are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (*Kaplan-Meier*) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (*log-rank* test).

E. Modelling Hazard [4 Lectures]

The *hazard* function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size...) on the time-to-event distribution (*proportional hazards* modelling). Model checking procedures are introduced with an emphasis on excess event (*Martingale*) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (*competing risk* analysis) and (ii) the cause of death is unknown (*net survival*). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar time (e.g. 2017) are addressed: *period* survival analysis.

Additional Information

Statistics in Medicine (3 units)

Pre-requisites

Undergraduate-level statistics and probability: including analysis and interpretation of data, maximum likelihood estimation, hypothesis testing, basic stochastic processes.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. A few books to complement the course material are listed below.

1. Armitage P, Berry G, Matthews JNS, *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001. [A good introductory companion to the whole course]
2. van den Hout, A, *Multi-State Survival Models for Interval-Censored Data.* Chapman and Hall, 2016 [Module A]
3. Keeling, M. J., & Rohani, P. *Modeling Infectious Diseases in Humans and Animals.* Princeton University Press, 2008 [Module A]
4. Burgess S, Thompson SG, *Mendelian Randomization: Methods for Using Genetic Variants in Causal Estimation* Chapman and Hall, 2015 [Module B]
5. Senn, S. *Statistical Issues in Drug Development.* Wiley, 2007. [Module C]
6. Jennison C, Turnbull B, *Group Sequential Methods with Applications to Clinical Trials.* Chapman and Hall, 2000. [Module C]
7. Cox DR, Oakes D, *Analysis of Survival Data.* Chapman and Hall, 1984 [Modules D, E, F: the classic text]
8. Collett D, *Modelling Survival Data in Medical Research.* CRC Press, 2015 [Modules D, E, F: modern, applied, supports and extends lectures.]

9. Aalen OO, Borgen Ø, Gjessing HK, *Survival and Event History Analysis: A Process Point of View.* Springer, 2008 [Modules D, E, F: excellent modern approach]

Additional support

[Modules A, B, C] A two-hour example class, supported by question sheets and solutions, will be given at the start of Lent term.

[Modules D, E, F] A two-hour example class, supported by question sheets and solutions, will be given in each of the Lent and Easter Terms. A two-hour revision class will be held just before the examination.

Functional Data Analysis (L16)

John Aston

Functional Data Analysis (FDA) is the study of the statistics of curves and surfaces. In most statistical analyses, the object of study is either a number (one dimensional univariate statistics) or a vector (finite dimensional multivariate statistics). In FDA the object of study is usually considered to be infinite dimensional, often but not always, with some inherent smoothness characteristics. In this course, we will examine both the theory behind such data analysis and the myriad of different places that these data arise.

The course will likely cover the following topics:

- Definition of Functional Data
- Derivatives in Functional Data
- Functional Linear Models
- Functional Non-Parametric Data Analysis

Pre-requisites

Basic knowledge of statistics, probability, linear algebra and real analysis.

Literature

1. Ramsay, J.O. and Silverman B.W. *Functional Data Analysis* 2nd edition, Springer, 2005.
2. Horvath, L. and Kokoszka, P. *Inference for Functional Data with Applications*, Springer, 2012
3. Kokoszka, P. and Reimherr, M. *Introduction to Functional Data Analysis*, Chapman and Hall, 2017

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Astrostatistics (L24)

Kaisey Mandel

This course will cover applied statistical methods necessary to properly interpret today's increasingly complex datasets in astronomy. Particular emphasis will be placed on principled statistical modeling of astrophysical data and statistical computation of inferences of scientific interest. Statistical techniques, such as Bayesian inference, sampling methods, hierarchical models, Gaussian processes, and model selection, will be examined in the context of applications to modern astronomical data analysis. Topics and examples will be motivated by case studies across astrophysics and cosmology.

Pre-requisites

Students of astrophysics, physics, statistics or mathematics are welcome. Astronomical context will be provided when necessary. Students without a previous statistics background should familiarise themselves with the material in Feigelson & Babu, Chapters 1-4, and Ivezić, Chapters 1, 3-5, by the beginning of the course. (Note that the two textbooks cover many of the same topics). These texts are freely available online to Cambridge students via the library website.

Literature

1. E. Feigelson and G. Babu. *Modern statistical methods for astronomy: with R applications.* Cambridge University Press, 2012.
2. Z. Ivezić, A. Connolly, J. VanderPlas & A. Gray. *Statistics, Data Mining, and Machine Learning in Astronomy.* Princeton University Press, 2014.
3. C. Schafer. *A Framework for Statistical Inference in Astrophysics.* 2015, Annual Review of Statistics and Its Application, 2: 141-162.
4. E. Feigelson, et al. *21st Century Statistical and Computational Challenges in Astrophysics.* 2020, Annual Review of Statistics and Its Application, 8.
   `https://arxiv.org/abs/2005.13025`
5. C. Bishop. *Pattern Recognition & Machine Learning.* Springer-Verlag, 2006.
   Also available at:
   `https://www.microsoft.com/en-us/research/people/cmbishop/#!prml-book`
6. D. MacKay. *Information Theory, Inference, and Learning Algorithms.* Cambridge University Press, 2003. Also available at:
   `http://www.inference.org.uk/mackay/itila/book.html`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistical Learning in Practice (L24)

Alberto J. Coca

Statistical learning is the process of using data to guide the construction and selection of models, which are then used to predict future outcomes. In this course, consisting of roughly 12 lectures and 12 practical classes, we will examine some of the most successful and widely used statistical methodologies in modern applications. The practical classes will deal with an introduction to R, exploratory data analysis and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models for regression and classification
- Model selection and regularisation
- Mixed effects models and quasi-likelihood methods
- Linear discriminant analysis and support vector machines
- Introduction to neural networks
- Introduction to time series

Pre-requisites

Elementary probability theory. Maximum likelihood estimation, hypothesis tests and confidence intervals. Linear models.

Previous experience with R is helpful but not essential.

Literature

1. Agresti, A. (2015) *Foundations of Linear and Generalized Linear Models.* Wiley.
2. Brockwell, P.J. and Davis, R.A. (1996) *Introduction to time series and forecasting.* Springer.
3. Dobson, A.J. and Barnett A. (2008) *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC.
4. Faraway, J. J. (2005) *Extending the linear model with R: generalized linear, mixed effects and non-parametric regression models.* CRC press.
5. Hastie, T., Tibshirani, R. and Friedman, J. (2009) *The Elements of Statistical Learning.* Second Edition. Springer.
6. Shumway, R. H., and Stoffer, D. S. (2010) *Time Series Analysis and Its Applications: with R Examples.* Springer Science & Business Media.

Additional support

This course includes practical classes, in which statistical methods are introduced in a practical context and students carry out analysis of datasets using R. In the practical classes, the students have the opportunity to discuss statistical questions with the lecturer. Four examples sheets will be provided and there will be four associated examples classes (the last one probably to be held in Easter term). There will be a revision class in the Easter Term.

Statistics in Medicine (3 units)

Statistics in Medical Practice (M12)

Statistics in Medicine (3 units)

Lecturers from the MRC Biostatistics Unit

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B. Causal Inference [4 Lectures] **(S. Burgess)**

It is well known that "correlation is not causation". But how then do you assess causal claims? Is it possible to show that X is a cause of Y? What does it even mean to say that X is a cause of Y? In this module, we introduce definitions of causal concepts, starting with the work of Rubin, Pearl, and Robins, and discuss practical approaches for assessing causal claims from observational data.

C. Design and Analysis of Randomised Trials [4 Lectures] **(M. Pilling, D. Robertson & S. Seaman)**

Sample size estimation for clinical trials; group-sequential designs and treatment effect estimation following a group-sequential trial. Adaptive and multi-stage designs. Types of randomisation procedures. Non-parametric and parametric response-adaptive procedures. Handling missing data: classification of missingness mechanisms, maximum likelihood, and multiple imputation.

Analysis of Survival Data (L12)

Statistics in Medicine (3 units)

P. Treasure

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

D. Time-to-Event Analysis [4 Lectures]

'Survival analysis' is generalised to *time-to-event* analysis. The implications of event times which are unknown or in the future (*censored* data) are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (*Kaplan-Meier*) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (*log-rank* test).

E. Modelling Hazard [4 Lectures]

The *hazard* function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size...) on the time-to-event distribution (*proportional hazards* modelling). Model checking procedures are introduced with an emphasis on excess event (*Martingale*) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (*competing risk* analysis) and (ii) the cause of death is unknown (*net survival*). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar year (e.g. 2019) are addressed: *period* survival analysis.

Additional Information

Statistics in Medicine (3 units)

Pre-requisites

Undergraduate-level statistics and probability: including analysis and interpretation of data, maximum likelihood estimation, hypothesis testing, basic stochastic processes.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. A few books to complement the course material are listed below.

1. Armitage P, Berry G, Matthews JNS, *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001. [A good introductory companion to the whole course]

2. van den Hout, A, *Multi-State Survival Models for Interval-Censored Data.* Chapman and Hall, 2016 [Module A]

3. Keeling, M. J., & Rohani, P. *Modeling Infectious Diseases in Humans and Animals.* Princeton University Press, 2008 [Module A]

4. Burgess S, Thompson SG, *Mendelian Randomization: Methods for Using Genetic Variants in Causal Estimation* Chapman and Hall, 2015 [Module B]

5. Senn, S. *Statistical Issues in Drug Development.* Wiley, 2007. [Module C]

6. Jennison C, Turnbull B, *Group Sequential Methods with Applications to Clinical Trials.* Chapman and Hall, 2000. [Module C]

7. Cox DR, Oakes D, *Analysis of Survival Data.* Chapman and Hall, 1984 [Modules D, E, F: the classic text]

8. Collett D, *Modelling Survival Data in Medical Research.* CRC Press, 2015 [Modules D, E, F: modern, applied, supports and extends lectures.]

9. Aalen OO, Borgen Ø, Gjessing HK, *Survival and Event History Analysis: A Process Point of View.* Springer, 2008 [Modules D, E, F: excellent modern approach]

Additional support

[Modules A, B, C] A two-hour example class, supported by question sheets and solutions, will be given at the start of Lent term.

[Modules D, E, F] A two-hour example class, supported by question sheets and solutions, will be given in each of the Lent and Easter Terms. A two-hour revision class will be held just before the examination.

Particle Physics, Quantum Fields and Strings

The courses on Symmetries, Fields and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of the course Symmetries, Fields and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
Second	*Essential*:	Quantum Mechanics, Methods, Complex Methods.
	Helpful:	Electromagnetism.
Third	*Essential*:	Principles of Quantum Mechanics, Classical Dynamics.
	Very helpful:	Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Field Theory (M24)

N.Dorey

Quantum Field Theory is the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics.

Interactions betweeen fields are examined next, using the interaction picture, Dyson's formula and Wick's theorem. A 'short version' of these techniques is introduced: Feynman diagrams. Decay rates and interaction cross-sections are introduced, along with the associated kinematics and Mandelstam variables.

Spinors and the Dirac equation are explored in detail, along with parity and γ^5. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions.

Finally, quantum electrodynamics (QED) is developed. A connection between the field strength tensor and Maxwell's equations is carefully made, before gauge symmetry is introduced. Lorentz gauge is used as an example, before quantisation of the electromagnetic field and the Gupta-Bleuler condition. The interactions between photons and charged matter is governed by the principal of minimal coupling. Finally, an example QED cross-section calculation is performed.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Literature

1. D. Tong, *Lectures on Quantum Field Theory*
   `http://www.damtp.cam.ac.uk/user/tong/qft.html` videos of lectures and printed lecture notes have a large overlap with the current course

2. B.C. Allanach, *Cross Sections and Decay Rates* printed lecture notes `3P11` from
   `http://www.damtp.cam.ac.uk/user/examples/indexP3.html`

3. T. Lancaster and S.J. Blundell, *Quantum field theory for the gifted amateur*, Oxford University Press (2015) is an introductory text that Part III students have been finding useful.

4. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996) is a classic, and also covers aspects of the Standard Model.

5. A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2010) gives a modern take with a lot of physical intuition, possibly taking the subject into topics more theory-specialised and advanced than the references above.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. One revision lecture will be given in Easter term.

Symmetries, Fields and Particles (M24)

Nick Dorey

Lie groups and Lie algebras are important in the construction of quantum field theories which describe interactions between known particles. Gauge theories, which describe many of the interactions in the Standard Model, rely on them. After some other preliminaries, we introduce representations in terms of square matrices. The group of rotations in three-dimensional space $SO(3)$ is covered, along with $SU(2)$ and the connection to angular momentum. Relativistic symmetries are discussed: in particular, the Lorentz and Poincaré groups and quantum fields. Lie groups and Lie algebras are covered in more generality, focusing on $SU(3)$ as a useful example. An overview of the results of the Cartan classification of simple Lie algebras is included. Finally, gauge theory is introduced.

Pre-requisites

Linear algebra including direct sums and tensor products of vector spaces. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices.

Literature

1. *Lie Algebras in Particle Physics*, H. Georgi, Westview Press, 1999.
2. *Representations and Physics* 2nd edition, Taylor and Francis, 1998.

Additional support

Four examples sheets will be provided and associated examples classes will be given by graduate students. There will be one office hour per week during term to ask questions of the lecturer. There will be a one-hour revision class in the Easter Term.

Statistical Field Theory (M16)

Christopher Thomas

This course introduces the renormalization group, focusing on statistical systems such as spin models with further connections to quantum field theory.

After introducing the Ising Model, Landau's mean field theory is introduced and used to describe phase transitions. The extension to the Landau-Ginzburg theory reveals broader aspects of fluctuations whilst consolidating connections to quantum field theory. At second order phase transitions, also known as 'critical points', renormalisation group methods play a starring role. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems.

Pre-requisites

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Literature

1. J M Yeomans *Statistical Mechanics of Phase Transitions.* Clarendon Press (1992).
2. M Le Bellac, *Quantum and Statistical Field Theory* Oxford University Press (1991).
3. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992).
4. M Kardar, *Statistical Physics of Fields*, Cambridge University Press (2007).
5. D Amit and V Martin-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005).
6. L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press (1996).
7. N Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Addison-Wesley (1992).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will also be a revision class in Easter Term.

Non-Equilibrium Statistical Field Theory (M8)

Non-Examinable (Part III Level)

Johannes Pausch

This course introduces Master Equations – an ODE for time dependent probability distributions – as a model for reaction systems on lattices. After briefly exploring its basic properties and models such as random walkers as well as branching and coagulation processes, they are transformed into a second-quantized form using bosonic ladder operators and coherent states. The meaning and properties associated with shifts of the creation and annihilation operators are explored as well as the interpretation of observables. Then, the second quantized equation is formally solved in a path integral. At this point, Doi-Peliti field theory starts and a few example models are presented: diffusion, branching, coagulation as well as coupled reactions. Feynman diagrams, loop corrections, n-point correlation functions are all concepts which are introduced and Part III students taking *SFT* will hopefully enjoy a non-equilibrium point of view on them.

Next, the lecture jumps back to pre-field-theory models and introduces Langevin Equations with a few example processes such as the Ornstein-Uhlenbeck process. This is done in order to derive a different field theory, called Response field formalism, based on work by Martin, Siggia, Rose, Janssen and DeDominicis. The previously described examples are now revisited as field theories and, if time allows, Doi-Peliti field theory is brought back to be combined with Response Field theories in the role of prior distributions.

Pre-requisites

It will be very beneficial to take the Part III course *Statistical Field Theory* (*SFT*) alongside this course. Although both courses talk about different statistical field theories, they explore many of the same concepts.

Literature

1. N.G. van Kampen, *Stochastic Processes in Physics and Chemistry* Elsevier, 1992
2. Uwe Täuber *Critical Dynamics* 1st edition. Cambridge University Press, 2014.
3. John Cardy, *Lecture Notes on Field Theory and Nonequilibrium Statistical Mechanics*, Available at

   `https://www-thphys.physics.ox.ac.uk/people/JohnCardy/`
4. Gunnar Pruessner *Lecture Notes on Non-equilibrium Statistical Mechanics* Available at

   `http://wwwf.imperial.ac.uk/~pruess/publications/Gunnar_Pruessner_field_theory_notes.pdf`

Additional support

Since this is a non-examinable course, there will be no example classes and no revision sessions. However, voluntary example sheets might be provided.

Advanced Quantum Field Theory (L24)

M B Wingate

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalization of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantizing a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study renormalization. Wilson's picture of renormalization is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the renormalization group (RG) flow. The course explores renormalization systematically using dimensional regularization in perturbative loop integrals. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as "asymptotic freedom," this phenomenon revolutionized our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrize possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful. There is some overlap with Statistical Field Theory.

References

1. Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).
2. Srednicki, M., *Quantum Field Theory*, CUP (2007).
3. Schwarz, M., *Quantum Field Theory and the Standard Model*, CUP (2014).
4. Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).

Additional support

There will be four problem sheets handed out during the course. Classes for each of these sheets will be arranged during Lent term (the 4th class will be scheduled for Easter term). There will also be a general revision class during Easter term.

The Standard Model (L24)

Fernando Quevedo

The Standard Model of particle physics is, by far, the most successful application of quantum field theory (QFT). At the time of writing, it accurately describes all experimental measurements involving strong, weak, and electromagnetic interactions. The course aims to demonstrate how this model, a QFT with gauge group $SU(3) \times SU(2) \times U(1)$ and fermion fields for the leptons and quarks, is realised in nature. It is intended to complement the more general Advanced QFT course.

We begin by describing the important role of symmetries in relativistic quantum physics and quantum field theory. We start with spacetime symmetries including representations of the Poincaré group and discrete symmetries (C,P,T). Then move to non-abelian gauge symmetries. Ideas of spontaneous symmetry breaking are applied to discuss Goldstone's theorem and the Higgs mechanism. We later apply these concepts to describe the weak interactions and their unification with electromagnetic interactions and also Quantum Chromodynamics (QCD) that describes strong interactions in terms of an $SU(3)$ gauge theory.

We put all this together to define the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content (spin-half leptons and quarks, spin-one gauge bosons and spin-zero Higgs boson).

Throughout the lectures, general consistency and fundamental properties such as the structure of gauge anomalies and asymptotic freedom will be highlighted. Also, phenomenological properties of the standard model will be introduced such as the Cabibbo-Kobayashi-Maskawa (CKM) mixing, the Glashow-Iliopoulos-Maiani (GIM) mechanism, neutrino oscillations, etc.

Both very high-energy experiments and very precise experiments are currently striving to observe effects that cannot be described by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model. General open questions for the standard model will be discussed at the end of the course.

General Outline

1. Introduction and History
2. Spacetime Symmetries
3. Internal Symmetries
4. Broken Symmetries
5. Weak Interactions: Electroweak unification
6. Strong Interactions: QCD
7. The Standard Model and Effective Field Theories.

Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or to be familiar with the material covered in them. It would be advantageous to attend the Advanced QFT course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

Literature

1. M.D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press (2014).
2. S. Weinberg, *The Quantum Theory of Fields, Volume 1,2*, Cambridge University Press (1995).
3. C.P. Burgess and G. Moore, *The Standard Model: A Primer,* Cambridge University Press (2007).
4. C.P. Burgess *Effective Field Theories,* Cambridge University Press (2020).
5. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1995).
6. F. Halzen and A.D. Martin, *Quarks and Leptons: An Introductory Course in Modern Particle Physics*, Wiley (1984).
7. I.J.R. Aitchison and A.J.G. Hey, *Gauge Theories in Particle Physics*, CRC Press (two volumes or earlier 1989 edition in one volume).
8. J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model*, Cambridge University Press (2014).

9. H. Georgi, *Weak Interactions and Modern Particle Theory*, Benjamin/Cummings (1984).

10. T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics*, Oxford University Press (1984).

11. M. Thomson, *Modern Particle Physics*, Cambridge University Press (2013).

Additional support

Four example sheets will be provided and four associated examples classes will be given. There will also be a revision class in Easter Term.

String Theory (L24)

R A Reid-Edwards

String theory is the quantum theory of interacting one-dimensional extended objects (strings). What makes the theory so appealing is that it is a quantum theory that contains gravitational interactions and therefore provides the first tentative steps towards a full quantum theory of gravity. It has become clear that string theory is also much more than this. It has become a framework in which to understand problems in quantum field theory, to ask meaningful questions about what we expect from a quantum theory of gravity, and as a crucible for new ideas in mathematics.

This course provides an introduction to String Theory. We begin by generalising the worldline of a particle to the two-dimensional surface swept out by a string. The quantum theory of the embedding of these surfaces in spacetime is governed by a two-dimensional quantum field theory and we shall study the simplest example - the bosonic string - in detail.

An introduction to relevant ideas in Conformal Field Theory (CFT) will be given. The quantisation of the string will be studied, its spectrum obtained, and the relationship between states on the two dimensional CFT and fields in spacetime will be discussed. We will see the necessity of the critical dimension of spacetime.

The path integral approach to the theory will be discussed. Fadeev-Popov and BRST methods will be introduced to deal with the redundenceies that appear in the theory. Vertex operators will be introduced and scattering amplitudes will be computed at tree level. Perturbation theory at higher loops and the role played by moduli space of Riemann surfaces will be sketched.

The course will focus on closed strings but time permitting, open strings and the role of D-branes may be discussed. There may also be some discussion of more stringy phenomena such as symmetry enhancement and duality.

Pre-requisites

Knowledge of the Quantum Field Theory course in Michaelmas term is assumed. Advanced Quantum Field Theory will complement this course but will not be assumed.

Literature

1. Polchinski, *String Theory: Vol. 1: An Introduction to the Bosonic String*, CUP 1998

2. Green, Schwarz and Witten, *Superstring Theory: Vol. 1:Introduction* CUP 1987.

3. Lust and Theisen, *Lecture Notes in Physics: Superstring Theory*, Springer 1989. (Note there is also a more recent expanded edition written with Blumenhagen).

4. David Tong, *String Theory*, arXiv:0908.0333

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Supersymmetry (L16)

David Skinner

This course provides an introduction to the role of supersymmetry in quantum field theory, with the emphasis on mathematics rather than phenomenology. We study representations of the super Poincaré algebra in $d = 4$. We introduce superfields and construct supersymmetric actions for gauge and matter theories. The associated quantum theories are often easier to study than their non-supersymmetric cousins and some observables can even be computed exactly via localization. We also study Seiberg's non-renormalization theorems and phases of SYM theories.

Further topics may include the Witten index for SQM on a Riemannian manifold, and its relation to the Atiyah–Singer index theorem, $\mathcal{N} = 2$ theories in $d = 2$, their chiral rings and the associated A and B models, and Seiberg–Witten theory from extended supersymmetry in $d = 4$.

Pre-requisites

You will need to familiar with the material in both the QFT and General Relativity course from Michaelmas. In particular, we will assume knowledge of differential geometry to the level of Prof. Reall's notes, available at

`http://www.damtp.cam.ac.uk/user/hsr1000/lecturenotes_2012.pdf`

or the Lent term Part III course on Applications of Differential Geometry to Physics. It is also strongly recommended that you attend the Lent AQFT course in parallel with this one; the material on path integrals introduced in that course will be needed for this one.

Literature

1. K. Hori, S. Katz, C. Vafa *et al.* *Mirror Symmetry* Clay Math Monographs, AMS (2003).
2. P. Deligne, E. Witten *et al.*, *Quantum Fields and Strings: A Course for Mathematicians* vols. 1&2, AMS (1999).
3. J. Terning, *Modern Supersymmetry*, International Series of Monographs on Physics, OUP (2009).
4. E. Witten, *Supersymmetry and Morse Theory*, J. Diff. Geom. **17**, (1982) no. 4, 661-692. Also available at

   `https://projecteuclid.org/euclid.jdg/1214437492`

5. E. Witten, *Phases of $\mathcal{N} = 2$ theories in two dimensions*, Nucl. Phys. **B403** (1993) 159-222. Also available at

   `https://www.sciencedirect.com/science/article/pii/055032139390033L`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Classical and Quantum Solitons (E16)

N.S. Manton

Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantisation, solitons give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, vortices of the abelian Higgs model in two dimensions, and Skyrmions in three dimensions. Quantised Skyrmions give us a model for protons and neutrons and larger nuclei like the alpha particle, where the topological charge is the conserved baryon number.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Literature

1. N. Manton and P. Sutcliffe, *Topological Solitons.* C.U.P., 2004 (Chapters 1,3,4,5,7,9).
2. E.J. Weinberg, *Classical Solutions in Quantum Field Theory.* C.U.P., 2012 (Chapters 1,2,3,4,8).
3. R. Rajaraman, *Solitons and Instantons.* North-Holland, 1987.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Physics Beyond the Standard Model (E8)

Maria Ubiali

This graduate course gives a brief overview on the successes and theoretical problems of the Standard Model (SM). It discusses applications of Effective Field Theory (EFT) ideas and techniques to the study of particle physics Beyond the Standard Model (BSM).

After introducing the basic concepts of EFTs and reviewing the SM from an EFT perspective, the course will give examples of how the precise measurements performed at the Large Hadron Colliders can be used as indirect probes of BSM Physics. A special emphasis will be given to the SMEFT and HEFT frameworks. The course will finally give an overview of the most promising Ultra-Violet completions of the Standard Model.

Pre-requisites

Background knowledge of Standard Model and Quantum Field Theories is highly desirable.

Literature

1. A. V. Manohar, *Effective field theories*, hep-ph/9606222
2. D. B. Kaplan, *Five lectures on effective field theory*, nucl-th/0510023.
3. B. Gripaios, *Lectures on Physics Beyond the Standard Model*, arXiv:1503.0263 [hep-ph]
4. S. Willenbrock and C. Zhang, *Effective Field Theory Beyond the Standard Model* arXiv:1401.0470 [hep-ph]

5. I. Brivio, M. Trott, *The Standard Model as an Effective Field Theory*, arXiv:1706.08945 [hep-ph]

Relativity and Cosmology

These courses provide a thorough introduction to General Relativity and Cosmology. The Michaelmas term courses introduce these subjects, which are then developed in more detail in the Lent term courses on Black Holes and Advanced Cosmology. A non-examinable course explores the application of spinor techniques in General Relativity.

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/undergrad/course`

Year		*Courses*
First	*Essential*:	Vectors & Matrices, Diff. Eq., Vector Calculus, Dynamics & Relativity.
Second	*Essential*:	Methods, Quantum Mechanics, Variational Principles.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Essential*:	Classical Dynamics.
	Very helpful:	General Relativity, Stat. Phys., Electrodynamics, Cosmology.
	Helpful:	Further Complex Methods, Asymptotic methods.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Cosmology (M24)

Blake Sherwin

This course discusses what we know (and don't know) about the evolution of our universe, from inflationary quantum fluctuations in the first fraction of a second to the formation of galaxies and structures today. It also seeks to illustrate how cosmology can serve as a uniquely powerful laboratory for understanding fundamental physics.

In detail, the course will cover the following topics:

1. Geometry and dynamics of our Universe
2. Inflation
3. Thermal history
4. Cosmological perturbation theory
5. Structure formation
6. Cosmic microwave background basics
7. Initial conditions from inflation

Pre-requisites

Although the course is intended to be as self contained as possible, knowledge of relativity, quantum mechanics and statistical mechanics will be very useful. General relativity and quantum field theory courses may allow a deeper understanding of some of the material covered.

Literature

1. S. Dodelson, *Modern Cosmology*
2. E. Kolb and M. Turner, *The Early Universe*
3. S. Weinberg, *Cosmology*

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

General Relativity (M24)

Harvey Reall

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

This is a second course on General Relativity, albeit one that could just about be followed without prior exposure to the subject. The first half of the course will give an introduction to differential geometry, the mathematics that underlies curved spacetime. The second half of the course will discuss the physics of gravity.

Pre-requisites

Familiarity with Newtonian gravity, special relativity, finite-dimensional vector spaces and the Euler-Lagrange equations is essential. Knowledge of the relativistic formulation of Maxwell's equations is highly desirable.

Most students attending this course have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before then you should read an introductory book (e.g. Hartle or Rindler) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Literature

1. *Gravity: An introduction to Einstein's General Relativity*, J.B. Hartle, Addison-Wesley, 2003.
2. *Relativity: Special, General, and Cosmological*, 2nd ed., W. Rindler, OUP, 2006.
3. *General Relativity*, R.M. Wald, Chicago UP, 1984.
4. *Gravitation*, C.W. Misner, K.S. Thorne and J.A. Wheeler, W.H. Freeman, 1973.
5. *Spacetime and geometry: an introduction to General Relativity*, S.M. Carroll, Addison-Wesley, 2004.
6. *Advanced General Relativity*, J.M. Stewart, CUP, 1993.
7. *Gravitation and Cosmology*, S. Weinberg, Wiley, 1972.

Our approach is based on Wald and Misner, Thorne and Wheeler. Carroll's book is a very readable introduction at about the same level. Stewart's book is based on a previous version of this course. Weinberg's book gives a good discussion of the Equivalence Principle.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Black Holes (L24)

J. E. Santos

A black hole is a region of spacetime that is causally disconnected from the rest of the Universe. These objects appear to be pervasive in Nature, and their properties have direct implications for the recent advances in gravitational wave astronomy. Besides being astrophysically relevant, black holes also play a fundamental role in quantum theory and are a natural arena to study and test any consistent quantum theory of gravity.

The following topics will be discussed:

1. Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
2. The initial value problem, strong cosmic censorship.
3. Causal structure, null geodesic congruences, Penrose singularity theorem.
4. Penrose diagrams, asymptotic flatness, weak cosmic censorship.
5. Reissner-Nordström and Kerr black holes.
6. Energy, angular momentum and charge in curved spacetime.
7. Positivity of energy theorem.
8. The laws of black hole mechanics. The analogy with laws of thermodynamics.
9. Quantum field theory in curved spacetime. The Hawking effect and its implications.

Pre-requisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

1. H. S. Reall, *Part 3 Black Holes: lecture notes* available at www.damtp.cam.ac.uk/user/hsr1000
2. R. M. Wald, *General relativity*, University of Chicago Press, 1984.
3. S. W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
4. V. P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.
5. N. D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be distributed during the course. Four examples classes will be held to discuss these. A revision class will be held in the Easter term.

Field Theory in Cosmology (L24)

Enrico Pajer and Tobias Baldauf

This course discusses applications of classical, statistical and quantum field theory to cosmology. The course comprises three interconnected topics:

- Cosmological inflation and primordial quantum perturbations (QFT in curved spacetime)
- The matter and galaxy distribution in the Large Scale Structure of the Universe (statistical field theory)
- The physics of the Cosmic Microwave Background (classical and statistical field theory)

The goals of the course are: to discuss open problems in cosmology and describe their intimate relation to fundamental high energy physics; to provide the basic knowledge to understand modern research literature in cosmology; to explore how field theory provides a unifying formalism to describe disparate physical processes from the birth of the Universe to the highly non-linear cosmic web.

More specifically, after a general introduction to open current research and open problems in cosmology, we review inflation and introduce the Effective Field Theory of cosmological perturbations and its connection to field theories with non-canonical interactions. Then we present the so-called "in-in" or Schwinger-Keldysh formalism to compute cosmological correlators and discuss some simple examples, focusing on the leading non-Gaussian statistic, the bispectrum. After showing that cosmological perturbations become classical, we review some basic properties of stochastic fields and correlation functions. The equations determining the dynamics of Large Scale Structure are then introduced together with the concept of renormalization. As an application, we derive a prediction for the matter and galaxy power spectrum at next-to-leading order. Finally, we introduce the Boltzmann equations for the coupled photon-baryon "fluid" and use them to compute the observed temperature anisotropies in the Cosmic Microwave Background.

Pre-requisites

Some familiarity with introductory Quantum Field Theory and General Relativity, as for example provided by the respective Michaelmas courses, is highly recommended. Basic knowledge of introductory Cosmology is essential. Students who did not attend the Michaelmas course on Cosmology may still follow this course after reviewing the relevant course notes.

Literature

Lecture notes including references will be provided by the lecturers.

Additional support

Four example sheets will be provided and four associated example classes will be given. There will be a one-hour revision class in Easter Term.

Applications of Differential Geometry to Physics(L16)

Maciej Dunajski

This is a course designed to develop the Differential Geometry required to follow modern developments in Theoretical Physics. The following topics will be discussed.

- Geometry of Lie Groups
  1. Manifolds
  2. Vector fields and one–parameter groups of transformations
  3. Group action on manifolds
  4. Metrics on Lie Groups and Kaluza Klein theories.
- Classical mechanics
  1. Symplectic and Poisson structures
  2. Geodesic flow, Killing vectors, Killing Tensors.
  3. Null Kaluza–Klein reductions
  4. Integrable Systems
- Fibre bundles and instantons
  1. Principal bundles and vector bundles.
  2. Connection and Curvature
  3. Instantons

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

References

[1] Arnold. V. Mathematical Methods of Classical Mechanics. Springer.

[2] Dunajski, M. Solitons, Instantons, and Twistors, *Oxford Graduate Texts in Mathematics*, Oxford University Press, 2009.

[3] Eguchi, T., Gilkey, P. and Hanson. A. J. Physics Reports 66 (1980) 213-393

Applications of Analysis in Physics (L16)

Non-Examinable (Graduate Level)

Claude Warnick

This course is aimed at students who are studying physics and are interested in learning some of the more advanced analysis that underpins much of modern theoretical physics. We will emphasise widely applicable concepts and avoid technical details of proofs, while signposting where students can find them. We will aim to cover:

- Background: Hilbert and Banach spaces; distributions; Fourier transform and Sobolev spaces.
- Compactness: spectra of self-adjoint compact operators; the direct method of the calculus of variations.
- PDEs on manifolds: Laplace/wave equation on a Riemannian/Lorentzian manifold.
- Topology and PDEs: index theorems, heat trace.

Pre-requisites

We assume some basic background analysis knowledge: roughly second year undergraduate level. We also assume some differential geometry at a level similar to that of the GR course.

Literature

1. M. Reed, B. Simon, *Methods of Mathematical Physics Vols. 1, 2.* Elsevier, 1981.
2. S. Rosenberg, *The Laplacian on a Riemannian Manifold.* CUP 1997.

Additional support

Problems will be set, and there will be an opportunity to discuss these with the lecturer.

Gauge/Gravity Duality (E16)

Aron Wall

Gauge/Gravity duality (also known as AdS/CFT) is an amazing duality that relates theories of quantum gravity (with a negative cosmological constant) to certain quantum field theories living in a smaller dimensional spacetime. This is the most precise known realization of the holographic principle, the idea that all information in the universe is encoded somehow at the boundary of the universe. These lectures will describe in detail the "dictionary" used to relate observables on the bulk side to observables on the boundary side.

Topics covered: Anti-de Sitter spacetime; conformal field theory; wave equations in AdS, and their relationship to CFT operators and sources; the duality between black holes and thermal states; holographic entanglement entropy. If time permits: recent developments concerning bulk reconstruction, and the black hole information puzzle.

Pre-requisites

Required: General Relativity, Black Holes, Advanced Quantum Field Theory

Helpful: Some basic aspects of quantum information theory and conformal symmetry will play an important role in this course, but the relevant aspects will be reviewed in a self-contained manner.

Not Required: String Theory, Supersymmetry. Although most of the specific known examples of AdS/CFT come from superstring theories, these aspects will not be emphasized in these lectures. (There will probably be one lecture on Juan Maldacena's original derivation of black hole entropy in string theory, but it will not be on the exam.)

Literature

More information about the course, including a list of relevant review articles, is available here:

`http://www.damtp.cam.ac.uk/user/aw846/AdSCFT.html`

(If you read through the lecture notes from the previous unexaminable version of this course from 2018-2019, please note that Topic 5: Large N Gauge Theories and Examples of AdS/CFT will be mostly excluded.)

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Astrophysics

Introduction to Astrophysics courses

These courses provide a broad introduction to research in theoretical astrophysics; they are taken by students of both Part III Mathematics and Part III Astrophysics. The courses are mostly self-contained, building on knowledge that is common to undergraduate programmes in theoretical physics and applied mathematics. For specific pre-requisites please see the individual course descriptions.

Structure and Evolution of Stars (M24)

A.N.Żytkow

Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These equations have to be supplemented by a description of the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed. Some familiarity with the principles of hydrodynamics, thermodynamics, quantum mechanics, atomic and nuclear physics will be assumed.

Approximate solutions of the equations will be described. Polytropic gas spheres, homology principles, the virial theorem will be presented. The evolution of a star will be discussed, starting from the main-sequence, following the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

There will be a brief discussion of helioseismology, stellar rotation and mass loss from stars.

The only way in which we may test stellar structure and evolution theory is through the comparison of theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

Pre-requisites

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics although a detailed knowledge of all of these is not expected.

Preliminary Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Literature

1. Eldridge, J.J. and Tout, C.A. *The Structure and Evolution of Stars*, World Scientific, 2019.
2. Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution, Second Edition*, Springer-Verlag, 2012.
3. Iben, I. *Stellar Evolution Physics, Vol. 1 and 2*, Cambridge University Press, 2013.
4. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
5. Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

Additional support

There will be four example sheets each of which will be discussed during an examples class. There will be a one-hour revision class in the Easter Term.

Galaxy Formation (M24)

N Wyn Evans

This course describes our current state of knowledge of galaxy formation and evolution in a cold dark matter cosmology. We will start with structure formation in the non-linear regime, the formation and evolution of dark matter haloes and Press-Schechter theory. We will cover physical processes (shock heating, radiative cooling and star formation) as well as dynamical transformations (dynamical friction, tidal shocking, accretion and mergers) that are responsible for the shapes and properties of the galaxies we see today. We will end with a study of the formation and current day attributes of disk galaxies (Sersic profiles, thin and thick disks, stellar haloes) and elliptical galaxies (fast/slow rotators, major/minor mergers, Faber-Jackson relation). Recent discoveries on the structure of the Local Group and the Milky Way galaxy will be used as illustrative examples of formation processes throughout the course.

There is a complementary course by Prof. V. Belokurov in Lent Term on the present-day life and evolution of the galaxies.

Pre-requisites

This Part III course assumes that you have taken undergraduate courses in cosmology, relativity and dynamics.

Literature

1. J. Binney and S. Tremaine *Galactic Dynamics* 2nd edition, Princeton University Press, 2008
2. J. Bland-Hawthorn, K. Freeman *The Origin of the Galaxy and the Local Group*, Springer, 2014
3. A. Loeb *How Did the First Stars and Galaxies Form*, Princeton, 2010 (Background reading)
4. M. Longair, *Galaxy Formation* 2nd edition, Springer, 2008
5. H. Mo, F. van den Bosch and S. White, *Galaxy Formation and Evolution*, Cambridge Universitry Press, 2010
6. S. Phillips, *The Structure and Evolution of Galaxies*, Wiley, 2005
7. L. Sparke, J. Gallagher, *Galaxies in the Universe*, 2nd edition, Cambridge University Press, 2007 (Background reading)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Planetary System Dynamics (M24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar

System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of 1000s of extrasolar planets and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems.

Specific topics to be covered include:

1. Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation
2. Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits
3. Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift
4. Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits
5. Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide
6. Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces
7. Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments
8. Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems
9. Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap

Pre-requisites

This course is self-contained.

Literature

1. Murray C. D. and Dermott S. F.,*Solar System Dynamics.* Cambridge University Press,1999.
2. Armitage P. J., *Astrophysics of Planet Formation.* Cambridge University Press, 2010.
3. de Pater I. and Lissauer J. J., *Planetary Sciences.* Cambridge University Press, 2010.
4. Valtonen M. and Karttunen H., *The Three-Body Problem.* Cambridge University Press, 2006.
5. Seager S., *Exoplanets.* University of Arizona Press, 2011.
6. Perryman M., *The Exoplanet Handbook.* Cambridge University Press, 2011.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extrasolar Planets: Atmospheres and Interiors (M24)

Nikku Madhusudhan

The field of extrasolar planets (or 'exoplanets') is one of the most dynamic frontiers of modern astronomy. Exoplanets are planets orbiting stars beyond the solar system. Thousands of exoplanets are now known with a wide range of sizes, temperatures, and orbital parameters, covering all the categories of planets in the solar system (gas giants, ice giants, and rocky planets) and more. The field is now moving into a new era of Exoplanet Characterization, which involves understanding the atmospheres, interiors, and formation mechanisms of exoplanets, and ultimately finding potential biosignatures in the atmospheres of rocky exoplanets. These efforts are aided by both high-precision spectroscopic observations as well as detailed theoretical models of exoplanets.

The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and stratospheres, equilibrium/non-equilibrium chemistry, atmospheric dynamics, clouds/hazes, etc) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, as well as parametric models and retrieval techniques) (3) exoplanetary interiors (equations of state, mass-radius relations, and internal structures of giant planets, super-Earths, and rocky exoplanets), and (4) relating atmospheres and interiors to planet formation. Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed. The course will also include a discussion on detecting biosignatures in rocky exoplanets, the relevant theoretical constructs and expected observational prospects with future facilities.

Pre-requisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self-contained and basic concepts will be introduced as required.

Literature

1. Seager, S., Exoplanet Atmospheres: Physical Processes, Princeton Series in Astrophysics (2010).
2. Exoplanets, University of Arizona Press (2011), ed. S. Seager.
3. de Pater, I. and Lissauer J., Planetary Sciences, Cambridge University Press (2010).
4. Chapters on exoplanetary atmospheres and interiors in the book Protostars and Planets VI, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning. Available publicly on astro-ph arXiv (e.g., arXiv:1402.1169, arXiv:1401.4738).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The Life and Death of Galaxies (L24)

Vasily Belokurov

This course will provide the observational perspective on the evolution of galaxies and will complement the theoretical Part III course "Galaxy Formation".

SEDs. Panchromatic view of galaxies. Dependance of the galactic structure on wavelength. Components of the galactic SED. Stars. Dust. Nebular emission. Stromgren sphere. Spectral lines. Initial mass function: its shape and observational constraints. Links between SEDs (continuum, absorption and emission lines) and star-formation history, the current (recent) star-formation, stellar and gas contents. Age-metallicity degeneracy.

Star-formation. Star-formation rates. Metal-enrichment. Yield. Closed box, instantaneous mixing, one zone model. G-dwarf problem. Inflows and Outflows. Mass-metallicity relation. Star-Formation Law. Cloud collapse and fragmentation. Star formation on sub-galactic scales. Quenching. Star-formation history of the Universe. Integrated light: Look back vs Archaeology. Halo and galaxy assembly in the Cold Dark Matter Universe. Galaxy formation efficiency as a function of galaxy mass. SFH with resolved stellar populations. Stellar Feedback. Dwarf galaxies.

Dynamics. Dynamics of elliptical galaxies. Distribution Functions. Collisionless Boltzman Equation (CBE). Integrals of CBE. Jeans Equations. Virial Theorem. Poisson Equation. Slow and Fast rotators. Dynamical Friction. Fundamental and Mass planes. Relaxation and phase mixing. Violent relaxation. Dynamics of spiral galaxies. Rotation curves. Dark matter. Navarro-Freak-White density profiles. Galactic halos. Mass measurements with gravitational lensing. Dark matter structure and sub-structure in the Local Group. Dwarf galaxies as dark matter laboratories.

Structure and evolution. Structure of galaxies. Two types of galaxies: dead and alive. Light distribution. Density de-projection. Sersic profile and its modifications. Detailed structure of elliptical galaxies. Cusp scouring. 3D shapes of ellipticals. Galaxy luminosity function. Schechter function. Connection between galaxy type/structure and environment. Mergers. Tides. Stellar streams. Accretion signatures in the Local Group. Formation and evolution of elliptical galaxies.

Active Galactic Nuclei and Quasars. Evidence for (S)MBHs in the centres of galaxies. Energy production. BH growth. Co-evolution of central BHs and galaxies. M-sigma relation. AGN feedback.

Pre-requisites

Knowledge of Galactic Dynamics (e.g. Part II Astrophysics Course "Stellar dynamics and Structure of Galaxies") would come in handy. It is preferable (but not required) that you have attended "Galaxy Formation" course in Michaelmas.

Literature

1. Binney, J., and Tremaine, S. *Galactic Dynamics* Second Edition. Princeton University Press, 2008
2. Mo, H., van den Bosch, F., and White, S. *Galaxy Formation and Evolution* Cambridge University Press, 2010
3. Sparke, L., and Gallagher, J. *Galaxies in the Universe* Second Edition. Cambridge University Press, 2007
4. Longair, M. *Galaxy Formation* Second Edition. Springer, 2008

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Astrophysical Fluid Dynamics (L24)

Gordon Ogilvie

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as their stability and their response to tidal forcing.

Provisional synopsis

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Stellar oscillations. Introduction to asteroseismology and astrophysical tides.
- Local dispersion relation. Internal waves and instabilities in stratified rotating astrophysical bodies.

Pre-requisites

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of vector calculus, fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas.* Cambridge University Press.
2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.
3. Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows.* Cambridge University Press.
4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics.* University Science Books.
5. Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics.* Imperial College Press.
6. Ogilvie, G. I. (2016). *Lecture Notes: Astrophysical Fluid Dynamics.* J. Plasma Phys. **82**, 205820301.

References [1], [3] and [6] are available online at

`https://www.cambridge.org/core/`

Additional support

Four example sheets will be provided and four associated classes will be given by the lecturer. Extended notes supporting the lecture course are available from reference [6] in the list above. There will be a revision class in Easter Term.

Dynamics of Astrophysical Discs (L16)

Henrik Latter

Disks are ubiquitous in astrophysics and participate in some of its most important processes. Most, but not all, feed a central mass: by facilitating the transfer of angular momentum, they permit the accretion of material that would otherwise remain in orbit. As a consequence, disks are essential to star, planet, and satellite formation. They also regulate the growth of supermassive black holes and thus indirectly influence galactic structure and the intracluster medium. Although astrophysical disks can vary by ten orders of magnitude in size and differ hugely in composition, all share the same basic dynamics and many physical phenomena.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of extrasolar planets.

Provisional synopsis:

- Occurrence of discs in various astronomical systems, basic physical and observational properties.
- Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.
- Viscous evolution of an accretion disc.
- Vertical disc structure, thin-disc approximations, thermal instability in cataclysmic variables.
- The shearing sheet, symmetries, shearing waves.
- Incompressible dynamics: hydrodynamic stability, vortices and dust dynamics in protoplanetary disks.
- Compressible dynamics: density waves, gravitational instability and 'gravitoturbulence' in planetary rings and protoplanetary discs.
- Satellite-disc interaction, impulse approximation, gap opening by embedded planets.
- Magnetorotational instability, 'dead zones' in protoplanetary discs, magnetised outflows and jets

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is helpful for the topic of magnetorotational instability.

Literature

Astrophysical background can be found in reference [1] below, while the classical theory is summarised in the two review articles [2,3].

1. Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.
2. Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.
3. Papaloizou, J. C. B. Lin, D. N. C. (1995), Annu. Rev. Astron. Astrophys. 33, 505.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Binary Stars (L16)

Christopher Tout

A binary star is a gravitationally bound system of two component stars. Such systems are common in our Galaxy and a substantial fraction interact in ways that can significantly alter the evolution of the individual stellar components. Many of the interaction processes lend themselves to useful mathematical modelling when coupled with an understanding of the evolution of single stars.

In this course we begin by exploring the observable properties of binary stars and recall the basic dynamical properties of orbits by way of introduction. This is followed by an analysis of tides, which represent the simplest way in which the two stars can interact. From there we consider the extreme case in which tides become strong enough that mass can flow from one star to the other. We investigate the stability of such mass transfer and its effects on the orbital elements and the evolution of the individual stars. As a prototypical example we examine Algol-like systems in some detail. Mass transfer leads to the concept of stellar rejuvenation and blue stragglers. As a second example we look at the Cataclysmic Variables in which the accreting component is a white dwarf. These introduce us to novae and dwarf novae as well as a need for angular momentum loss by gravitational radiation or magnetic braking. Their formation requires an understanding of significant orbital shrinkage in what is known as common envelope evolution. Finally we apply what we have learnt to a number of exotic binary stars, such as progenitors of type Ia supernovae, X-ray binaries and millisecond pulsars.

Pre-requisites

The Michaelmas term course on Structure and Evolution of Stars is very useful but not absolutely essential. Knowledge of elementary Dynamics and Fluids will be assumed.

Literature

1. Pringle J. E. and Wade R. A., *Interacting Binary Stars.* CUP.

Reading to complement course material

1. Eggleton P. P., *Evolutionary Processes in Binary and Multiple Stars.* CUP.

Additional support

Three examples sheets will be provided and three associated two-hour examples classes will be given. There will be a two-hour revision class in the Easter Term.

Astrophysical black holes (L16)

Debora Sijacki

Black holes are one of the most fascinating objects lying at the interface of mathematics, physics and astronomy. From the astrophysical stand point they give rise to extremely rich and complex phenomena occurring from sub-parsec to Mega-parsec scales and covering the full electromagnetic spectrum. A large body of state-of-the-art current and upcoming observational facilities and theoretical models is aimed at investigating black hole properties and their link with the larger scale environment, making this an exciting and fast paced research field. With the recent gravitational wave detections of merging black hole binaries the field has experienced further stimulus, as black holes have become unique multi-messengers to explore cosmology, gravity in the strong regime, high energy phenomena and complex (magneto)hydrodynamic flows.

This course will cover a range of concepts pertinent to astrophysical (supermassive) black holes highlighting both observational and theoretical advances in the field. The aim of the course is to give an overview of the possible formation and growth channels of these objects and to discuss various mechanisms through which black holes interact with their surroundings, with the provisional synopsis as follows:

- Basic concepts; observational evidence for dormant and non-dormant objects (SgrA*)
- AGN properties and classification
- Formation pathways for supermassive black holes
- Black hole growth overview
- Fuelling mechanisms from kpc to sub-pc scales
- Bondi-Hoyle solution and limitations
- Accretion disk models: thin, slim and thick discs
- Outflows: basic concepts; collimated wind and jet phenomena
- Energy-, momentum- and radiation pressure-driven outflow solutions
- Impact of outflows on host properties
- Brief overview of black hole binaries and hardening processes

Pre-requisites

Good knowledge of material covered in Part II courses: Astrophysical fluid dynamics, stellar dynamics and structure of galaxies, structure and evolution of stars is required. Some knowledge of (thermo)dynamics, electromagnetism and galaxy formation is advantageous but not strictly necessary.

Literature

1. Frank, J., King, A. R., & Raine, D., *Accretion Power in Astrophysics*, Cambridge University Press, 2002.
2. Netzer, H., *The Physics and Evolution of Active Galactic Nuclei*, Cambridge University Press, 2013.
3. Misner C. W., Thorne K. S., & Wheeler J. A., *Gravitation*, W. H. Freeman and company, 1973.
4. Pringle, J. E., & King, A. R., *Astrophysical Flows*, Cambridge University Press, 2007.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Astrostatistics (L24)

Kaisey Mandel

This course will cover applied statistical methods necessary to properly interpret today's increasingly complex datasets in astronomy. Particular emphasis will be placed on principled statistical modeling of astrophysical data and statistical computation of inferences of scientific interest. Statistical techniques, such as Bayesian inference, sampling methods, hierarchical models, Gaussian processes, and model selection, will be examined in the context of applications to modern astronomical data analysis. Topics and examples will be motivated by case studies across astrophysics and cosmology.

Pre-requisites

Students of astrophysics, physics, statistics or mathematics are welcome. Astronomical context will be provided when necessary. Students without a previous statistics background should familiarise themselves with the material in Feigelson & Babu, Chapters 1-4, and Ivezić, Chapters 1, 3-5, by the beginning of the course. (Note that the two textbooks cover many of the same topics). These texts are freely available online to Cambridge students via the library website.

Literature

1. E. Feigelson and G. Babu. *Modern statistical methods for astronomy: with R applications.* Cambridge University Press, 2012.

2. Z. Ivezić, A. Connolly, J. VanderPlas & A. Gray. *Statistics, Data Mining, and Machine Learning in Astronomy.* Princeton University Press, 2014.

3. C. Schafer. *A Framework for Statistical Inference in Astrophysics.* 2015, Annual Review of Statistics and Its Application, 2: 141-162.

4. E. Feigelson, et al. *21st Century Statistical and Computational Challenges in Astrophysics.* 2020, Annual Review of Statistics and Its Application, 8.
   `https://arxiv.org/abs/2005.13025`

5. C. Bishop. *Pattern Recognition & Machine Learning.* Springer-Verlag, 2006.
   Also available at:

   `https://www.microsoft.com/en-us/research/people/cmbishop/#!prml-book`

6. D. MacKay. *Information Theory, Inference, and Learning Algorithms.* Cambridge University Press, 2003. Also available at:

   `http://www.inference.org.uk/mackay/itila/book.html`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Computation, Information and Foundations

Quantum Computation (M16)

Richard Jozsa

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

This course will be a 'second' course in the subject, following the Part II course Quantum Information and Computation (see below in prerequisites) that was introduced in 2017-2018.

In this course we will aim to cover the following topics:
- The hidden subgroup problem and quantum Fourier transform on a group;
- The quantum phase estimation algorithm and applications;
- Amplitude amplification and applications;
- Quantum simulation for local hamiltonians;
- The Harrow-Hassidim-Lloyd quantum algorithm for systems of linear equations.

If time permits we may also discuss (or substitute) further topics such as: Introduction to Clifford operations; Classical simulation properties of Clifford circuits (Gottesman-Knill theorem); Measurement based quantum computing; The Pauli based model of quantum computing (Bravyi, Smith and Smolin 2016).

Pre-requisites

This course will assume a prior basic acquaintance with quantum computing, to the extent presented in the course notes for the Cambridge Part II course Quantum Information and Computation available at `http://www.qi.damtp.cam.ac.uk/part-iii-quantum-computation`
In particular you should be familiar with Dirac notation and principles of quantum mechanics, as presented in the course notes sections 2.1, 2.2 and 2.3. You should also have a basic acquaintance with quantum computation to the extent of the second half of the course notes, pages 47 to 86 (Chapters 6-11). *It would be desirable for you to look through this material before the start of the course.*

Literature

Further useful literature includes the following.

1. Nielsen, M. and Chuang, I., *Quantum Computation and Quantum Information.* CUP, 2000.
2. John Preskill *Lecture Notes on Quantum Information Theory* (especially Chapter 6) available at `http://www.theory.caltech.edu/people/preskill/ph219/`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

Quantum Information Theory (L24)

Sergii Strelchuk

Quantum Information Theory (QIT) lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the mathematical theory of acquisition, storage, transmission and processing of information.

QIT is the study of how these tasks can be accomplished, using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical counterparts. These new features can be exploited, not only to improve the performance of certain information-processing tasks but also to accomplish tasks which are impossible or intractable in the classical realm.

The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of QIT. Topics in this part of the course will include a brief discussion of data compression, transmission of data through noisy channels, Shannon's theorems, entropy and channel capacity. The quantum part of the course will commence with a study of open systems and a discussion of how they necessitate a generalization of the basic postulates of quantum mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs, the Kraus Representation Theorem, the Choi-Jamilkowski isomorphism, quantum data compression limit, and random coding arguments.

We will further focus on data compression, reliable transmission of information over noisy communication channels, and introduce accessible information and coherent information. In particular, we will discuss the Holevo bound on the accessible information, the Holevo-Schumacher-Westmoreland (HSW) Theorem, and key properties of coherent information leading to surprising superadditivity effects for quantum channel capacities.

Pre-requisites

Familiarity with the Part II course *Quantum Information and Computation* or equivalent is essential.

Knowledge of basic quantum mechanics will be assumed.

Elementary knowledge of Probability Theory, Vector Spaces and Linear Algebra will be useful.

Literature

1. M. A. Nielsen and I. L. Chuang *Quantum Computation and Quantum Information.* Cambridge University Press, 2002.

2. M. M. Wilde *From Classical to Quantum Shannon Theory* Cambridge University Press, 2013.

3. J. Preskill, *Lecture notes on Quantum Information Theory*, Acta Applicandae, **56**, 1-98 (1999). Also available at `http://www.theory.caltech.edu/~preskill/ph229/#lecture`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Philosophy of Physics

The courses in Philosophy of Physics are open to all students doing Part III, but are formally listed as graduate courses. This means there is no exam at the end of May for any such course; but a Part III student can get credit for them by doing their submitted Part III essay in association with one of the courses. More generally, the Philosophy of Physics courses are intended as a refreshing and reflective companion to the other Part III courses, especially the courses in theoretical physics.

Philosophical Aspects of Quantum Fields (M8)

J. Butterfield and B. Roberts: Non-Examinable (Part III Level)

Quantum field theory has for many decades been the framework for several basic and outstandingly successful physical theories. Nowadays, it is being addressed by philosophy of physics (which has traditionally concentrated on conceptual questions raised by non-relativistic quantum mechanics and relativity). This course will introduce this literature, and prepare for the sequel course in the Lent Term. The content will be moulded by students' interests. But we plan to discuss such topics as: (i) quantization theory, (ii) particle vs. field, including localization, (iii) discrete symmetries, especially time-reversal.

We also expect, in the first half of the course, to review: (a) the mathematical structure of quantum theories in general, at the level of the books by Hannabuss, Jordan and Prugovecki; (b) some foundational issues, using the books by Araki, Clifton and Landsman (which is Open Access); (c) ideas of operator algebras, using the books by Emch, Haag and Ruetsche.

Overall, we expect: (i) to mostly use the books by Folland, and by de Faria and de Melo; and (ii) to prepare for the Lent Term course which will consider such topics such as quantum field theory on curved spacetime.

Pre-requisites

There are no formal prerequisites. But of course, previous familiarity with quantum mechanics will be essential.

Preliminary Reading

This list of reading gives an overview of the course's topics, and is approximately in order of increasing difficulty.

1. S. Weinberg (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: http://arxiv.org/abs/hep-th/9702027; and in T. Cao, (ed.) *The Conceptual Foundations of Quantum Field Theory.* Cambridge University Press, 1999.

2. D. Wallace (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151** (1):33-80, 2006. Available online at: http://arxiv.org/pdf/quant-ph/0112148v1

3. D. Wallace (2001), 'Emergence of particles from bosonic quantum field theory'. Available online at: http://arxiv.org/abs/quant-ph/0112149

4. L. Ruetsche, *Interpreting Quantum Theories*: especially up to Chapter 9. Oxford University Press, 2011.

Literature

To give an idea of the literature, this list includes only books. On this list, our main resource will be the books by Folland, and by de Faria and de Melo. For mathematical background, we will draw on the books by Hannabuss, Jordan and Prugovecki. For foundational issues, we will draw on the books by Araki, Clifton and Landsman (the last being freely downloadable, and an invaluable resource for the whole course). For operator algebras, we will also use the books by Emch, Haag and Ruetsche. A recent advanced monograph is Rejzner (2016).

1. E. de Faria and W. de Melo. *Mathematical Aspects of Quantum Field Theory*: up to Chapter 6. Cambridge University Press, 2010.
2. G. Folland. *Quantum Field Theory: a tourist guide for mathematicians*: up to Chapter 6. American Mathematical Society, 2008.
3. K. Hannabuss. *An Introduction to Quantum Theory*: up to Chapter 11. Oxford University Press, 1997.
4. T. Jordan. *Linear Operators for Quantum Mechanics*: especially Chapters 3 to 5. John Wiley 1969; Dover 2006.
5. E.Prugovecki. *Quantum Mechanics in Hilbert Space*: especially Parts III, IV. Academic Press 1981; Dover 2006.
6. H. Araki. *Mathematical Theory of Quantum Fields*: up to Chapter 4. Oxford University Press, 1999.
7. R. Clifton. *Quantum Entanglements*, edited by J. Butterfield and H. Halvorson: Chapters 6 to 9. Oxford University Press 2004.
8. N. Landsman. *Foundations of Quantum Theory.* Springer 2017: especially Chapters 5, 6,7,9,10. Open access: downloadable at: https://link.springer.com/book/10.1007/978-3-319-51777-3
9. G. Emch. *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*: especially Chapter 1. John Wiley 1972; Dover 2009.
10. R. Haag. *Local Quantum Physics: fields, particles, algebras*: Chapters I, II, III and V.4: Springer 1992.
11. K. Rejzner. *Perturbative Algebraic Quantum Field Theory: an introduction for mathematicians*: up to Chapter 5. Springer 2016.

Additional support

One or two Part III essays will be offered in conjunction with this course.

Further Topics in Philosophy of Quantum Field Theory (L8)

J. Butterfield and B. Roberts: Non-Examinable (Part III Level)

This is a sequel to the Michaelmas Term course, 'Philosophical Aspects of Quantum Fields'. But that course is not a formal pre-requisite. The content of the course will be moulded by students' interests. But we expect to cover the following topics, often using ideas from algebraic quantum theory:
(a) the CPT theorem: roughly, that a Lorentz-invariant quantum field theory must be invariant under the combined operations of charge-conjugation, parity and time-reversal;
(b) the Unruh effect: roughly, that an observer accelerating through the vacuum state of a free quantum field on Minkowski spacetime sees—not *no* particles—but a thermal bath of particles (at a temperature that depends on the observer's acceleration);
(c) quantum field theory on curved spacetime, especially generalizations of the Unruh effect to curved spacetime, and thermal radiation from black holes (the Hawking effect).

We may also discuss recent philosophical literature about testing black hole radiation on analogue systems.

Pre-requisites

There are no formal prerequisites. But of course, some familiarity with quantum field theory will be essential; and familiarity with the Michaelmas Term course, 'Philosophical Aspects of Quantum Fields', will be helpful.

Preliminary Reading

This is the same list as for the Michaelmas Term course, 'Philosophical Aspects of Quantum Fields'. It is approximately in order of increasing difficulty.

1. S. Weinberg (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: http://arxiv.org/abs/hep-th/9702027; and in T. Cao, (ed.) *The Conceptual Foundations of Quantum Field Theory*. Cambridge University Press, 1999.

2. D. Wallace (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151** (1):33-80, 2006. Available online at: http://arxiv.org/pdf/quant-ph/0112148v1

3. D. Wallace (2001), 'Emergence of particles from bosonic quantum field theory'. Available online at: http://arxiv.org/abs/quant-ph/0112149

4. L. Ruetsche, *Interpreting Quantum Theories*: especially up to Chapter 9. Oxford University Press, 2011.

Literature

For the CPT theorem, item 1 will be our main source. For both the Unruh effect and radiation from black holes, item 2 is a good introduction. For details of the Unruh effect, we will mainly use: (i) for the physics, the books listed in item 3, which are approximately in order of increasing difficulty, and which also deal with black hole evaporation; (ii) for the philosophy, the articles listed in item 4. For philosophical discussion of radiation from black holes, our main source will be Wallace's papers in item 5.

1. Swanson, N. (2019), 'Deciphering the algebraic CPT theorem', *Studies in History and Philosophy of Modern Physics* **68** 106-125. Available at: http://philsci-archive.pitt.edu/16138/

2. Lambert, P. (2013), 'Introduction to black hole evaporation', *Proceedings of Science*, gr-qc: 1310.8312

3. (1): Carroll, S. (2019), *Spacetime and Geometry*, Cambridge University Press; Chapter 9.
(2): N. Birrell and P. Davies *Quantum Fields in Curved Space*, Cambridge University Press 1984, Chapters 1 to 4.
(3): Fulling, S. *Aspects of quantum field theory in curved spacetime*, LMS Student Texts 17, Cambridge University Press 1989, up to Chapter 6.
(4): Parker, L. and Toms, D. (2009), *Quantum Field Theory in Curved Spacetime*, Cambridge University Press; Chapters 1-4.
(5): Wald, R. (1994), *Quantum field theory in curved spacetime and black hole thermodynamics*, Chicago University Press

4. (1): Clifton, R. and H. Halvorson (2001), 'Are Rindler quanta real? Inequivalent particle concepts in quantum field theory', *British Journal for Philosophy of Science*, **52**, pp 417-470. especially Sections 1, 2.1, 2.2, 3.1, 3.2. Available online at: http://arxiv.org/abs/quant-ph/0008030. Reprinted as Chapter 9 in R. Clifton *Quantum Entanglements*, ed. J. Butterfield and H. Halvorson, Oxford University Press 2004.
(2): Clifton, R. and H. Halvorson (2001a), 'Entanglement and opens systems in algebraic QFT', *Studies in History and Philosophy of Physics* **32**, 1-31.
(3): Earman, J. (2011), 'The Unruh Effect for Philosophers', *Studies in History and Philosophy of Physics* **42** 81-97

5. (1): Wallace, D. (2018), 'The case for black hole thermodynamics Part I: Phenomenological thermodynamics', *Studies in History and Philosophy of Modern Physics* **64** 52-67.
(2): Wallace, D. (2019), 'The case for black hole thermodynamics Part II: Statistical mechanics', *Studies in History and Philosophy of Modern Physics* **66** 103-117.
(3): Wallace, D. (2017), 'Why Black Hole Information Loss is Paradoxical', available at https://arxiv.org/abs/1710.03783.

Additional support

One or two Part III essays will be offered in conjunction with this course.

Applied and Computational Analysis

Inverse Problems (M24)

Yury Korolev & Jonas Latz

Inverse problems arise whenever there is a need to infer quantities of interest from indirectly measured data. Inverse problems are ubiquitous in science; they arise in physics, biology, medicine, engineering, finance and computer science (e.g., in machine learning and computer vision). Many imaging problems, such as reconstruction of medical images (computer tomography, magnetic resonance imaging, positron-emission tomography) and deblurring or denoising of microscopy and astronomy images, are also instances of inverse problems. Inverse problems typically share a feature that makes them challenging to solve in practice: they lack continuous dependence on the data and, therefore, small errors in the measurements can lead to large errors in naive reconstructions, making them useless. To deal with this issue, special *regularisation* and *Bayesian* techniques have been developed to overcome the instability by using additional a priori information about the unknown, such as smoothness or sparsity in some basis.

In this course we will present mathematical theory and algorithms for solving inverse problems using regularisation and Bayesian methods, from the classical foundations to modern state-of-the-art methods. We will apply theory and algorithms to inverse problems in imaging and engineering.

Pre-requisites

This course assumes basic knowledge in linear algebra, analysis and probability theory (e.g. Linear Analysis or Analysis of Functions and Probability and Measure). Additional knowledge in convex analysis is beneficial, but not mandatory.

Literature

1. H. W. Engl, M. Hanke and A. Neubauer. *Regularization of Inverse Problems.* Vol. 375, Springer Science & Business Media, 1996, ISBN: 9780792341574.

2. O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier and F. Lenzen. *Variational Methods in Imaging.* Applied Mathematical Sciences, Springer New York, 2008, ISBN: 9780387309316.

3. M. Dashti and A.M. Stuart, *The Bayesian approach to inverse problems,* in: *Handbook of Uncertainty Quantification.* Springer, 2015.

4. A.M. Stuart, *Inverse problems: a Bayesian perspective.* Acta Numerica, **19**, 451-559 (2010).

5. M. Benning, M. Burger *Modern Regularization Methods for Inverse Problems*, Acta Numerica, **27**, 1-111 (2018).

6. S.L. Cotter, G.O. Roberts, A.M. Stuart, D. White, *MCMC Methods for Functions: Modifying Old Algorithms to Make Them Faster.* Statistical Science, **28**(3), 424-446 (2013).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Distribution Theory and Applications (M16)

Dr. A. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will breifly look at Sobolev spaces in $\mathbf{R}^n$ and their description in terms of the Fourier transform of tempered distrbutions. The material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. The final section of the course will be concerned with Hörmander's oscillatory integrals.

Pre-requisites

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Methods or Complex Analysis).

Preliminary Reading

1. F.G. Friedlander & M.S. Joshi, *Introduction to the Theory of Distributions*, Cambridge Univ Pr, 1998.
2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge Univ Pr, 1958.
3. G.B. Folland, *Introdution to Partial Differential Equations*, Princeton Univ Pr, 1995.

Literature

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols* I-II, Springer Verlag, 1985.
2. M. Reed & B. Simon, *Methods of Modern Mathematical Physics: Vols* I-II, Academic Press, 1979.
3. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be made available. There will be a revision class in the Easter Term.

Numerical Solution of Differential Equations (L24)

Arieh Iserles

The course describes modern algorithms for the solution of ordinary and partial differential equations, inclusive of finite difference and finite element methods, with an emphasis on broad mathematical principles underlying their construction and analysis.

Pre-requisites

Although prior knowledge of *some* numerical analysis and of abstract function spaces is advantageous, it will not be taken for granted. Reasonable understanding of basic concepts of analysis (complex analysis and analytic functions, basic existence and uniqueness theorems for ODEs and PDEs, elementary facts about PDEs) and of linear algebra is a prerequisite.

Literature

1. U. Ascher, *Numerical Methods for Evolutionary Differential Equations*, SIAM, 2008.

2. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations* (2nd edition), Cambridge University Press, 2006.

Additional support

An extensive printed handout, covering the entire material of the course, will be provided in the first week. There will be examples' classes, starting from the third week, as well as a revision supervision in the Easter Term.

Nonlinear Functional Analysis and Applications(M24)

Edriss S. Titi

In these self-contained lectures I will introduce and develop some of the basic analytical tools in nonlinear functional analysis. The main focus will be on the implementation of these methods in investigating and solving certain nonlinear problems, with emphasize on nonlinear partial differential equations. The course will be useful to all students with interest in nonlinear analysis, applied and computational mathematics, and differential geometry.

Topics to be covered:

1. Review of Basic Function Spaces.
2. Some Fixed Point Theorems.
3. Galerkin Method.
4. Monotone Iterations and Monotone Operators.
5. Differential Calculus in Banach Spaces.
6. Variational Methods
7. Palais-Smale Condition and Moutain Pass Lemma
8. Pohozaev's Identity

Pre-requisites

This course expects students to know Real Analysis and basic Linear Functional Analysis.

Literature

1. E. Zeidler, *Nonlinear Functional Analysis and its Applications, Parts I, II, III.* Springer–Verlag.

2. S. Kesavan, *Nonlinear Functional Analysis and its Applications: A First Course.* Hindustan Book Agency (India), 2004.

Additional comments

This will be non-examinable graduate course. However, PhD students might choose to have an oral exam if they wish. There will be no examples sheets associated with this course. Nonetheless, some problems will be given once in a while for those who are interested to have hands on the material, in particular PhD students who choose to have an oral exam.

Continuum Mechanics

The four courses in the Michaelmas Term are intended to provide a broad educational background for any student preparing to start a PhD in fluid dynamics. The courses in the Lent Term are more specialized and in some cases (see the course descriptions) build on the Michaelmas Term material.

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice, familiarity with the continuum assumption, the material derivative, the stress tensor and the Navier-Stokes equation will be assumed, as will basic ideas concerning incompressible, inviscid fluid mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable. Previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is desirable for some courses. No previous knowledge of solid mechanics, Earth Sciences, or biology is required.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses

Year	*Courses*
First	Differential Equations, Dynamics and Relativity, Vector Calculus, Vectors & Matrices.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves, Asymptotic Methods.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Pre-requisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Preliminary Reading

1. D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7
2. G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.
3. L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 & 5.

Literature

1. J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).
2. S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)
3. C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).
4. O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Perturbation Methods (M16)

S.J. Cowley

This course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in some cases can provide accurate predictions when the parameter or coordinate has only moderately large or small values.

Some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. The advantage of uniformly valid expansions for comparison with experiment and numerical solutions will be covered. [7]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. Further examples will be given of asymptotics beyond all orders. This section will include a brief introduction to the summation of [divergent] series, e.g. covering Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, and Domb-Sykes plots. [6]

- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKB[JLG]' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied, potentially including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium). [3]

Pre-requisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward differential equations and partial differential equations and evaluate simple integrals.

Literature

Relevant Textbooks

1. Bender, C.M. & Orszag, S., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to 'Stokes' lines as 'anti-Stokes' lines, and vice versa. The course will use Stokes' convention.*

2. Hinch, E.J., *Perturbation Methods*, Cambridge University Press (1991). *This is the book of the course; some view it as somewhat terse.*

3. Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975). *This is the original book on perturbation methods; somewhat dated, but still a useful read.*

Reading to Complement Course Material

1. Berry, M.V., *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).

2. Boyd, J.P., *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, **56**, 1-98 (1999). Also available at

   `http://hdl.handle.net/2027.42/41670` and
   `http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf`.

3. Kevorkian, J. & Cole, J.D., *Perturbation Methods in Applied Mathematics*, Springer (1981).

Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.

Non-Newtonian Fluid Mechanics (M16)

Eric Lauga

Standard courses in fluid mechanics are concerned with the dynamics of Newtonian flows. In the Newtonian limit, viscous stresses depend linearly on the instantaneous deformation rate of the fluid. However, in many instances relevant to industry as well as natural and physical sciences, a wide variety of fluids display non-Newtonian behaviour. In fact, we are all familiar with these fluids in our daily life. For example, in the kitchen, while water and olive oil are Newtonian, mayonnaise and ketchup are non-Newtonian fluids. Similarly, in the bathroom, toothpaste, shampoo and shaving cream are materials which can be made to flow like liquids but also share many properties with elastic solids. Most biological fluids are also non-Newtonian, in particular blood. In this course, we give an introduction to the mathematical modelling of flowing Non-Newtonian fluids.

After introducing the experimental phenomenology of non-Newtonian flows, we will present the mathematical frameworks to tackle:
(i) Generalised Newtonian fluids with instantaneous but nonlinear stress-deformation responses;
(ii) Linear viscoelastic fluids that have a memory of their past deformation;
(iii) Nonlinear viscoelastic fluids displaying normal stress differences and resistance to extension;
(iv) Yield-stress fluids that can only deform if applied stresses exceed critical values;
(v) Viscoelastic instabilities.

Throughout the course, mathematical modelling will be motivated and compared with experiments. At the end of the course, students will be equipped with the necessary skills to carry out independent research in complex fluids and rheology relevant to a wide range of scientific problems, for both fundamental research and industry.

Pre-requisites

Undergraduate fluid dynamics, vector calculus and mathematical methods.

Literature

1. National Committee for Fluid Mechanics Film on "Rheological Behavior of Fluids" at: `http://web.mit.edu/hml/nc`
2. F. A. Morrison (2001) Understanding Rheology, Oxford University Press.
3. R. B. Bird, C. F. Curtiss, R. C. Armstrong, and O. Hassager (1987) Dynamics of Polymeric Liquids, Vol. 1: Fluid Mechanics, 2nd ed, Wiley.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

Fluid Dynamics of Climate (M24)

P.H. Haynes & J.R. Taylor

Understanding the Earth's climate and predicting its future evolution is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the climate system, helping to make the planet hospitable for life. The dynamical complexity of this fluid motion and the wide range of space and time scales involved is one of the most difficult aspects of climate prediction.

This course, focusing on the large-scale behaviour of stratified and rotating flows, provides an introduction to the fluid dynamics necessary to build mathematical models of the climate system. The course begins by considering flows which evolve on a timescale which is long compared with a day, where the Earth's rotation plays an important role. The rotation is felt through the Coriolis force (a fictitious force arising from use of a frame of reference rotating with the Earth) which causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics, particularly in combination with stable density stratification. Canonical models are introduced and studied to illustrate phenomena such as adjustment to a state of geostrophic balance, where Coriolis force balances pressure gradient, new wave modes that can communicate dynamical information on both regional and global scales, and new hydrodynamic instabilities that lead to atmospheric weather systems and ocean eddies.

The course then moves on apply these basic ideas to important aspects of the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the structure and hence the effects of eddies and weather systems, the dynamics of ocean gyres and boundary currents like the Gulf Stream, the dynamics of the meridional (north/south) circulation in the ocean and atmosphere and the associated transport of heat and of chemical and biological tracers and special dynamics of tropical regions which give rise to phenomena such as El Nino.

Pre-requisites

Undergraduate fluid dynamics

Reading to complement course material

1. Vallis, G.K. Atmospheric and Oceanic Fluid Dynamics (2nd edition). Cambridge University Press. (2017).
2. Gill, A.E., Atmosphere-Ocean Dynamics. Academic Press (1982).
3. Marshall, J. and R.A. Plumb. Atmosphere, Ocean, and Climate Dynamics. Academic Press. (2008).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Fluid Dynamics of the Environment (L24)

S.B. Dalziel

Understanding the environment and predicting the impact of human activity on it are critical challenges in our time. Whether we are concerned about climate change, pollution or the spread of infectious aerosol droplets within our buildings, fluid dynamics plays a vital role. This course explores the basic fluid dynamics necessary to build mathematical models of the environment in which we live, focusing on problems which occur over sufficiently small time and length scales to be largely unaffected by the earth's rotation.

The course begins by considering fluid flow in the presence of (typically small) density variations. If the fluid is stably stratified, 'internal gravity waves' can occur since the stratification provides a restoring force when fluid parcels are displaced vertically. The course highlights some of the rich and surprising dynamics of these waves. For example, internal gravity waves radiate energy vertically as well as horizontally, and their interaction with boundaries can focus this energy and cause mixing far from where the energy was input.

Density variations within fluids can also drive the flow and the course will consider two important related cases, where the flow is either tall and thin or long and shallow. Both cases allow substantial simplification of the governing equations by integrating them over the smaller dimension. One example of the first case is the flow from a localised source of buoyancy that can lead to the formation of a tall, thin buoyant plume moving vertically away from the source. Volcanic eruption clouds and the thermal plumes we each produce are just two examples of this wide-spread mechanism. Turbulence, entrainment and mixing play an important part, as does the stratification of the fluid into which they move. Examples of long and shallow flows include gravity currents, again driven by density differences, but this time in the presence of boundaries or large changes in the density of a stratified ambient fluid. In such cases, waves on the interface that develops play an important role in governing the behaviour of the flow. Such flows develop, for example, when you open the door on a cold day, or as a particle-laden pryoclastic flow from a volcanic eruption. When confined by geometry, the combination of both these limiting cases plays an important role in determining the ventilation of a room and hence one of the potential mechanisms for transmission of airborne infections.

Pre-requisites

Undergraduate fluid dynamics is desirable.

Literature

Reading to complement course material

1. B. R. Sutherland, Internal gravity waves, Cambridge University Press (2010).
2. J. S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).
3. J. Pedlosky, Geophysical Fluid Dynamics, Springer (1987).

Additional support

In addition to the lectures, four examples sheets will be provided and four associated examples classes will run in parallel to the course. There will be a revision class in the Easter Term.

Theoretical Physics of Soft Condensed Matter (L24)

Mike Cates, Ronojoy Adhikari, Rob Jack

Soft Condensed Matter refers to liquid crystals, emulsions, molten polymers and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, generally with noise so that the equations of motion are stochastic PDEs. The form of these equations is helpfully constrained by the requirement that the Boltzmann distribution is regained in the steady state (when this indeed holds, i.e. for systems in contact with a heat bath but not subject to forcing). Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a 'smectic liquid crystal' (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field (and several steps short of the renormalization group, whose application to classical field theories is discussed in other courses but not this one).

Important models of soft matter include diffusive ϕ^4 field theory ('Model B'), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates 'Model H', a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth. We will also explore the continuum theory of nematic liquid crystals, which spontaneously break rotational but not translational symmetry, focusing on topological defects and their associated mathematical structure such as homotopy classes.

A section of the course will present the mechanical equations for low-dimensional soft materials informed by concepts of topology and differential geometry. We will first identify kinematic variables suitable for the description of one-dimensional materials like filaments and two-dimensional materials like membranes, and then consider dynamical conservation laws, emphasising their topological character. We will move on to material-specific relations that close the conservation laws, emphasising their geometric character. These general principles will be illustrated by examples of specific one- and two-dimensional materials.

Finally, the course will analyse soft-matter systems whose microscopic dynamics does not have time-reversal symmetry, such as self-propelled colloidal swimmers. We will discuss how the absence of time-reversal symmetry leads to qualitative changes in dynamical behaviour, both for averaged quantities and for fluctuations. This part of the course will describe some general results, particularly fluctuation theorems, and their consequences for the observation of rare events. The implications of these results in specific soft-matter systems may also be discussed.

Note on lectures

Approximately 16 lectures will be given by Prof. Cates followed by about 4 lectures each from Dr Adhikari and Dr Jack respectively.

Pre-requisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements in part the following Michaelmas Term courses although none are prerequisites: Statistical Field Theory; Non-Newtonian Fluid Dynamics; Slow Viscous Flow; Quantum Field Theory.

In previous years the audience has included a mix of students whose main specialism is either fluid dynamics or field theory. People with these differing backgrounds may find different parts of the course easier or harder, but the intention is to create a roughly level playing field.

Preliminary Reading

1. D. Tong *Lectures on Statistical Physics*

   `http://www.damtp.cam.ac.uk/user/tong/statphys.html`

   Before embarking on this course you do need to understand the equation $F = -k_B T \ln Z$ and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Familiarity with the Landau theory of phase transitions is desirable. We will not need much abstract thermodynamics (e.g. Maxwell relations) but you do need to know the zeroth, first and second laws. The above lecture notes are an excellent resource for revising and reviewing the key material.

2. M. E. Cates and E. Tjhung *Theories of binary fluid mixtures: from phase-separation kinetics to active emulsions.* J. Fluid Mech. (2018), **836**, pp1-66.

   `https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/theories-of-binary-fluid-mixtures-from-phaseseparation-kinetics-to-active-emulsions/5BD133CB20D89F47E724D77C296FEF80/share/106fd30f307db12134745de39fd568fbbaa3f9d2`

This JFM perspectives article has significant overlap (perhaps 40%) with the course but takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics when needed. It gives a good flavour of the types of problem we will address and some of the methodologies involved. However we will not have time to cover much of the material it contains on active systems.

Literature

So far there are no books that treat this material at the right level. But it may be worth looking at:

1. P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics.* Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, topological defects all feature in the description of condensed matter systems at $\hbar = 0$. More for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will also be a two-hour revision class in the Easter Term.

Unofficial lecture notes

Unofficial lecture notes were taken several years ago when this course was 16 lectures long and lectured by Prof. Cates alone – effectively the first 16 lectures of this course:

`https://dec41.user.srcf.net/h/III_L/theoretical_physics_of_soft_condensed_matter`

These notes are far from perfect but are mentioned here so that all students are equally aware of their existence.

Hydrodynamic Stability (L16)

Rich Kerswell

Developing an understanding of how "small" perturbations grow, saturate and modify fluid flows is central to addressing many challenges of interest in fluid mechanics. Furthermore, many applied mathematical tools of much broader relevance have been developed to solve hydrodynamic stability problems, and hydrodynamic stability theory remains an exceptionally active area of research, with several exciting new developments being reported over the last few years.

In this course, an overview of some of these recent developments will be presented. After a brief introduction to the general concepts of flow instability, presenting a range of examples, the major content of this course will be focussed on the broad class of flow instabilities where velocity "shear" and fluid inertia play key dynamical roles. Such flows, typically characterised by sufficiently"high" Reynolds number Ud/ν, where U and d are characteristic velocity and length scales of the flow, and ν is the kinematic viscosity of the fluid, are central to modelling flows in the environment and industry.

A hierarchy of mathematical approaches will be discussed to address a range of stability problems, from more classical concepts of normal mode growth on laminar parallel shear flows and their subsequent weakly nonlinear behaviour, to transient but significant growth of infinitesimal perturbations and finite amplitude instabilities.

Pre-requisites

Undergraduate fluid mechanics, linear algebra, complex analysis and asymptotic methods.

Literature

1. F. Charru *Hydrodynamic Instabilities* CUP 2011.
2. P. G. Drazin & W. H. Reid *Hydrodynamic Stability* 2nd edition. CUP 2004.
3. P. J. Schmid & D. S. Henningson, *Stability and transition in shear flows.* Springer, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be an hour and a half revision class in the Easter Term.

Fluid dynamics of the solid Earth (L16)

Dr. Jerome A. Neufeld

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth's core is formed by the solidification of a mixture of molten iron and various lighter elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth's magnetic field. On million year timescales, the solid mantle convects, and as it upwells to the surface it partially melts leading to the volcanism. At the surface, convection drives the motion of brittle plates which are responsible for the Earth's topography as can be felt and imaged through the seismic record. In the Earth's surface, fluids flow through porous rocks as in porous, groundwater aquifers which feed streams and rivers which erode the solid surface. On the Earth's surface, similar physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of sea ice to the flow of glacial ice as over land and as ice shelves over the ocean.

This course will use the wealth of observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials. The focus will be on the generation and solution of mathematical models motivated by observational data.

Desirable Previous knowledge

A basic understanding of viscous fluid dynamics. Mathematical methods, particularly the solution of ordinary and partial differential equations.

Literature

1. M.G. Worster. *Solidification of Fluids.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
2. H.E. Huppert. *Geological fluid mechanics.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
3. D.L. Turcotte, G. Schubert. Geodynamics, second edition. CUP (2002)

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Computation Methods in Fluid Mechanics (M16)

John Hinch

The aim of this Graduate course is to provide an overview of some of the computational methods used to solve the partial differential equations that arise in fluid dynamics and related fields. The idea is to provide a feel for the computational methods rather than study them in depth.

The course will start with a four-lecture introduction to the numerical solution of the Navier-Stokes equations at moderate Reynolds number; the issues and difficulties will be highlighted.

Next some general issues will be covered in greater detail.

- Discretisations: finite difference, finite element and spectral.
- Time-stepping: explicit, implicit, multi-step, splitting, symplectic.
- Solution of Linear Systems: packages, LU and QR decompositions, sparse matrices, conjugate gradients, eigenproblems.

The remaining lectures will focus on specific issues selected from the following.

- Demonstration of the software package FreeFem++.
- Methods for hyperbolic equations.
- Representation of surfaces.
- Boundary Integral/Element Method.
- Fast Poisson Solvers: Multigrid, Fast Fourier, Domain Decomposition.
- Fast Multipole Method.
- Nonlinear considerations.
- Particle Methods.
- Wavelets.

Pre-requisites

Attendance at an introductory course in Numerical Analysis that has covered (at an elementary level) the solution of ordinary differential equations and linear systems will be assumed. Some familiarity with the Navier-Stokes equations and basic fluid phenomena will be helpful (as covered by a first course in Fluid Dynamics).

Literature

1. E.J.Hinch *Think Before You Compute: a Prelude to Computational Fluid Dynamics* CUP 2020

Additional support

MatLab code will be provided for the first four lectures.

Math Biology

Mathematical phyllotaxis (L4)

Non-Examinable (Graduate Level)

Dr J. Swinton

Fibonacci numbers in plants, such as in sunflower spiral counts, have long fascinated mathematicians. Most analyses are variants of a Standard Model in which organs are treated as point nodes successively placed on a cylinder according to a given function of the previous node positions, not too close or too far away from the existing nodes. These models usually lead to lattice solutions. As a parameter of the model, like the diameter of the cylinder, is changed, the lattice can transition to another, more complex lattice, with a different spiral count. It can typically be proved that these transitions move lattice counts to higher Fibonacci numbers. While mathematically compelling, empirical validation of the Standard Model is as yet weak.

The course will begin with a gallery of examples of Fibonacci patterning and a survey of the quantitative datasets available. We will give a brief history of mathematical approaches, including a partially successful attempt by Alan Turing. We study the mathematics of lattices on cylinders and classify lattice space using a fractal decomposition with close links to number theory. We will see the general properties a model will need to have to lead generically to Fibonacci structure. We will then introduce a range of biological models and survey the links between models and data, from the statistical to the molecular.

This non-examinable course will consists of four lectures and an examples class. It is strongly recommended for anyone who intends to offer the Mathematical Phyllotaxis essay.
Pre-requisites There are no pre-requisites for this course.

Preliminary reading There is no preliminary reading for this course.

Literature

- Roger V. Jean. *Phyllotaxis: A Systemic Study in Plant Morphogenesis.* Cambridge University Press, 1994
- Roger V. Jean and Denis Barabé. *Symmetry in Plants.* World Scientific, 1998.
- Jonathan Swinton. *A Textbook of Mathematical Phyllotaxis.* Deodands Ltd, 2020. This text is available online to students on request to jonathan@swintons.net.