Mathematical Tripos
Part III Lecture Courses in 2019-2020

Department of Pure Mathematics & Mathematical Statistics

Department of Applied Mathematics & Theoretical Physics

Notes and Disclaimers.

- Students may take any combination of lectures that is allowed by the timetable. The examination timetable corresponds to the lecture timetable and it is therefore not possible to take two courses for examination that are lectured in the same timetable slot. There is ***no*** requirement that students study only courses offered by one Department.
- The code in parentheses after each course name indicates the term of the course (M: Michaelmas; L: Lent; E: Easter), and the number of lectures in the course. Unless indicated otherwise, a 16 lecture course is equivalent to 2 credit units, while a 24 lecture course is equivalent to 3 credit units. Please note that certain courses are *non-examinable*, and are indicated as such after the title. Some of these courses may be the basis for Part III essays.
- At the start of some sections there is a paragraph indicating the desirable previous knowledge for courses in that section. On one hand, such paragraphs are not exhaustive, whilst on the other, not all courses require all the pre-requisite material indicated. However you are strongly recommended to read up on the material with which you are unfamiliar if you intend to take a significant number of courses from a particular section.
- The courses described in this document apply only for the academic year 2019-20. Details for subsequent years are often broadly similar, but *not* necessarily identical. The courses evolve from year to year.
- Please note that while an attempt has been made to ensure that the outlines in this booklet are an accurate indication of the content of courses, the outlines do *not* constitute definitive syllabuses. The lectures and associated course materials as offered in this academic year define the syllabus. Each course lecturer has discretion to vary the material covered.
- Some courses have no writeup available at this time, in which case you will see "No description available" in place of a description. Course descriptions will be added to the online version of the Guide to Courses as soon as they are provided by the lecturer. Until then, the descriptions for the previous year (available at http://www.maths.cam.ac.uk/postgrad/mathiii/courseguide.html) may be helpful in giving a rough idea of course content, but beware of the comments in the preceding item on what defines the syllabus.

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Algebra

Algebra(M24)

Christopher Brookes

The primary aim of the course is to give an introduction to the theory of commutative Noetherian algebras and modules, a theory that is an essential ingredient in algebraic geometry, algebraic number theory and representation theory. In particular we shall learn about (commutative) polynomial and power series algebras.

I shall also include a small amount of introductory material about non-commutative algebras.

Topics I hope to fit in will be

Examples, tensor products. Ideal theory for commutative Noetherian algebras, localisations. Artinian algebras (commutative and non-commutative), Artin-Wedderburn theorem. Integral dependence. Dimension theory. Filtrations and associated graded algebras. Injective and Projective modules; Ext and Tor. Derivations and differential operators. Hochschild (co-)homology.

Pre-requisites

It will be assumed that you have attended a first course on ring theory, eg IB Groups, Rings and Modules. Experience of other algebraic courses such as II Representation Theory, Galois Theory or Number Fields will be helpful but not necessary.

Literature

1. M.F. Atiyah and I.G. Macdonald, Introduction to commutative algebra, Addison-Wesley, 1969.
2. N. Bourbaki, Commutative algebra, Elements of Mathematics, Springer, 1989.
3. I. Kaplansky, Commutative rings, University of Chicago Press, 1974.
4. H. Matsumura, Commutative ring theory, Cambridge Studies 8, Cambridge University Press, 1989.
5. M.Reid, Undergraduate Commutative Algebra, LMS student texts 29, Cambridge University Press, 1995.
6. R.Y. Sharp, Steps in commutative algebra, LMS Student Texts 19, Cambridge University Press, 1990.

   The basic introductory text for commutative algebra is Atiyah and Macdonald but it doesn't go into much detail and many results are left to the exercises. Sharp fills in some of the detail but neither book goes far enough. Both Kaplansky and Matsumura cover additional material though Matsumura is a bit tough as an introduction. Reid's book is a companion to one on algebraic geometry and that influences his choice of topics and examples. Bourbaki is encyclopaedic.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Lie Algebras and their representations (M24)

Ian Grojnowski

This course is an introduction to the basic properties of finite dimensional complex Lie algebras and of their representations.

Lie algebras are 'infinitesimal symmetries'; linearisations of groups. They are ubiquitous in many branches of mathematics: in topology, in arithmetic and algebraic geometry, and in theoretical physics (string theory, exactly solvable models in statistical mechanics),...

One reason for their importance is that the finite dimensional complex representations of the simple Lie algebras are exactly the same as those of the corresponding groups. So instead of needing to study the topology and geometry of the simple Lie groups, or the algebraic geometry of the simple algebraic groups, we can use nothing other than linear algebra and still completely describe these representations.

(Later you will want to study the topology and geometry as well, of course!)

We will cover the following topics:

Definitions, motivations, and basic structure theory.

Root systems, Weyl groups, the finite simple Lie algebras.

Classification of finite dimensional representations, Verma modules, Weyl character formula.

Crystals, Littelmann paths.

If there is time, we will finish by discussing affine Lie algebras, the basic representation, Boson-Fermion correspondence, and theta functions.

Desirable Previous Knowledge

None other than linear algebra, but the part II course on representation theory (or equivalent) will be useful as background.

Reading to complement course material

1. V. Kac, Infinite dimensional Lie algebras, Cambridge University Press
2. M. Kashiwara. On crystal bases. *in* Representations of groups (Banff, AB, 1994), 155–197, CMS Conf. Proc., 16,
3. N. Jacobson. Lie algebras

Additional support

Exercises will be provided in class, and three examples classes will be given. There will be a one-hour revision class in the Easter Term.

$\mathcal{D}$-modules (M16)

Non-Examinable (Graduate Level)

Andreas Bode

$\mathcal{D}$-modules are modules over the sheaf of differential operators on an algebraic variety. They give one of the most fundamental instances of non-commutative algebra entering algebraic geometry.
In this course we will develop the basic language of $\mathcal{D}$-modules and see how it connects to areas like differential equations and representation theory. The final goal of the course is the Riemann–Hilbert correspondence, which shows that a category of $\mathcal{D}$-modules encodes topological information about the

underlying space: the category of regular holonomic $\mathcal{D}$-modules is equivalent to the category of perverse sheaves.

I plan to cover the following topics:
Basic definitions and examples. Operations (tensor product, direct and inverse image) and Kashiwara's equivalence.
Coherent and holonomic $\mathcal{D}$-modules. Characteristic varieties and Bernstein's inequality.
The six functor formalism. Minimal extensions and simple holonomic modules.
Constructible sheaves and perverse sheaves. De Rham and solution complexes.
Beilinson–Bernstein localization.
The Riemann–Hilbert correspondence.

Pre-requisites

Basic algebraic geometry (varieties, sheaves, operations of $\mathcal{O}$-modules) and some basic category theory. Some knowledge of Lie algebra representations might help to appreciate the representation-theoretic applications.

Literature

1. J. Bernstein *Algebraic theory of D-modules.* Available at

   `http://www.math.columbia.edu/~khovanov/resources/Bernstein-dmod.pdf`

2. A. Borel et at, *Algebraic D-modules*, Academic Press, 1987.

3. R. Hotta, K. Takeuchi, T. Tanisaki, *D-modules, Perverse Sheaves, and Representation Theory*, Birkhäuser, 2008.
   The course will essentially cover the first eight chapters of HTT, minimizing however the analytic part of the theory.

Profinite Groups (L24)

Gareth Wilkes

Profinite groups are groups which are the limit of a family of finite groups in a certain sense, and arise naturally in algebraic geometry, geometric group theory and Galois theory. As such they link together several regions of mathematics, being at the same time infinite groups and complete metric spaces, and exhibit properties of the theory of both finite and infinite groups. This course will introduce profinite groups and their basic properties. We will go on to explore their relations with classical groups and certain algorithmic questions: for instance, the question of deciding whether two given infinite groups are isomorphic or not.

We will also introduce the immensely powerful and versatile theory of group cohomology. As well as exhibiting further links between group theory and topology, cohomology provides computable invariants which bring out the finer structure of a group. We introduce this theory both for classical groups and profinite groups, and discuss the similarities and differences of the two theories.

Expected course synopsis (subject as always to time):

- Inverse and direct limits of groups. Definitions and basic properties of profinite groups. Gaschutz's Theorem.
- Pro-p groups and the p-adic integers; Hensel's lemma; Serre's theorem on uniqueness of topology.
- Residual finiteness properties of groups. Profinite completions. Algorithmic solutions of e.g. the word problem. Distinguishing discrete groups through their finite quotients ('profinite rigidity' questions).

- Elementary group cohomology: definitions for both discrete groups and profinite groups. Comparison of theory for discrete groups vs pro-p groups. Extensions and semi-direct products. Groups of cohomological dimension one.

Pre-requisites

Part IB Groups, Rings and Modules, Part IB Metric and Topological Spaces and Part II Algebraic Topology.

Literature

Notes will be made available on the lecturer's webpage `https://www.dpmms.cam.ac.uk/~grw46/`.

1. K. Brown, *Cohomology of Groups.* Springer 1982.
2. L. Ribes and P. Zalesskii, *Profinite Groups.* Springer 2000.
3. J-P. Serre, *Galois Cohomology.* Springer 2013.
4. J.S. Wilson, *Profinite Groups.* Clarendon Press 1998.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Modular Representation Theory (L24)

Stuart Martin

Modular representation theory, the study of representations over fields of characteristic other than zero, was initiated by L.E. Dickson (1902). Such representations arise naturally in number theory, the theory of error-correcting codes, combinatorics and in topology.

The first really major developments came with Richard Brauer, who from 1935 onwards exploited this rich and virtually untapped area of mathematics. Brauer's methods were mainly character-theoretic. What is now called the theory of Brauer characters determines the composition factors of the representation but, unlike the situation over $\mathbf{C}$, not the equivalence type. One of Brauer's main goals was finding numerical constraints on the orders and internal structure of finite simple groups, and his methods were later used by Glauberman and others in the Classification program.

The next revolution in the theory came with Sandy Green, who considered the modules themselves. His techniques were completely different to those of Brauer, and his main goals lay in understanding the modules, rather than Brauer, who worked mostly with characters and blocks (a set of equivalence classes of characters). The aim of this course is to give a flavour of both classical techniques of Brauer and the more modern techniques of Green. We'll discuss Brauer characters, defect groups, blocks, decomposition numbers as well as projective and injective modules. The course culminates in a proof of Brauer's first main theorem, which establishes a correspondence between blocks with a certain defect group D and blocks of the normaliser $N_G(D)$ with defect group D.

- Review of assumed and basic material: group algebras, representations and modules, reducibility and decomposability, Maschke's theorem, tenors and homs, Frobenius reciprocity, Mackey's theorem, maximal and primitive ideals, Jacobson radical, complete reducibility, semisimple rings, Artin-Wedderburn structure theorem.
- Modular character theory: p-singular and p-regular elements, Brauer characters, Grothen-dieck groups.

- Irreducible, projective and injective modules: DVRs, (splitting) p-modular systems, forms, decomposition numbers and the decomposition matrix, counting modular irreducible modules.
- Projective indecomposable modules: the Higman criterion, primitive orthogonal idempotents, idempotent refinement.
- Block theory I: the socle and the head, Cartan matrix, lifting projectives, central idempotents.
- Central characters: the central character of an ordinary irreducible mod p determines the block.
- Block theory II: defect groups. relative projectivity and blocks of defect 0.
- The Brauer morphism and the Brauer correspondence; Brauer's first main theorem. If time, (statements of) the Alperin conjecture, comments on local–global conjectures, cyclic defect theory, representation type.

One or two sheets of examples will be provided backed up by classes.

Desirable Previous Knowledge

Basic group theory; ordinary representations/character theory from the Part II course; Sylow theory for finite groups; commutative algebra from Part III courses (rings, ideals, completions, local rings, primality, Artin-Wedderburn theorem); Lie algebras from the Part III course (weight spaces, root systems, Verma and Weyl modules); some categorical nonsense.

Introductory Reading

1. C.W. Curtis, Pioneers of representation theory: Frobenius, Burnside, Schur and Brauer (AMS 1999)
2. I.M. Isaacs, Character theory of finite groups (Dover reprint 1994)
3. G.D. James & M.W. Liebeck, Representations and characters of groups (CUP, 2nd edn 2001)
4. J.P. Serre, Linear Representations of Finite Groups (Springer GTM 42)
5. R.P. Langlands, Representation theory: its rise and its role in number theory. Proceedings of the Gibbs Symposium (New Haven, 1989).

Reading to complement course material

1. J.L. Alperin, Local representation theory (CUP 1986)
2. D.J. Benson, Representations and cohomology, Vol 1 (CUP 1991)
3. K. Erdmann & T. Holm, Algebras and representation theory (Springer 2018)
4. P. Landrock, Finite group algebras and their modules (CUP 1984)
5. P. Schneider, Modular representation theory of finite groups, (Springer 2013)
6. P. Webb, A course in finite group representation theory (CUP 2016). Chapters 6–12.

Algebraic Geometry

Algebraic Geometry (M24)

Mark Gross

This will be a basic course introducing the tools of modern algebraic geometry. The most relevant reference for the course is the book of Hartshorne and the notes of Vakil.

The course will begin with a quick review of the theory of varieties as presented in the Part II algebraic geometry course (see e.g., the book of Reid for this background) and then proceeding to sheaves and the notion of an abstract variety. We then turn to an introduction to scheme theory, explaining why we want schemes and what they will do for us. We define schemes and introduce projective schemes. From there, we will pass to coherent sheaves, and introduce a number of tools, such as sheaf cohomology, necessary for any practicing algebraic geometer, with applications to problems in projective geometry.

Pre-requisites

Basic theory on rings and modules will be assumed. It is strongly recommended that students either have had a previous course on Commutative Algebra or had a quick read of the book on Commutative Algebra by Atiyah and MacDonald, and/or the elementary text by Reid on Algebraic Geometry.

Literature

Introductory Reading

1. M. Reid, *Undergraduate Algebraic Geometry*, Cambridge University Press (1988) (preliminary reading).
2. M. Atiyah and I. MacDonald, *Introduction to Commutative Algebra*, Addison–Wesley (1969) (basic text also for the commutative algebra we'll need).

Reading to complement course material

1. U. Görtz, T. Wedhorn, *Algebraic Geometry I*, Vieweg+Teubner, 2010.
2. R. Hartshorne, *Algebraic Geometry*, Springer (1977) (more advanced text).
3. R. Vakil, *The rising sea. Foundations of Algebraic Geometry*, available at http://math.stanford.edu/∼vakil/216blog/index.html

Additional support

Four examples sheets will be provided and four associated examples classes will be given.

Toric geometry (M24)

Dhruv Ranganathan

This course will provide an introduction to the theory of toric varieties in algebraic geometry. Toric varieties possess intricate geometry, but can often be understood through a combinatorial framework. Various constructions in modern algebraic geometry can be made explicit on them, which makes them fertile for the development and testing of general theory. Toric varieties form a fundamental building block in a number of modern research areas, including mirror symmetry and tropical geometry.

The course will begin with examples and constructions, and the basic classification theorem of toric varieties and their morphisms in terms of polyhedral combinatorics. A detailed discussion of toric surface theory will be followed by the fundamental properties and constructions: compactness, smoothness, quotients, blowups, and resolution of singularities in the toric context. All of this theory will be applied to give a proof of the toric semistable reduction theorem, one of cornerstones of modern moduli theory. The final part of the course will cover line bundles and sheaves on toric varieties, and their cohomological properties.

The course will include a large number of examples, designed to demonstrate the fundamental concepts in modern algebraic geometry.

Pre-requisites

The basic pre-requisite is Part II algebraic geometry and the basic theory of rings and modules, which will be assumed. Some familiarity with algebraic topology and manifolds will be helpful. The course will be optimized for students who take the Part III algebraic geometry course in parallel, as the toric situation will provide a large number of examples in which to see those concepts in action.

A student who has not taken a course in schemes or who will not take the Part III algebraic geometry course in parallel will need to do additional background reading as the course progresses in order to follow the lectures, and is encouraged to write to the lecturer before enrolling in the course.

Preliminary Reading

1. Atiyah and MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley (1969). (Chapters 1–3, 9,10 contain what we will need)
2. Reid. *Undergraduate algebraic geometry.* Cambridge: Cambridge University Press, 1988.

Literature

1. Fulton, William. *Introduction to toric varieties.* Princeton University Press, 1993.
2. Cox, David A., John B. Little, and Henry K. Schenck. *Toric varieties.* American Mathematical Soc., 2011.

Additional support

Detailed lecture notes will be made available online.

Four examples sheets will be provided, and three associated examples classes will be given. There will be a revision class during the easter term.

Complex Manifolds (L24)

Ruadhaí Dervan

Complex manifolds are the holomorphic analogue of smooth manifolds. Special cases of complex manifolds include Riemann surfaces and smooth algebraic varieties, so the theory of complex manifolds is much more rigid than differential geometry, and the tools used are quite different. The main goal of this course is to cover the basic theory of complex manifolds. The course will culminate in a result explaining which compact complex manifolds are actually smooth projective varieties (however, no knowledge of algebraic geometry will be assumed).

A preliminary outline of the course is as follows:

- Basic concepts of complex manifolds, holomorphic vector bundles, holomorphic tangent and cotangent bundles.

- Brief description of sheaf cohomology, with deduction of de Rham and Dolbeault cohomology for complex manifolds.
- Hermitian metrics, connections, curvature and Chern classes for complex vector bundles. Case of holomorphic vector bundles.
- Harmonic forms: the Hodge theorem and Serre duality.
- Compact Kähler manifolds. Hodge and Lefschetz decompositions on cohomology, Kodaira vanishing, Kodaira embedding theorem.

Pre-requisites

A knowledge of basic Differential Geometry at the level of the Michaelmas term course is essential. Part III Algebraic Geometry would be helpful for motivation for certain aspects of the course, but is not necessary. A working knowledge of complex analysis in one variable is also necessary; a review of the relevant material is given in Chapter 1 of the Huybrechts book mentioned below.

1. J. P. Demailly *Complex analytic and differential geometry.* Available as a pdf at

   `https://www-fourier.ujf-grenoble.fr/~demailly/documents.html`

2. P. Griffiths and J. Harris, *Principles of Algebraic Geometry.* Wiley, 1978.
3. D. Huybrechts, *Complex Geometry - an introduction*, Springer, 2004.
4. C. Voisin, *Hodge theory and complex algebraic geometry. I*, Cambridge Studies in Advanced Mathematics, 2007.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Cubic hypersurfaces (L16)

Non-Examinable (Graduate Level)

Daniel Huybrechts[1]

The course will be an introduction into modern aspects of cubic hypersurfaces (mainly of dimension three and four). Main emphasize will be put on Hodge theoretic and geometric aspects but we will also touch upon derived categories.

Pre-requisites

This course assumes a good understanding of basic methods in complex and algebraic geometry. Standard references are listed below. Towards the end of the course we will also need certain homological tools, e.g. derived categories of coherent sheaves on varieties and the Fourier–Mukai formalism. (Depending on the background of the audience, those could be reviewed quickly.)

[1] visiting from Bonn, holding G.C. Steward Fellowship at Gonville and Caius College

Literature

1. Ph. Griffiths, J. Harris *Principles of Algebraic Geometry.* Wiley 2nd edition (1994).
2. R. Hartshorne *Algebraic geometry.* GTM 52 Springer (1977).
3. D. Huybrechts *Complex geometry - an introduction.* Universitext Springer (2004).
4. D. Huybrechts *Fourier–Mukai transforms in algebraic geometrx.* Oxford Mathematical Monographs (2006).
5. C. Voisin *Hodge Theory and Complex Algebraic Geometry, I & II.* CUP (2007).

Additional support

There are notes of a similar course `http://www.math.uni-bonn.de/people/huybrech/Notes.pdf` which will be expanded and augmented during the lectures.

Analysis

Analysis of Partial Differential Equations (M24)

Dr Warnick

This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is nowadays a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on developing the theory and methods of the modern approach to PDE theory. Emphasis will be given to functional analytic techniques, relying on a priori estimates rather than explicit solutions. The course will primarily focus on approaches to linear elliptic and evolutionary problems through energy estimates, with the prototypical examples being Laplace's equation and the heat, wave and Schrödinger equations.

The following concepts will be studied: well-posedness; the Cauchy problem for general (nonlinear) PDE; Sobolev spaces; elliptic boundary value problems: solvability and regularity; evolutionary problems: hyperbolic, parabolic and dispersive PDE.

Pre-requisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Preliminary Reading

The following article gives an overview of the field of PDEs:

1. Klainerman, S., *Partial Differential Equations*, Princeton Companion to Mathematics (editor T. Gowers), Princeton University Press, 2008.

Literature

1. Some lecture notes from a previous lecturer of the course are available online at: `http://cmouhot.wordpress.com/teachings/`.

   The following textbooks are excellent references:

2. Evans, L. C., *Partial Differential Equations*, Springer, 2010.
3. Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.
4. John, F., *Partial Differential Equations*, Springer, 1991.

Additional Information

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Introduction to nonlinear analysis (M24)

Pierre Raphaël

This class in an introduction to the basic analytic tools needed for the mathematical study of nonlinear problems arising from mathematical physics. A particular emphasis will be made on the classical description of fundamental non linear waves discovered in the 19th century: solitons or solitary waves. The exact role of these bubbles of energy in many systems is still mysterious and the subject of an intense research activity. We will review the main classical methods at hand for the mathematical description of these objects:
(i) the Lyapounov-Schmidt theorem and the construction of nonlinear bifuration branches;
(ii) the nonlinear differential equations approach: local solutions, Lyapounov functional and phase portraits;
(iii) the variational approach: minimization of functionals in Hilbert and Banach spaces.

We will then discuss the dynamical properties of these objects and in particular the stability property for some canonical non linear models.

1. Basic tool box of modern analysis.

– Lebesgue's dominated convergence theorem, L^p spaces, Hölder and Young inequalities, the density of smooth functions in $L^p(\mathbb{R}^d)$.
– Hilbert spaces: weak convergence and compact operators.
– A crash course on distributions: definition, examples and basis properties. Derivative in the sense of distributions.
– Continuous Fourier analysis: Fourier in L^1, L^2 and tempered distributions.
– Sobolev spaces $H^s(\mathbb{R}^d)$: definition, Hilbertian structure.
– The Sobolev embedding Theorem and compactness.

2. A canonical model: the harmonic oscillator.

– The Cauchy Lipschitz theorem for ode's: local existence, global existence for linear equations and the blow up criterion.
– Inverting the Laplace operator: ode vs pde approach.
– Inverting the harmonic oscillator: the ode approach in dimension one.
– Inverting the harmonic oscillator: the pde approach, Lax Milgram theorem and variational methods in Hilbert spaces.
– Compactness of the resolvent, the spectral theorem and Legendre polynomials.

3. The Lyapounov-Schmidt approach.

– Computing the spectrum for a perturbed model: formal expansion.
– The Lyapounov-Schmidt approach and the implicit function theorem.
– Starting the bifurcation branch for a non linear model.

3. Phase portrait: solitons for the nonlinear Schrödinger equation.

– Phase portrait in dimension 1.
– Lyapounov monotonicity formulas for higher order non linearities.

4. An introduction to nonlinear variational methods.

– Spherically symmetric Sobolev embedddings.
– The ground state solution to the nonlinear Schrödinger equation.
– Extremal interpolation inequalities and best constant problems.
– Vortices in fluid dynamics: the travelling ring of smoke.

5. The stability of solitons problem

– The linear Schrödinger equation in $\mathbb{R}^d$ and dispersion of linear waves.
– The nonlinear Schrödinger equation in $\mathbb{R}$: local existence and blow up criterion.
– The stability problem for the ground state solitary wave.

Pre-requisites

Basic notions of Hilbert spaces (Hilbertian basis, weak convergence, compact operators). Basic notion of integration (Lebesgue's dominated convergence). Continuous Fourier transform (in L^1).

Literature

1. J.-M. Bony : Intégration et analyse hilbertienne, *cours de l'École Polytechnique*, 2006.

2. J.-M. Bony : Cours d'analyse, théorie des distributions et analyse de Fourier, Éditions de l'École Polytechnique.

3. H. Brézis : *Analyse fonctionnelle. Théorie et applications*, Masson, 1984.

4. Danchin, R.; Raphaël, P., Analyse nonlinéaire, sur la stabilité des ondes solitaires, Ecole Polytechnique 2016, https://math.unice.fr/ praphael/Teaching.html

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Introduction to Geometric Measure Theory (L24)

Spencer Becker-Kahn

Geometric Measure Theory began with the use of analysis and measure theory to generalize many fundamental geometric concepts from the classical setting of smooth surfaces to much wider classes of sets. The motivation to do so came primarily from the study of area-minimizing surfaces in higher dimensions but the powerful (and sometimes technical) frameworks that were developed were also inspired by and have been applied to PDE, harmonic analysis, algebraic topology, differential geometry and a host of other geometric variational problems. This is very much an analysis-based course though and we will introduce the subject by first building up the necessary tools from analysis and measure theory (Radon measures, covering theorems, a deeper study of Lipschitz functions) and then introducing rectifiable sets and integral varifolds (the name 'varifold' comes from the words 'variational' and 'manifold', so-called because of their applicability in the geometric calculus of variations).

Pre-requisites

Essential: The core parts of real analysis, functional analysis and measure theory at an advanced undergraduate level. Being comfortable with the material in the first half of the Part II course 'Analysis of Functions' (which has as prerequisites both 'Linear Analysis' and 'Probability & Measure') is definitely an appropriate background. Occasionally some computations will require non-trivial linear algebra, but a standard undergraduate background will suffice.

Helpful: In the latter part of the class there will be some points at which it will be helpful to know what an "embedded C^1 submanifold of $\mathbf{R}^n$" is.

No background in differential geometry or PDE is necessary.

Literature

1. L. C. Evans & R. F. Gariepy *Measure Theory and Fine Properties of Functions.* Chapman and Hall/CRC, 2015.

2. L. Simon *Lectures on Geometric Measure Theory.* The Australian National University, Mathematical Sciences Institute, Centre for Mathematics & its Applications, 1983. Draft of revised version available at:

   `https://web.stanford.edu/class/math285/ts-gmt.pdf`

Evans and Gariepy [1] will be most useful for the first half of the class and parts of Simon's notes [2] will be most useful for the latter parts of the class.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Elliptic Partial Differential Equations (L 24)

Neshan Wickramasekera

This course will provide an introduction to the theory of linear second order elliptic partial differential equations. Second order elliptic equations play a fundamental role in many areas of mathematics including geometric analysis and mathematical physics. Linear elliptic theory provides the foundation for studying a number of non-linear problems arising in these and other fields such as minimal submanifolds, harmonic maps and certain evolution problems in geometry and mathematical physics.

The course will provide a rigorous treatment, based on a priori estimates, of both classical and weak solutions to linear elliptic equations, focusing on the question of existence and uniqueness of solutions to the Dirichlet problem and the question of regularity of solutions. Specific topics include harmonic functions, maximum principles for general second order equations, Schauder estimates (via L. Simon's scaling argument), the continuity method for existence of solutions, divergence form operators, De Giorgi–Nash–Moser estimates and the Harnack theory and, as time permits, a discussion of some aspects of the quasilinear theory centered around the prototypical minimal surface equation.

Pre-requisites

The Part III course *Analysis of PDEs* is useful (but not essential).

Literature

1. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order.*

2. L. Simon, *Schauder estimates by scaling.* Calc. Var. & PDE, **5**, (1997), 391–407.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Metric Embeddings (L24)

András Zsák

In the area of metric embeddings, one is mostly concerned with the following problem. Given metric spaces X and Y, is there a bi-Lipschitz embedding of X into Y, and what is the best distortion of such embeddings? In most situations of interest, X is a finite metric space and Y is a Banach space, particularly L_1, L_2, or a more general L_p space. Other types of embeddings, uniform and coarse embeddings, are also important. The aim of this course is to demonstrate the richness of this theory and the variety of the techniques (analytic, combinatorial and probabilistic) through a number of topics and major results in the field. We will also indicate the connections to other areas of mathematics (optimization, graph theory, computer science, geometry, Banach space theory). We aim to cover as many of the following topics as time permits.

Basic definitions and examples

Fréchet embeddings, Aharoni's theorem

L_1-embeddings and combinatorial optimization

Euclidean distortion, Dvoretzky's theorem, Bourgain's embedding theorem

Obstructions to embeddability: Poincaré inequalities, expander graphs

Dimension reduction in L_2, the Johnson–Lindenstrauss lemma

Impossibility of dimension reduction in L_1, Diamonds and Laakso graphs

Local theory of Banach spaces, Ribe's rigidity theorem, the Ribe programme, metric characterization of some Banach space properties

Metric theory of type and cotype, non-linear Dvoretzky theorem

Coarse embeddings into c_0 and ℓ_2; coarse embeddings of ℓ_2 into Banach spaces

Pre-requisites

Undergraduate level analysis, general topology, probability and functional analysis.

Literature

1. Mikhail I. Ostrovskii *Metric Embeddings.* de Gruyter, 2013.
2. Jiří Matoušek *Lecture notes on metric embeddings.* online notes, 2013.

`http://kam.mff.cuni.cz/~matousek/ba-a4.pdf`

Additional support

Up to four examples sheets will be provided and associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Approximate group actions and Ulam stability (L16)

Non-Examinable (Graduate Level)

Oren Becker

The following is an example of a problem in group theoretic Ulam stability: Given two permutations $A, B \in \mathrm{Sym}(n)$ such that AB and BA are almost equal, can we always find $A', B' \in \mathrm{Sym}(n)$ such that A' is close to A, B' is close to B, and $A'B' = B'A'$? Equivalently, given an approximate action of $\mathbb{Z}^2$ on a finite set, is it necessarily close to a genuine action?

This question was formulated and given an affirmative answer by Arzhantseva and Paunescu. Accordingly, we say that $\mathbb{Z}^2$ is *stable in symmetric groups*, or *stable* for short. The topic of this course is the more general question: Which finitely generated groups are stable?

I aim to cover the following topics:

- The classification of amenable stable groups in terms of invariant random subgroups;
- Stability of surface groups;
- Stability and property (T);
- Relations to sofic groups and (non-)approximability problems;
- Applications to property testing and computer science.

Pre-requisites

Most of the topics require only basic knowledge in group theory. Familiarity with basic notions from geometric group theory would be useful.

Literature

1. A. Thom. *Finitary approximations of groups and their applications.* Proceedings of the ICM 2018.
2. M. De Chiffre, L. Glebsky, A. Lubotzky, A. Thom. *Stability, cohomology vanishing, and non-approximable groups.* Available at

   `https://arxiv.org/abs/1711.10238`

3. N. Lazarovich, A. Levit, Y. Minsky. *Surface groups are flexibly stable*

   `https://arxiv.org/abs/1901.07182`

4. G. Elek, *Finite graphs and amenability*, J. Funct. Anal. 263 (9) (2012) 2593–2614.
5. O. Becker, A. Lubotzky, A. Thom. *Stability and Invariant Random Subgroups*, Duke Math. J. (to appear). Also available at

   `https://arxiv.org/abs/1801.08381`

Combinatorics

Combinatorics (M16)

Béla Bollobás

What can one say about a graph or a collection of subsets of a finite set satisfying certain conditions in terms of containment, intersection and union? In the past fifty years or so, a good many fundamental results have been proved about such questions: in the course we shall present a selection of these results and their applications, with emphasis on the use of algebraic and probabilistic arguments. Beauty is the first test: there is no permanent place in the world for ugly mathematics.

The topics to be covered are likely to include the following.

Isoperimetric inequalities: vertex-isoperimetric inequalities and edge-isoperimetric inequalities; the BTBT theorem about projections; sumsets: the Cauchy-Davenport theorem, the EGZ theorem and the Moser-Scherk theorem.

Alon's Combinatorial Nullstellensatz and its applications, including the Chevalley–Warning theorem, and proofs of the Erdős–Heilbronn and Kemnitz conjectures.

Time permitting, we shall also cover some or all of the following topics.

The theorems of Sperner, EKR, LYMB, Katona, Frankl and Füredi.

Correlation inequalities, including those of Harris, van den Berg and Kesten, and the Four Functions Inequality.

Pre-requisites

The main requirement is mathematical maturity, but familiarity with the basic graph theory course in Part II would be helpful.

Additional support

Almost all the material in the course will be self-contained; some parts of the course will be supported by printed notes. There will be three examples sheets and three associated examples classes. In addition, we shall have a one-hour revision class in the Easter Term.

Ramsey Theory (M16)

Prof. I. B. Leader

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. A typical example is van der Waerden's theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions.

The flavour of the course is combinatorial. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. We shall cover a number of 'classical' Ramsey theorems, such as Gallai's theorem and the Hales-Jewett theorem, as well as some more recent developments. There will also be several indications of open problems. We hope to cover the following material.

Monochromatic Systems

Ramsey's theorem (finite and infinite). Canonical Ramsey theorems. Colourings of the natural numbers; focusing and van der Waerden's theorem. Combinatorial lines and the Hales-Jewett theorem. Applications, including Gallai's theorem.

Partition Regular Equations

Definitions and examples. The columns property; Rado's theorem. Applications. (m, p, c)-sets and Deuber's theorem. Ultrafilters; the Stone-Čech compactification. Idempotent ultrafilters and Hindman's theorem.

Infinite Ramsey Theory

Basic definitions. Not all sets are Ramsey. Open sets and the Galvin-Prikry lemma. Borel sets are Ramsey. Applications.

Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey's theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness.

Literature

B. Bollobás, *Combinatorics*, C.U.P. 1986

R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*, John Wiley 1990

Topics in discrete Fourier analysis (L16)

W. T. Gowers

The focus of this course will be on problems where one wishes to solve an equation of the form $a \circ b = c$, where $\circ$ is some binary operation (usually a group operation) and a, b and c lie in specified sets, as well as generalizations of such problems. Here are three examples.

1. A theorem of Roth, the first non-trivial case of Szemerédi's theorem, states that for every $\delta > 0$ there exists a positive integer n such that every subset A of $\{1, 2, \ldots, n\}$ of size at least δn contains an arithmetic progression of length 3, which we can interpret as saying that there is a triple (a, b, c) such that $a + b = c$ with $a, b \in A$ and $c \in \{2x : x \in A\}$.

2. A theorem of Furstenberg and Sárközy states that for every $\delta > 0$ there exists a positive integer n such that every subset A of $\{1, 2, \ldots, n\}$ contains elements a, c such that $c - a$ is a non-zero perfect square. Here we are trying to write $a + b = c$ with $a, c \in A$ and b an element of the set $\{1, 4, 9, \ldots, \lfloor\sqrt{n}\rfloor^2\}$.

3. A theorem of mine states that if n is the order of the group $\mathrm{PSL}(2, p)$ and A, B, C are any subsets of the group of size at least $100n^{8/9}$, then the equation $ab = c$ can be solved with $a \in A, b \in B$ and $c \in C$.

All these results, and many others, can be proved using discrete Fourier analysis of one kind or another. The basic Fourier theory required is very simple, but the main aim of the course is to explain why it is useful and how to use it, which is not always simple.

Prerequisites

There are few formal prerequisites for the course: for instance, if you have a good grasp of the content of the pure mathematical courses from the first two years of a typical university course, then that will be easily enough. Some prior acquaintance with Fourier analysis will help you to appreciate links with other parts of mathematics – in particular analysis – but it is not necessary.

Literature

There isn't really a book that fits the course, but some of the topics covered are included in the book Additive Combinatorics, by Terence Tao and Van Vu.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extremal Graph Theory (L16)

Andrew Thomason

Turán's theorem, giving the maximum size of a graph that contains no complete r-vertex subgraph, is an example of an extremal graph theorem. Extremal graph theory is an umbrella title for the study of how graph and hypergraph properties depend on the values of parameters. This course builds on the material introduced in the Part II Graph Theory course, which includes Turán's theorem and an introduction to the Erdős-Stone theorem.

The first half of the course will discuss the fundamental structural results of the subject, namely the Erdős-Stone theorem and Szemerédi's Regularity Lemma, together with some of their consequences. The second half of the course will consider the application of simple probabilistic arguments to the solution of extremal problems. It will include a discussion of the semi-random method (showing, for example, the existence of almost perfect Steiner systems).

Pre-requisites

An awareness of the basic concepts, techniques and results of graph theory, as afforded by the Part II Graph Theory course (such as Turán's theorem, Ramsey's theorem, Hall's theorem and so on) will be assumed, though not relied upon. Familiarity with the most elementary aspects of Probability Theory (such as the inequalities of Markov and Chebychev) will also be useful.

Literature

No book covers the course but the following can be helpful.

1. B. Bollobás, *Modern graph theory*, Graduate Texts in Mathematics **184**, Springer-Verlag, New York (1998), xiv+394 pp.
2. J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, in "Combinatorics, Paul Erdős is Eighty" (D. Miklós, V.T. Sós and T. Szőny, eds.), Bolyai Mathematical Society Studies, Vol. 2, János Bolyai Mathematical Society, Budapest (1996) 295–352.
3. N. Alon and J. Spencer, The Probabilistic Method, Wiley, 3rd ed. (2008)

Additional support

Example sheets will be supplied, and three hours of examples classes will be given. There will be a further revision class in the Easter Term.

Algebraic Methods in Combinatorics (L12)

Non-Examinable (Graduate Level)

Natasha Morrison

Linear algebraic methods are some of the most beautiful and powerful techniques in combinatorics. In this course I will present some the most appealing applications of these techniques. Topics I hope to cover include: intersecting family theorems, Fisher's inequality, Frankl-Wilson type theorems, two-families theorems with applications to saturation and weak saturation.

Pre-requisites

I will assume some very basic knowledge of linear algebra. It is helpful, but not essential, to have taken Combinatorics or Part II Graph theory.

Literature

1. L. Babai and P. Frankl, *Linear Algebra Methods in Combinatorics.* 1988.
2. J. Matoušek, Thirty-three Miniatures: Mathematical and Algorithmic Applications of Linear Algebra. 2010.

Geometry and Topology

Algebraic Topology (M24)

Jacob Rasmussen

Algebraic topology assigns algebraic invariants (groups and homomorphisms) to topological spaces and continuous maps between them. The most important example of such an invariant is ordinary homology theory, which is part of the basic language of geometry today. This course will cover homology and cohomology, together with applications to the topology of manifolds and vector bundles. The emphasis will be on learning to compute and use these invariants in a variety of examples. A tentative syllabus is as follows:

- *Homology.* Singular homology and cohomology. Eilenberg-Steenrod axioms and cellular homology. The Hurewicz homomorphism.
- *Cohomology and products.* Cohomology and the universal coefficient theorem. The Künneth theorem and cup products.
- *Vector Bundles.* Vector bundles and principal bundles. The long exact sequence of a fibration. The Euler class and the Thom isomorphism.
- *Topology of Manifolds.* Handlebodies. The fundamental class of an oriented manifold. Poincare duality. The Leftshetz fixed-point theorem.

Pre-requisite Mathematics

The only required background is basic point-set topology, but those who have not had a previous course in algebraic topology should expect to work hard to keep on top of the material in the first eight lectures, which will be quite rapid. The material in the Michaelmas term Differential Geometry course will be useful as well.

Literature

1. A. Hatcher, *Algebraic Topology*, CUP (2002).
2. D. Fuchs and A. Fomomenko, *Homotopical Topology*, (2nd ed.) Springer, 2016.
3. J.P. May, *A Concise Course in Algebraic Topology*, University of Chicago Press (1999).
4. R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Springer (1982).
5. J.W. Vick, *Homology Theory*, Springer (1994).

Hatcher is the standard text, and most closely matches the course syllabus. Vick is a more terse alternative, May more advanced. Bott and Tu and Fuchs and Fomenko offer different perspectives on some of the topics in the course.

Additional support

Four examples sheets will be provided, each with an associated examples class. There will be a two-hour revision class in the Easter Term.

Differential Geometry (M24)

A. Kovalev

This course is intended as an introduction to modern differential geometry. It can be taken with a view of further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are e.g. in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, matrix Lie groups. Tangent vectors, the tangent and cotangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Exterior algebra of differential forms. Orientability of manifolds. Partition of unity and integration on manifolds, Stokes' Theorem. De Rham cohomology.
- *Vector Bundles.* Structure group, principal bundles. The example of Hopf bundle. Bundle morphisms. Three views on connections: vertical and horizontal subspaces, Christoffel symbols, covariant derivative. The curvature form and second Bianchi identity.
- *Riemannian Geometry.* Connections on manifolds, torsion. Riemannian metrics, the Levi–Civita connection. Geodesics, the exponential map, Gauss' Lemma. Decomposition of the curvature of a Riemannian manifold, Ricci and scalar curvature, low-dimensional examples. The Hodge star and Laplace–Beltrami operator. Statement of the Hodge decomposition theorem (with a sketch-proof, time permitting).

Pre-requisites

An essential pre-requisite is a working knowledge of linear algebra (including dual vector spaces and bilinear forms) and of multivariate calculus (e.g. differentiation in several variables and the inverse function theorem). Exposure to some ideas of classical differential geometry would be useful.

Literature

[1] R.W.R. Darling, *Differential forms and connections.* CUP, 1994.

[2] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry.* Springer-Verlag, 1990.

[3] V. Guillemin, A. Pollack, *Differential topology.* Prentice-Hall Inc., 1974.

[4] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer, 1983.

Roughly, half of the course material is taken from [4]. The book [3] covers the required topology. On the other hand, [1] which has a chapter on vector bundles and on connections assumes no knowledge of topology. Both [1] and [2] have a lot of worked examples. There are many other good differential geometry texts, e.g. the five volume series by M. Spivak.

Additional support

The lectures will be supplemented by four example classes, the fourth class to take place at the beginning of Lent Term will also fulfill a revision function. Printed notes will be available from
`https://www.dpmms.cam.ac.uk/~agk22/teaching.html`

Mapping class groups (M16)

Henry Wilton

Let S be a compact, smooth, orientable surface. The group of orientation-preserving self-diffeomorphisms of S, $\mathrm{Diff}^+(S)$, is too large to conveniently study, so we instead pass to the quotient

$$\mathrm{Mod}(S) = \mathrm{Diff}^+(S)/\mathrm{Diff}_0(S)$$

by factoring out the path component of the identity element. The resulting group – the *mapping class group* of S – is both tractable to study, and encodes a great deal of information about the topology and geometry of S. Mapping class groups are ubiquitous, appearing in subjects as diverse as algebraic geometry, combinatorial group theory, symplectic geometry, dynamics and 3-manifold topology. The goal of this course is to introduce the tools that are used to study them, and to prove some fundamental results.

Pre-requisites

Part II Algebraic Topology is essential. Part II Riemann Surfaces is useful. Part III Algebraic Topology, taken concurrently, is useful.

Literature

1. B. Farb and D. Margalit *A primer on mapping class groups.* Princeton Mathematical Series, 49. Princeton University Press, Princeton, NJ, 2012. xiv+472 pp.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Symplectic Topology (L24)

Ailsa Keating

The study of symplectic manifolds originated as an extension of classical mechanics; it has since developed into a field of its own right, with connections to e.g. low-dimensional topology, algebraic geometry, and theoretical physics. The course will focus on the core foundations of symplectic topology, with an emphasis on explicit geometric techniques and examples, and end with an introduction to J-holomorphic curves,which are at the heart of modern symplectic topology.

Time allowing, topics are expected to include:

- Symplectic linear algebra. Hamilton's equations, cotangent bundles. Lagrangian submanifolds. Symplectic submanifolds. Moser's trick, Darboux and Weinstein neighbourhood theorems.
- Symplectic circle actions and moment maps, symplectic reduction.
- Surgery constructions: blow ups, symplectic fibre sums. Lefschetz pencils. Gompf's theorem on fundamental groups of symplectic 4-manifolds.
- Almost complex structures and compatible triples. Some properties of Kaehler manifolds.
- Basic properties of pseudo-homolorphic curves. Gromov non-squeezing. Symplectic capacities, Eliashberg's theorem on C^0-closure of *Symp* in *Diff*.

Pre-requisites

Essential: Algebraic Topology and Differential Geometry, at the level of the Part III Michaelmas courses. Basic concepts from Algebraic Geometry (at the level of the Part II course) will be useful.

Literature

1. D. McDuff and D. Salamon, *Introduction to symplectic topology*, 3rd edition. Oxford University Press, 2017.
2. D. McDuff and D. Salamon, *J-holomorphic curves and symplectic topology*, 2nd edition. American Mathematical Society, 2012.
3. A. Cannas da Silva, *Lectures on symplectic geometry*, Springer-Verlag, 2001.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Complex Dynamics (L24)

Holly Krieger

This course will introduce the study of iteration of rational functions of one complex variable. We will cover the local theory and the global theory, and introduce key modern ideas in the field to form a basis for further study.

Pre-requisites

Essential: IB Complex Analysis, IB Metric and Topological Spaces, II Riemann Surfaces, II Probability and Measure (the measure theory bit).

Helpful: IB Geometry, II Algebraic Topology, II Dynamical Systems.

Literature

Milnor and Douady-Hubbard (below) are the primary resources. At some points of the course, particularly in the final two weeks, there will be material presented which is not covered in textbooks.

1. J. Milnor, *Dynamics in One Complex Variable.* Any version is fine, including the early online version available on the arXiv:

   `https://arxiv.org/pdf/math/9201272.pdf`

2. A. Douady and J. H. Hubbard, *Exploring the Mandelbrot Set - Orsay Notes*, Available at

   `http://pi.math.cornell.edu/~hubbard/OrsayEnglish.pdf`

Additional resources for students seeking more detailed (in the first instance) or more accessible (in the second instance) presentations of mostly the same material:

1. A. Beardon, *Iteration of Rational Functions.* Springer, 1991.
2. L. Carleson, T. W. Gamelin, *Complex Dynamics.* Springer, 1993.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The lecturer may be contacted by email at `hkrieger@dpmms.cam.ac.uk`.

Geometric Inverse Problems (L24)

Non-Examinable (Graduate Level)

Gabriel P. Paternain

The study of geometric inverse problems is typically motivated by inverse problems in PDEs, geophysics and medical imaging. The main goal is the reconstruction of geometric structures (metrics, connections, vector bundles etc.) from either boundary measurements or local measurements. The course will describe recent developments in the area with an emphasis on the 2D picture.

The first part of the course will include a thorough discussion of the geodesic X-ray transform for functions and tensors when the background manifold is simple (i.e. it has strictly convex boundary, no conjugate points and is simply connected). Then we shall move on to attenuated X-ray transforms, including versions for connections and Higgs fields (systems).

The second part of the course will address non-linear geometric inverse problems. A full proof of boundary rigidity for simple surfaces will be given along with proofs for the recovery of a connection and a Higgs field from scattering data.

While a geometric background will be desirable, it is not strictly necessary, as the relevant tools in 2D can be provided on demand.

Pre-requisites

Part III (or Part II) Differential Geometry and Analysis of Partial Differential Equations will be helpful.

Preliminary Reading

1. P. Kuchment, *The Radon transform and medical imaging.* CBMS-NSF Regional Conference Series in Applied Mathematics, **85**. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2014.

Literature

1. J. Ilmavirta and F. Monard, *Integral geometry on manifolds with boundary and applications.* Available at

   `https://arxiv.org/pdf/1806.06088.pdf`

2. L. Pestov and G. Uhlmann, *Two dimensional compact simple Riemannian manifolds are boundary distance rigid.* Ann. of Math. **161** (2005) 1093–1110.
3. G.P. Paternain, M. Salo and G. Uhlmann, *The attenuated ray transform for connections and Higgs fields.* Geom. Funct. Anal. **22** (2012) 1460–1489.
4. *Lecture Notes.* In preparation; to be given during the course.

Additional support

The lecture notes will contain plenty of exercises and example classes can be set up if there is demand, bearing in mind the non-examinable nature of the course.

Geometric Group Theory (L16)

Non-Examinable (Graduate Level)

Ana Khukhro

Groups are algebraic objects which are well-suited to capturing notions of symmetry. As well as their intrinsic algebraic structure, groups have the ability to act on other mathematical objects such as sets or spaces. This action is often useful for learning more about either the object that the group is acting on, or the group itself. The Orbit-Stabiliser Theorem, which students will have already met, is a basic example of this phenomenon.

When a group acts on a metric space in a sufficiently nice way, one can often use the geometric properties of the space to deduce algebraic or analytic information about the group. In this way, one can build up a dictionary between algebra and geometry, which features beautiful and sometimes surprising connections between these two subjects. These connections can often be exploited to solve deep problems in other fields such as topology or analysis.

One metric space on which a group acts in a particularly pleasing way can be created using data from the group itself. Namely, by fixing a generating set of the group, one can construct a graph with vertex set equal to the set of elements of the group, with edges defined using multiplication by elements of the generating set. This graph, called a Cayley graph of the group, is not only a neat visualisation of the group, but is also an invaluable tool in modern group theory, since the geometric properties of this graph are profoundly connected to the group-theoretic properties of the group. Geometric group theory is the study of groups and spaces via these connections.

In this course, we will concentrate on some of the following aspects of this rich theory (time permitting):

- Cayley graphs; quasi-isometries; the Švarc–Milnor Lemma;
- a Smörgåsbord of geometric properties and invariants of groups, such as growth, ends, hyperbolicity, and connections to algorithmic group theory;
- analytic properties of groups, such as amenability, and connections to actions on Banach spaces.

Pre-requisites

A good knowledge of basic group theory is essential. Part II Algebraic Topology (or equivalent) is required, and some intuition in graph theory and geometry would be helpful. Some functional analysis (such as the Part II Linear Analysis course) will be useful for the last part of the course.

Literature

1. P. de la Harpe, *Topics in Geometric Group Theory*, Chicago Lectures in Mathematics, 2000.
2. C. Druţu and M. Kapovich, *Geometric Group Theory*, Colloquium Publications 63, 2018. Also available at

   `https://www.math.ucdavis.edu/~kapovich/EPR/ggt.pdf`

3. P. W. Nowak and G. Yu, *Large Scale Geometry*, EMS Textbooks in Mathematics, 2012.
4. M. Clay and D. Margalit (Editors), *Office Hours with a Geometric Group Theorist*, Princeton University Press, 2017.
5. M. R. Bridson and A. Haefliger, *Metric Spaces of Non-Positive Curvature*, Grundlehren der Mathematischen Wissenschaften 319, 1999.

Random groups and related topics (L16)

Non-Examinable (Graduate Level)

Emmanuel Breuillard

One classical way to define a group is via a presentation by generators and relations. Putting a probability measure on the set of presentations, one can make sense of the notion of random group and start asking questions about the properties of a typical group, or make use of the "probabilistic method" to construct groups with interesting properties. Of course various probabilistic models arise naturally, but by far the most studied in the last twenty years is the density model of Misha Gromov. This will be the main focus of the course. I will touch upon a variety of topics as random groups will serve as a pretext to discuss other important notions in modern infinite group theory, such as small cancellation theory, word hyperbolic groups, Burnside groups, Kazhdan's property (T), amenability, growth, expander graphs, random walks on groups, etc.

Pre-requisites

This course has no genuine prerequistes but assumes some familiarity with undergraduate topology, geometry, group theory and probability. It may be taken in parallel with the Part III course on Geometric Group Theory.

Literature

1. C. Drutu and M. Kapovich, *Geometric Group Theory*, Colloquium Publications 63, (2018).
2. M. Gromov, *Asymptotic invariants of infinite groups*, in Geometric group theory, ed. G. Niblo, M. Roller, Cambridge University Press, Cambridge (1993).
3. M. Gromov, *Random walk in random groups*, GAFA, Geom. Funct. Anal. 13 (2003), no. 1, 73–146.
4. Y. Ollivier, *A January 2005 invitation to random groups*, Ensaios Matemáticos 10. Sociedade Brasileira de Matemática, (2005) +100 pp.
5. B. Bekka, P. de la Harpe, A. Valette, *Kazhdan's property (T)*, New Mathematical Monographs, 11. Cambridge University Press, Cambridge, (2008).

Additional support

One or two Part III essays will be offered in connection to some of the topics of the course.

Logic

Category Theory (M24)

Prof. P.T. Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

Categories, functors and natural transformations. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]

Locally small categories. The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]

Adjunctions. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]

Limits. Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]

Monads. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

Filtered colimits. Finitary functors, finitely-presentable objects. Applications to universal algebra.

Regular categories. Embedding theorems. Categories of relations, introduction to allegories.

Abelian categories. Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.

Monoidal categories. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.

Fibrations. Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Pre-requisites

There are no specific pre-requisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. S. Mac Lane *Categories for the Working Mathematician.* Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.

2. S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.
3. T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.
4. E. Riehl *Category Theory in Context.* Dover Publications 2016. A new account of the subject by someone who first encountered it as a Part III student a dozen years ago.
5. F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Model Theory (L16)

Silvia Barbina

Model theory is a branch of mathematical logic. Initially, the focus was on how far a set of sentences in a first-order language determines the class of structures it describes. Later, the subject evolved in several directions. For example, part of the attention shifted to contexts where the subsets of a structure that are *definable* are also mathematically meaningful – for instance, the definable sets in an algebraically closed field are the constructible sets. This is one of several ways in which model theory interacts with other areas of mathematics. Connections have emerged to real and algebraic geometry, number theory and, more recently, combinatorics. This course introduces some basic model-theoretic tools and ideas up to initial notions in stability theory.

We aim to cover a selection of the following topics:

- preliminaries: structures, theories, elementarity (including elementary substructures, Tarski-Vaught test, downward Löwenheim-Skolem theorem)
- examples of relational structures (dense linear orders, the random graph)
- universal homogeneous models
- the compactness theorem for theories and for types
- saturation and the monster model
- omitting types theorem, possibly including ω-categorical theories and small theories
- preservation theorems and quantifier elimination
- strongly minimal structures
- imaginaries and the eq-expansion
- formulas without the order property (externally definable sets, stability and the number of types, stationary types and canonical bases).

Pre-requisites

The Part II course *Set Theory and Logic*, or an equivalent course, is essential.

Literature

References and notes will be provided during the course. A relevant model theory textbook is

1. Katrin Tent and Martin Ziegler, *A Course in Model Theory* (Lecture Notes in Logic 40). Cambridge University Press, 2012.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Infinite Games (L24)

Professor Benedikt Löwe

Infinite two-player perfect information games are connected to many topics in the foundations of mathematics: central concepts from analysis and topology can be reformulated in game-theoretic terms using infinite two-player perfect information games. Examples are the concepts of Lebesgue measurability, the property of Baire, as well as the perfect set property.

The central game-theoretic notion is the concept of *determinacy*: the full axiom of determinacy AD ("all infinite two-player perfect information games with natural number moves are determined") contradicts the axiom of choice AC, but definable fragments of AD can be proved in ZFC or extensions of ZFC. The axiom of determinacy itself yields an interesting alternative foundations of mathematics.

We shall treat several of the following topics:

Basic theory of determinacy. Applications in topology and measure theory. Incompatibility of AC and AD. Basics of descriptive set theory: the Borel hierarchy and the projective hierarchy.

Proving determinacy. Open determinacy. Low-level Borel games. Borel determinacy.

Proving determinacy from large cardinals. Introduction to large cardinals: inaccessible cardinals and measurable cardinals. Proving Π^1_1-determinacy from a measurable cardinal.

The axiom of determinacy. Combinatorial consequences: $\aleph_1$ and $\aleph_2$ are measurable. Infinite exponent partition relations.

Stronger axioms of determinacy. The axiom of real determinacy. Inconsistent extensions of the axiom of determinacy. Long games.

The Wadge hierarchy. Definition and structure theory of the Wadge hierarchy under AD.

A course webpage will be available at

`https://www.math.uni-hamburg.de/home/loewe/Lent2020/`.

Pre-requisites

The Part II course *Logic and Set Theory* or an equivalent course is essential. The Part Ib course *Metric and Topological Spaces* is useful.

Literature

1. Akihiro Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings.* Springer 2003 [Springer Monographs in Mathematics]

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Number Theory

Modular forms and L-functions (M24)

Professor A J Scholl

Modular Forms are classical objects that appear in many areas of mathematics, from number theory to representation theory and mathematical physics. Most famous is, of course, the role they played in the proof of Fermat's Last Theorem, through the conjecture of Shimura-Taniyama-Weil that elliptic curves are modular. One connection between modular forms and arithmetic is through the medium of L-functions, the basic example of which is the Riemann ζ-function. We will discuss various types of L-function in this course and give arithmetic applications.

Pre-requisites

Prerequisites for the course are fairly modest; from number theory, apart from basic elementary notions, some knowledge of quadratic fields is desirable. A fair chunk of the course will involve (fairly 19th-century) analysis, so we will assume the basic theory of holomorphic functions in one complex variable, such as are found in a first course on complex analysis (e.g. the 2nd year Complex Analysis course of the Tripos).

Literature

1. J. P. Serre, *A course in Arithmetic*, Graduate Texts in Maths. **7**, Springer, New York, 1973 (Chapter VII is an easy-going introduction to modular forms, and Chapter VI covers Dirichlet L-functions and the theorem on primes in arithmetic progresions.).

2. D. Bump, *Automorphic forms and representations*, Cambridge Studies in Adv. Maths. **55**, CUP, Cambridge, 1997 (Sections 1.1-1.6 of Chapter I are particularly relevant).

3. F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Maths. **228**, Springer, New York, 2005 (a good reference providing also an introduction to the algebraic theory of modular forms, although goes into a lot more detail than we will give in this course).

4. J. Milne, *Modular Functions and Modular Forms*, Lecture notes from a course, download available at `http://www.jmilne.org/math`.

Additional references for enthusiasts

5. T. Miyake, *Modular Forms*, Springer, Berlin, 1989 (a standard reference for classical theory of modular forms).

6. F. Diamond, J. Im, *Modular forms and modular curves*, in: *Seminar on Fermat's Last Theorem*, CMS Conf. Proc. 17, Amer. Math. Soc., Providence, RI, 1995, 39-133.

7. J. Coates, Shing-Tung Yau, *Elliptic curves, modular forms & Fermat's last theorem- Conference Proceedings*, International Press, Cambridge, MA, 1997 (in particular, the survey article by H. Darmon, F. Diamond, R. Taylor).

8. H. Hida, *Elementary theory of L-functions and Eisenstein series*, London Math. Soc. Student Texts **26**, CUP, Cambridge, 1993 (not so elementary introduction to arithmetic of modular forms).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a further revision class in the Easter Term.

Elliptic Curves (M24)

Tom Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points. The following topics will be covered, and possibly others if time is available.

Weierstrass equations and the group law. Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process.

Isogenies. The degree of an isogeny is a quadratic form. The invariant differential and separability. The torsion subgroup over an algebraically closed field.

Elliptic curves over finite fields. Hasse's theorem and zeta functions.

Elliptic curves over local fields. Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.

Elliptic curves over number fields. The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Pre-requisites

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. The first few lectures will include a review of the necessary geometric background, but some previous knowledge of algebraic curves (at the level of the Part II course *Algebraic Geometry* or the first two chapters of [3]) would be very helpful. Later in the course, some basic knowledge of the field of p-adic numbers will be assumed.

Preliminary Reading

1. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Literature

2. J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.
3. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Algebraic Number Theory (M24)

J. A. Thorne

Algebraic number theory lies at the foundation of much current research in number theory, from Fermat's last theorem to the proof of the Sato–Tate conjecture, and is a beautiful subject in its own right. This will be a second course in algebraic number theory, with an emphasis on local (p-adic) aspects of the theory.

Topics likely to be covered include:

Dedekind domains, localization, and passage to completion. The p-adic numbers.

Galois theory of Dedekind domains.

Ramification theory.

Class field theory (review of statements only).

Pre-requisites

Part II Galois Theory and Part IB Groups, Rings and Modules (or equivalent) are essential pre-requisites. Exposure to Part II Number Fields would be useful, but knowledge of this course will not be assumed.

Literature

1. S. Lang, *Algebraic number theory.* Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994.
2. H. P. F. Swinnerton-Dyer, *A brief guide to algebraic number theory.* London Mathematical Society Student Texts, 50. Cambridge University Press, Cambridge, 2001.
3. J.-P. Serre, *Local fields.* Graduate Texts in Mathematics, 67. Springer-Verlag, New York-Berlin, 1979.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Analytic Number Theory (L24)

Thomas F. Bloom

Analytic number theory studies the properties of integers using techniques from analysis, both real and complex. This course will give an introduction to the topic, focusing especially on the classical theory of the Riemann zeta function. A particular highlight is a proof of the Prime Number Theorem, which gives an asymptotic formula for the number of prime numbers, and which has an intimate relationship with the distribution of zeros of the zeta function. We will seek to understand the distribution of these zeros in some depth, including a discussion of the consequences of the infamous Riemann Hypothesis.

Topics will include:

- An introduction to Dirichlet series and the Riemann zeta function, including the functional equation and analytic continuation;
- Perron's formula and the proof of the Prime Number Theorem;
- Quantitative zero-free regions for the zeta function;
- An explicit formula for the prime counting function in terms of zeros of the zeta function;
- Zero density results;
- The Riemann Hypothesis and its consequences.

Pre-requisites

Knowledge of basic complex analysis will be assumed, up to and including the maximum modulus principle and evaluating contour integrals using the method of residues. The more specialised analytic tools required will be developed in the course.

As this course focuses exclusively on the integers, knowledge of any algebraic number theory will *not* be required, and this course may be taken independently of any other number theory courses.

Literature

1. H. Montgomery and R. C. Vaughan, *Multiplicative Number Theory. I. Classical Theory* Cambridge Studies in Advanced Mathematics, 2007.
2. E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd edition, Oxford Science Publications, 1986.
3. H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, 2004.

Additional support

Four examples sheets will be provided and four associated examples classes will be given, along with informal drop-in sessions. There will be a one-hour revision class in the Easter Term.

Probability and Finance

Advanced Probability (M24)

Sebastian Andres

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

Review of measure and integration: sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini's theorem.

Conditional expectation: Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.

Martingales: Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.

Stochastic processes in continuous time: Kolmogorov's criterion, regularization of paths; martingales in continuous time.

Weak convergence: Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem. Sums of independent random variables: Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.

Brownian motion: Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.

Poisson random measures: Construction and properties; integrals.

Lévy processes: Lévy-Khinchin theorem.

Pre-requisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed very briefly at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book or James Norris' lecture notes for 'Probability and Measure') to strengthen their understanding.

Literature

1. Lecture notes by James Norris, avalaible at

   `www.statslab.cam.ac.uk/~james/Lectures/ap.pdf`

2. D. Applebaum, Lévy processes (2nd ed.), Cambridge University Press 2009.
3. A. Bovier, Stochastic Processes, Lecture notes, available at

   `https://wt.iam.uni-bonn.de/bovier/lecture-notes/`

4. R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
5. O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.
6. J.R. Norris, Probability and Measure, Lecture notes, available at

   `www.statslab.cam.ac.uk/~james/Lectures/pm.pdf`

7. D. Williams, Probability with martingales, CUP 1991.

Additional support

Four example sheets will be provided along with examples classes. There will be a revision class in Easter term.

Advanced Financial Models (M24)

M.R. Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability and Stochastic Calculus & Applications.

- *Discrete-time models.* Arbitrage, martingale deflators, the fundamental theorem of asset pricing. Numéraires, equivalent martingale measures. Forwards, options, futures, bonds, interest rates. Attainable claims, market completeness. The Breeden–Litzenberger formula. Fourier pricing. Optimal stopping.
- *Continuous-time models.* Brief survey of Brownian stochastic calculus, Itô's formula, martingale representation theorem, Girsanov's theorem. Admissible strategies. Absolute and relative arbitrage. Existence of replicating strategies. Pricing and hedging via partial differential equations. Black–Scholes model. The implied volatility surface. Dupire's formula. Stochastic volatility models.

Pre-requisites

Familiarity with measure-theoretic probability will be assumed.

Literature

1. M. Baxter & A. Rennie. *Financial calculus: an introduction to derivative pricing.* Cambridge University Press, 1996
2. M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling.* Springer, 2006
3. D. Kennedy.*Stochasic Financial models.* Chapman & Hall, 2010
4. D. Lamberton & B. Lapeyre.*Introduction to stochastic calculus applied to finance.* Chapman & Hall, 1996
5. S. Shreve. *Stochastic Calculus for Finance: Vol. 1 and 2.* Springer-Finance, 2005

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Percolation and Related Topics (M16)

Geoffrey Grimmett

The percolation process is the simplest probabilistic model for a random medium in finite-dimensional space. It has a central role in the general theory of disordered systems arising in the mathematical sciences, and it has strong connections with statistical mechanics. Percolation has a reputation as a source of beautiful mathematical problems that are simple to state but seem to require new techniques for solution, and a number of such problems remain very much alive. Amongst connections of topical importance are the relationships to so-called Schramm–Loewner evolutions (SLE), and to the theory of phase transitions in physics.

The basic theory of percolation will be described in this course. The fundamental techniques, including correlation and/or concentration inequalities and their ramifications, will be covered. The related topics may include self-avoiding walks, and further models from interacting particle systems, and (if time permits) certain physical models for the ferromagnet such as the Ising and Potts models.

Pre-requisites

There are no essential pre-requisites beyond probability and analysis at undergraduate levels, but a familiarity with the measure-theoretic basis of probability will be helpful.

Literature

The following text will cover the majority of the course, and is available online.

1. Grimmett, G. R., *Probability on Graphs*, Cambridge University Press, 2nd edn, 2018; see `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`

Additional support

Three examples sheets will be provided with three associated examples classes.

Mixing times of Markov chains (M16)

Perla Sousi

An ergodic Markov chain converges to its equilibrium distribution as time goes to infinity. But how long should one wait until the distribution is "close" to the invariant one? How many times should one shuffle a deck of cards until the order becomes uniform? This question lies at the heart of the modern theory of mixing times for Markov chains. The classical theory of Markov chains studied fixed chains and the focus was on large time asymptotics of their distribution. Recently the need to analyse large spaces has increased and the focus has shifted on studying asymptotics of the mixing time as the size of the state space tends to infinity. The area of mixing times is at the interface of mathematics, statistical physics and theoretical computer science.

In this course we will develop the basic theory and some of the main techniques and tools from probability and spectral theory used to estimate mixing times. We will apply them to study the mixing time of several chains of interest. We shall also discuss the *cutoff* phenomenon which was first discovered by Diaconis in the context of card shuffling and it says that a Markov chain converges to equilibrium abruptly. This phenomenon seems to be widespread but it remains a challenging question to obtain criteria for cutoff for general classes of chains.

Pre-requisites

This course assumes almost no background, except for prior exposure to Markov chains at an elementary level.

Literature

1. D. Levin and Y. Peres and E. Wilmer *Markov chains and Mixing Times.* American Mathematical Society, 2008.

2. D. Aldous and J. Fill, *Reversible Markov Chains and Random Walks on Graphs.* book in preparation available online at https://www.stat.berkeley.edu/~aldous/RWG/book.html

3. R. Montenegro and P. Tetali, *Mathematical aspects of mixing times in Markov chains.* Foundations and Trends in Theoretical Computer Science: Vol. 1: No. 3, pp 237-354, 2006.

Random Walks and Uniform Spanning Trees (L16)

Tom Hutchcroft

A central part of modern probability theory aims to understand how the geometry of a space (in this course, a graph) influences the behaviour of random processes in that space, and vice versa. In this course, we will explore these connections by studying two closely related processes: random walks and the uniform spanning tree. This course could serve as an introduction to several topics at the forefront of current research in discrete probability, and may also be relevant to those with an interest in geometric group theory.

- Electrical networks. Recurrence and transience of random walk and their geometric nature. The Nash-Williams Criterion. Relation between isoperimetry and return probability decay. Amenability and nonamenability.
- Harmonic functions and their probabilistic interpretations. Relation between bounded harmonic functions, entropy, and the rate of escape of random walk. The Varopoulos-Carne inequality. Free and wired currents in infinite volume.
- Uniform spanning trees. Connections with random walks and electrical networks. Wilson's algorithm and the Aldous-Broder algorithm. The free and wired uniform spanning forests as infinite-volume limits of the uniform spanning tree. Pemantle's Theorem on the number of trees in the forest.

I may also cover some more advanced topics if time admits.

Pre-requisites

We will use the notions of conditional expectation, martingales, and the martingale convergence theorem as lectured in Advanced Probability. Familiarity with Markov chains as in the 1B course will be helpful but not essential.

Literature

1. R. Lyons and Y. Peres. *Probability on Trees and Networks.* Cambridge University Press, New York, 2016. Available at `http://pages.iu.edu/~rdlyons/`.
2. G. Pete. Probability and geometry on groups. http://www.math.bme.hu/ gabor/PGG.pdf, 2014.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Random planar geometry (L16)

Jason Miller

This course will be an introduction to two-dimensional random geometric structures, both discrete and continuous.

The first part of the course will be on random planar graphs. Recall that a *tree* is a connected graph without cycles. A *plane tree* is a tree together with an embedding into the plane. Since there are only a finite number of plane trees, one can pick one uniformly at random. We will discuss how random plane trees are related to random walks on $\mathbf{Z}$. We will also describe their continuous counterpart, the so-called *continuum random tree*, which is a random tree defined using Brownian motion. Plane trees turn out to serve as the basic building block for more elaborate geometric structures. One important example is a *planar map*, which is a graph together with an embedding into the plane so that no two edges cross. Since there are only a finite number of planar maps with a fixed number of faces, one can also talk about picking one uniformly at random. This is an example of a *random planar map*. The study of random planar maps goes back to work of Tutte in the 1960s in his attempt to prove the four color theorem. In recent years, random planar maps have been the subject of intense study, in part due to their deep connection with understanding different models in statistical mechanics (e.g., the percolation model and loop-erased random walk).

The second part of the course will be on the Schramm-Loewner evolution (SLE), which is a family of non-crossing curves in the plane indexed by a parameter $\kappa \geq 0$. SLE was introduced by Schramm in 1999 to describe the scaling limit of many different models in statistical mechanics (e.g., the percolation model and loop-erased random walk) in the same way that Brownian motion describes the scaling limit of simple random walk. SLE is defined in a very interesting way, combining ideas from complex analysis and probability. In this part of the course, we will introduce SLE and derive some of its basic properties.

Time permitting, we will discuss more advanced topics, such as the Gaussian free field and its connection with random planar maps and SLE.

Pre-requisites

Advanced probability. Stochastic calculus is a co-requisite.

Literature

1. G. Miermont *Aspects of random maps.* 2014 St Flour lecture notes.

   `http://perso.ens-lyon.fr/gregory.miermont/coursSaint-Flour.pdf`

2. J. Miller *Schramm-Loewner evolutions.*

   `http://statslab.cam.ac.uk/~jpm205/teaching/lent2019/sle_notes.pdf`

3. N. Berestycki and J.R. Norris, *Lectures on Schramm-Loewner evolutions.*

   `http://www.statslab.cam.ac.uk/~james/Lectures/sle.pdf`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistics and Operational Research

The courses in statistics form a coherent Masters-level course in statistics, covering statistical methodology, theory and applications. You may take all of them, or a subset of them. Core courses are Modern Statistical Methods and Applied Statistics in the Michaelmas Term.

All statistics courses for examination in Part III assume that you have taken an introductory course in statistics and one in probability, with syllabuses that cover the topics in the Cambridge undergraduate courses Probability in the first year and Statistics in the second year. It is helpful if you have taken more advanced courses, although not essential. For Applied Statistics and other applications courses, it is helpful, but not essential, if you have already had experience of using a software package, such as R or Matlab, to analyse data. The statistics courses assume some mathematical maturity in terms of knowledge of basic linear algebra and analysis. However, they are designed to be taken without a background in measure theory, although some knowledge of measure theory is helpful for Topics in Statistical Theory.

The desirable previous knowledge for tackling the statistics courses in Part III is covered by the following Cambridge undergraduate courses. The syllabuses are available online at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
First	*Essential*	Probability
Second	*Essential*	Statistics
	Helpful for some courses	Markov Chains
Third	*Helpful*	Principles of Statistics
	Helpful for applied statistics courses	Statistical Modelling
	For additional background	Probability and Measure

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation. If you have more time, then it would be helpful to review other courses as indicated.

Modern Statistical Methods (M24)

Sergio Bacallado

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data with a huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to study such datasets. We aim to cover the following topics.

- Kernel machines: Ridge regression, the kernel trick, kernel ridge regression, the support vector machine, the hashing trick.
- Penalised regression: Model selection, the Lasso, variants of the Lasso.

- High-dimensional covariance matrices: non-asymptotic error bounds in the operator norm and the effective rank, the spiked covariance model, PCA and sparse PCA, the graphical Lasso.
- Multiple testing and high-dimensional inference: the closed testing procedure and the Benjamini–Hochberg procedure, the debiased Lasso.

Pre-requisites

Basic knowledge of statistics, probability, linear algebra and real analysis. Some background in optimisation would be helpful but is not essential.

Literature

1. T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning.* 2nd edition. Springer, 2001.
2. P. Bühlmann, S. van de Geer, *Statistics for High-Dimensional Data.* Springer, 2011.
3. C. Giraud, *Introduction to High-Dimensional Statistics.* CRC Press, 2014.
4. T. Hastie, R. Tibshirani and M. Wainwright, *Statistical learning with sparsity: the lasso and generalizations.* CRC Press, 2015.
5. M. Wainwright, *High-dimensional statistics: A non-asymptotic viewpoint.* Cambridge University Press, 2019.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Causal Inference (M16)

Qingyuan Zhao

From its onset, modern statistics engages in the problem of inferring causality from data. A common mindset is that causal inference is only possible using randomised experiments, but developments in statistics and related fields have shown that this view is oversimplified and restrictive. We now have a much better understanding of the assumptions and methodologies that enable causal inference from non-experimental data. This course aims to cover some of the most fundamental ideas in causal inference, a vibrant research area where statistical theory meets scientific practice.

1. **Motivations:**
    - Path analysis and linear structural equation models (SEMs);
    - Randomised experiments: permutation tests, asymptotic inference for average treatment effect, regression adjustment.
2. **Languages for causality:**
    - Graphical models: Bayesian network, Markov properties, d-separation;
    - Nonparametric SEMs: multiple-world and single-world models.
    - Causal identification: back-door criterion, g-formula, front-door criterion.
    - Structure learning: the PC algorithm.
    - *Dynamic treatment regimes.
3. **Design and statistical inference:**

- Matching, permutation inference, sensitivity analysis;
- Inverse-probability weighting, outcome regression, doubly robust estimators;
- Instrumental variables: core assumptions and identification, estimation in linear model, weak instruments, bounds.
- *Other topics in causal inference: mediation analysis; differences-in-differences; regression discontinuity design; time-varying treatments (marginal structural models and structural nested mean models).

(* indicates a tentative topic that might be skipped)

Pre-requisites

This course assumes familiarity with undergraduate-level probability and statistics. Experience with statistical applications and programming in `R` is helpful but not required.

Literature

1. Pearl, J. (2011). Linear models: A useful "microscope" for causal analysis, *Journal of Causal Inference*, **1**(1): 155–170 (2013). Also available at:

   `https://ftp.cs.ucla.edu/pub/stat_ser/r409-corrected-reprint.pdf`.

2. Imbens, G. W. and Rubin, D. B. (2015) *Causal Inference in Statistics, Social, and Biomedical Sciences.* Cambridge University Press.
3. Lauritzen, S. L. (1996). *Graphical Models.* Clarendon Press.
4. Richardson, T. S. and Robins, J. M. (2013) *Single World Intervention Graphs.* Technical report, University of Washington. Available at

   `https://www.csss.washington.edu/Papers/wp128.pdf`.

5. Rosenbaum, P. R. (2002) *Observational Studies.* Springer-Verlag.
6. Hernán M. A. and Robins, J. M. (2019) *Causal Inference.* Chapman & Hall, forthcoming. Available at

   `https://www.hsph.harvard.edu/miguel-hernan/causal-inference-book/`.

7. Angrist, J. D. and Pischke, J. S. (2008) *Mostly harmless econometrics: An empiricist's companion.* Princeton University Press.

Additional support

Lecture notes will be provided. Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistics in Medicine (3 units)

Statistics in Medical Practice (M12)

Statistics in Medicine (3 units)

Lecturers from the MRC Biostatistics Unit

This part of the course includes three modules covering a range of statistical methods and their application in three areas of biostatistics.

A. Stochastic Models for Chronic and Infectious Diseases [4 Lectures] **(C. Jackson, A. Presanis & D. De Angelis)**

Continuous-time multi-state and Markov models: properties and quantities of interest, and fitting models to individual disease history data. Applications to modelling the onset and progression of chronic diseases. Multi-state modelling to estimate incidence of infectious diseases from population-level prevalence data. Backcalculation methods for the estimation of incidence of disease with long incubation periods. Dynamic modelling of infectious disease transmission.

B. Causal Inference [4 Lectures] **(S. Burgess)**

It is well known that "correlation is not causation". But how then do you assess causal claims? Is it possible to show that X is a cause of Y? What does it even mean to say that X is a cause of Y? In this module, we introduce definitions of causal concepts, starting with the work of Rubin, Pearl, and Robins, and discuss practical approaches for assessing causal claims from observational data.

C. Design and Analysis of Randomised Trials [4 Lectures] **(M. Pilling, D. Robertson & S. Seaman)**

Sample size estimation for clinical trials; group-sequential designs and treatment effect estimation following a group-sequential trial. Adaptive and multi-stage designs. Types of randomisation procedures. Non-parametric and parametric response-adaptive procedures. Handling missing data: classification of missingness mechanisms, maximum likelihood, and multiple imputation.

Analysis of Survival Data (L12)

Statistics in Medicine (3 units)

P. Treasure

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

D. Time-to-Event Analysis [4 Lectures]

'Survival analysis' is generalised to *time-to-event* analysis. The implications of event times which are unknown or in the future (*censored* data) are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (*Kaplan-Meier*) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (*log-rank* test).

E. Modelling Hazard [4 Lectures]

The *hazard* function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size...) on the time-to-event distribution (*proportional hazards* modelling). Model checking procedures are introduced with an emphasis on excess event (*Martingale*) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (*competing risk* analysis) and (ii) the cause of death is unknown (*net survival*). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar time (e.g. 2017) are addressed: *period* survival analysis.

Additional Information

Statistics in Medicine (3 units)

Pre-requisites

Undergraduate-level statistics and probability: including analysis and interpretation of data, maximum likelihood estimation, hypothesis testing, basic stochastic processes.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. A few books to complement the course material are listed below.

1. Armitage P, Berry G, Matthews JNS, *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001. [A good introductory companion to the whole course]

2. van den Hout, A, *Multi-State Survival Models for Interval-Censored Data.* Chapman and Hall, 2016 [Module A]

3. Keeling, M. J., & Rohani, P. *Modeling Infectious Diseases in Humans and Animals.* Princeton University Press, 2008 [Module A]

4. Burgess S, Thompson SG, *Mendelian Randomization: Methods for Using Genetic Variants in Causal Estimation* Chapman and Hall, 2015 [Module B]

5. Senn, S. *Statistical Issues in Drug Development.* Wiley, 2007. [Module C]

6. Jennison C, Turnbull B, *Group Sequential Methods with Applications to Clinical Trials.* Chapman and Hall, 2000. [Module C]

7. Cox DR, Oakes D, *Analysis of Survival Data.* Chapman and Hall, 1984 [Modules D, E, F: the classic text]

8. Collett D, *Modelling Survival Data in Medical Research.* CRC Press, 2015 [Modules D, E, F: modern, applied, supports and extends lectures.]

9. Aalen OO, Borgen Ø, Gjessing HK, *Survival and Event History Analysis: A Process Point of View.* Springer, 2008 [Modules D, E, F: excellent modern approach]

Additional support

[Modules A, B, C] A two-hour example class, supported by question sheets and solutions, will be given at the start of Lent term.

[Modules D, E, F] A two-hour example class, supported by question sheets and solutions, will be given in each of the Lent and Easter Terms. A two-hour revision class will be held just before the examination.

Astrostatistics (L24)

Kaisey Mandel

This course will cover applied statistical methods necessary to properly interpret today's increasingly complex datasets in astronomy. Particular emphasis will be placed on principled statistical modeling of astrophysical data and statistical computation of inferences of scientific interest. Statistical techniques, such as Bayesian inference, sampling methods, hierarchical models, Gaussian processes, and model selection, will be examined in the context of applications to modern astronomical data analysis. Topics and examples will be motivated by case studies across astrophysics and cosmology.

Pre-requisites

Students of astrophysics, physics, statistics or mathematics are welcome. Astronomical context will be provided when necessary. Students without a previous statistics background should familiarise themselves with the material in Feigelson & Babu, Chapters 1-4, and Ivezić, Chapters 1, 3-5, by the beginning of the course. (Note that the two textbooks cover many of the same topics). These texts are freely available online to Cambridge students via the library website.

Literature

1. E. Feigelson and G. Babu. *Modern statistical methods for astronomy: with R applications.* Cambridge University Press, 2012.
2. Z. Ivezić, A. Connolly, J. VanderPlas & A. Gray. *Statistics, Data Mining, and Machine Learning in Astronomy.* Princeton University Press, 2014.
3. C. Schafer. *A Framwork for Statistical Inference in Astrophysics.* 2015, Annual Review of Statistics and Its Application, 2: 141-162.
4. C. Bishop. *Pattern Recognition & Machine Learning.* Springer-Verlag, 2006.
   Also available at

   `https://www.microsoft.com/en-us/research/people/cmbishop/#!prml-book`
5. D. MacKay. *Information Theory, Inference, and Learning Algorithms.* Cambridge University Press, 2003. Also available at

   `http://www.inference.org.uk/mackay/itila/book.html`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Stochastic Calculus and Applications (L24)

R. Bauerschmidt

This course provides an introduction to Itô calculus.

- *Stochastic calculus for continuous processes.* Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita-Watanabe theorem, and Itô's formula.
- *Applications to Brownian motion and martingales.* Lévy characterization of Brownian motion, Dubins-Schwarz theorem, martingale representation, Girsanov theorem, and Dirichlet problems.
- *Stochastic differential equations.* Strong and weak solutions, notions of existence and uniqueness, strong Markov property, and relation to second order partial differential equations.
- *Applications and examples.*

Pre-requisites

Knowledge of measure theoretic probability as taught in Part III Advanced Probability will be assumed, in particular familiarity with continuous-time martingales and Brownian motion.

Literature

1. J.-F. Le Gall, *Brownian Motion, Martingales, and Stochastic Calculus.* Springer. 2016
2. D. Revuz and M. Yor, *Continuous martingales and Brownian motion.* Springer. 1999
3. I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus.* Springer. 1998
4. L.C. Rogers and D. Williams, *Diffusions, Markov Processes, and Martingales.* Cambridge. 2000

Topics in Statistical Theory (L16)

Yoav Zemel

This course will provide an introduction to the theory behind a selection of statistical problems that play a key role in modern statistics. Most undergraduate statistics courses are restricted to the study of parametric models; here we will no longer assume that our distributions belong to finite dimensional classes and will instead study fundamental nonparametric problems such the estimation of a distribution function, a density function or a regression function. We will consider the canonical machine learning problem of classification, and may also introduce the topical notion of Wasserstein distances and some of their uses in statistics. Minimax lower bounds are studied as a way of quantifying the intrinsic difficulty of a statistical problem, and provide limits on how well any estimator can perform in a given situation.

A tentative outline of the course is as follows:

- An introduction to nonparametric statistics: the basics of empirical process theory, Glivenko–Cantelli theorem, Dvoretzky–Kiefer–Wolfowitz theorem, order statistics, quantile estimation and associated asymptotic distribution theory.

- Kernel density estimation: histograms, bias and variance expansions, asymptotically optimal bandwidth, uniform nonasymptotic bounds.

- Nonparametric regression: kernel nonparametric regression, bias and variance expansions. Cubic splines, natural cubic smoothing splines, choice of smoothing parameter, other splines, equivalent kernel. Classification problems, the Bayes classifier, nearest neighbour classifiers.

- Minimax theory: notion of information-theoretic lower bounds, distance and divergence between distributions, optimal rates, Le Cam's two points lemma.

- Wasserstein distances: basic properties, subadditivity, application to bootstrap consistency.

Pre-requisites

A good background in undergraduate probability theory, though measure theory is not necessary; elements of linear algebra; a preliminary course in mathematical statistics can be helpful, but is not necessary. Though the material in the Modern Statistical Methods course will not be needed here, the two courses complement each other well.

Literature

No book will be explicitly followed, but some of the material is covered in:

1. L. Devroye, L. Györfi, G. Lugosi, *A Probabilistic Theory of Pattern Recognition.* Springer, 1996.
2. A. Tsybakov, *Introduction to Nonparametric Estimation.* Springer, 2009.
3. M. J. Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint.* Cambridge University Press, 2019.

Additional support

Three examples sheets will be provided and associated examples classes will be given. There will be a revision class in the Easter Term.

Statistical Learning in Practice (L24)

Alberto J. Coca

Statistical learning is the process of using data to guide the construction and selection of models, which are then used to predict future outcomes. In this course, consisting of roughly 12 lectures and 12 practical classes, we will examine some of the most successful and widely used statistical methodologies in modern applications. The practical classes will deal with an introduction to R, exploratory data analysis and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models for regression and classification
- Model selection and regularisation
- Mixed effects models and quasi-likelihood methods
- Linear discriminant analysis and support vector machines
- Introduction to neural networks
- Introduction to time series

Pre-requisites

Elementary probability theory. Maximum likelihood estimation, hypothesis tests and confidence intervals. Linear models.

Previous experience with R is helpful but not essential.

Literature

1. Dobson, A.J. and Barnett A. (2008) *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC.
2. Faraway, J. J. (2005) *Extending the linear model with R: generalized linear, mixed effects and non-parametric regression models.* CRC press.
3. Hastie, T., Tibshirani, R. and Friedman, J. (2009) *The Elements of Statistical Learning.* Second Edition. Springer.
4. Shumway, R. H., and Stoffer, D. S. (2010) *Time Series Analysis and Its Applications: with R Examples.* Springer Science & Business Media.

Additional support

This course includes practical classes, in which statistical methods are introduced in a practical context and students carry out analysis of datasets using R. In the practical classes, the students have the opportunity to discuss statistical questions with the lecturer. Four examples sheets will be provided and there will be four associated examples classes. There will be a revision class in the Easter Term.

Particle Physics, Quantum Fields and Strings

The courses on Symmetries, Fields and Particles, Quantum Field Theory, Advanced Quantum Field Theory and The Standard Model are intended to provide a linked course covering High Energy Physics. The remaining courses extend these in various directions. Knowledge of Quantum Field Theory is essential for most of the other courses. The Standard Model course assumes knowledge of the course Symmetries, Fields and Particles.

Desirable previous knowledge

Basic quantum mechanics, wave functions, amplitudes and probabilities. Quantisation in terms of commutation relations between coordinates q and corresponding momenta p. Schrödinger and Heisenberg pictures. Dirac bra and ket formalism.

Harmonic oscillator, its solution using creation and annihilation operators.

Angular momentum operators and their commutation relations. Determination of possible states $|jm\rangle$ from the basic algebra. Idea of spin as well as orbital angular momentum. Two body systems. Clebsch-Gordan coefficients for decomposition of products of angular momentum states.

Perturbation theory, degenerate case and to second order. Time dependent perturbations, 'Golden Rule' for decay rates. Cross sections, scattering amplitudes in quantum mechanics, partial wave decomposition.

Lagrangian formulation of dynamics. Normal modes. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

Basic knowledge of δ-functions (including in 3 dimensions) and Fourier transforms. Basic properties of groups and the idea of a matrix representation. Permutation group.

The desirable previous knowledge needed to tackle the Particle Physics, Quantum Fields and Strings courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/system/files/schedule16-17.pdf`

Year		*Courses*
Second	*Essential*:	Quantum Mechanics, Methods, Complex Methods.
	Helpful:	Electromagnetism.
Third	*Essential*:	Principles of Quantum Mechanics, Classical Dynamics.
	Very helpful:	Applications of Quantum Mechanics, Statistical Physics, Electrodynamics.

If you have not taken the courses equivalent to those denoted 'essential', then you should review the relevant material over the vacation.

Quantum Field Theory (M24)

B.C. Allanach

Quantum Field Theory is the marriage of quantum mechanics with special relativity and provides the mathematical framework in which to describe the interactions of elementary particles.

This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalar, spinor (Dirac), and vector (gauge) fields. The relativistic invariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem.

The quantisation of the basic non-interacting free fields is firstly developed using the Hamiltonian and canonical methods in terms of operators which create and annihilate particles and anti-particles. The associated Fock space of quantum physical states is explained together with ideas about how particles propagate in spacetime and their statistics.

Interactions betweeen fields are examined next, using the interaction picture, Dyson's formula and Wick's theorem. A 'short version' of these techniques is introduced: Feynman diagrams. Decay rates and interaction cross-sections are introduced, along with the associated kinematics and Mandelstam variables.

Spinors and the Dirac equation are explored in detail, along with parity and γ^5. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions.

Finally, quantum electrodynamics (QED) is developed. A connection between the field strength tensor and Maxwell's equations is carefully made, before gauge symmetry is introduced. Lorentz gauge is used as an example, before quantisation of the electromagnetic field and the Gupta-Bleuler condition. The interactions between photons and charged matter is governed by the principal of minimal coupling. Finally, an example QED cross-section calculation is performed.

Pre-requisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics.

Literature

1. D. Tong, *Lectures on Quantum Field Theory*
   `http://www.damtp.cam.ac.uk/user/tong/qft.html` videos of lectures and printed lecture notes have a large overlap with the current course

2. B.C. Allanach, *Cross Sections and Decay Rates* printed lecture notes `3P1l` from
   `http://www.damtp.cam.ac.uk/user/examples/indexP3.html`

3. T. Lancaster and S.J. Blundell, *Quantum field theory for the gifted amateur*, Oxford University Press (2015) is an introductory text that Part III students have been finding useful.

4. M.E. Peskin and D.V. Schroeder, *An Introduction to Quantum Field Theory*, Addison-Wesley (1996) is a classic, and also covers aspects of the Standard Model.

5. A. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, (2010) gives a modern take with a lot of physical intuition, possibly taking the subject into topics more theory-specialised and advanced than the references above.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. One revision lecture will be given in Easter term.

Symmetries, Fields and Particles (M24)

N. Dorey

This course introduces the theory of Lie groups and Lie algebras and their applications to high energy physics. The course begins with a brief overview of the role of symmetry in physics. After reviewing basic notions of group theory we define a Lie group as a manifold with a compatible group structure. We give the abstract definition of a Lie algebra and show that every Lie group has an associated Lie algebra corresponding to the tangent space at the identity element. Examples arising from groups of orthogonal and unitary matrices are discussed. The case of SU(2), the group of rotations in three dimensions is studied in detail. We then study the representations of Lie groups and Lie algebras. We discuss reducibility and classify the finite dimensional, irreducible representations of SU(2) and introduce the tensor product of representations. The next part of the course develops the general theory of complex simple Lie algebras. We define the Killing form on a Lie algebra. We introduce the Cartan-Weyl basis and discuss the properties of roots and weights of a Lie algebra. We cover the Cartan classification of simple Lie algebras in detail. We describe the finite dimensional, irreducible representations of simple Lie algebras, illustrating the general theory for the case of the Lie algebra of SU(3). The last part of the course discusses some physical applications. After a general discussion of symmetry in quantum mechanical systems, we review the approximate SU(3) global symmetry of the strong interactions and its consequences for the observed spectrum of hadrons. We introduce gauge symmetry and construct a gauge-invariant Lagrangian for Yang-Mills theory coupled to matter.

Pre-requisites

Linear algebra including direct sums and tensor products of vector spaces. Definition of a group. Special relativity and quantum theory, including orbital angular momentum theory and Pauli spin matrices. Basic ideas about manifolds, including coordinates, dimension, tangent spaces.

Literature

1. J. Fuchs and C. Schweigert, *Lie Algebras and Representations.* Cambridge University Press, 2003.
2. H.F. Jones, *Representations and Physics.* 2nd edition. Taylor and Francis, 1998.
3. H. Georgi, *Lie Algebras in Particle Physics.* Westview Press, 1999.

Additional support

A set of course notes will provided as handouts in the lectures. Printed notes of previous version of the course are also available on the Part III Examples and Lecture Notes webpage. Four examples sheets will be provided and four associated examples classes in moderate-sized groups will be given by graduate students.

Statistical Field Theory (M16)

Matthew McCullough

This course introduces the renormalization group, focussing on statistical systems such as spin models with further connections to quantum field theory.

After introducing the Ising Model, Landau's mean field theory is introduced and used to describe phase transitions. The extension to the Landau-Ginzburg theory reveals broader aspects of fluctuations whilst consolidating connections to quantum field theory. At second order phase transitions, also known as 'critical points', renormalisation group methods play a starring role. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems.

Pre-requisites

Background knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Literature

1. J M Yeomans *Statistical Mechanics of Phase Transitions.* Clarendon Press (1992).
2. M Le Bellac, *Quantum and Statistical Field Theory* Oxford University Press (1991).

3. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992).
4. M Kardar, *Statistical Physics of Fields*, Cambridge University Press (2007).
5. D Amit and V Martin-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005).
6. L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press (1996).

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Advanced Quantum Field Theory (L24)

M B Wingate

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess. As well as being the basic theoretical framework for describing elementary particles and their interactions (excluding gravity), QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course builds on the Michaelmas Quantum Field Theory course, using techniques of path integrals and functional methods to study quantum gauge theories. Gauge Theories are a generalization of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie Group at every point in space-time and yet still describe the same physics. Quantizing a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry. We discuss the construction of gauge theories and their most important observables, Wilson loops. Time permitting, we will explore the possibility that a classical symmetry may be broken by quantum effects. Such anomalies have many important consequences, from constraints on interactions between matter and gauge fields, to the ability to actually render a QFT inconsistent.

A further major component of the course is to study renormalization. Wilson's picture of renormalization is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the renormalization group (RG) flow. The course explores renormalization systematically, from the use of dimensional regularization in perturbative loop integrals, to the difficulties inherent in trying to construct a quantum field theory of gravity. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a non-Abelian gauge theory can effectively become small at high energies. Known as "asymptotic freedom," this phenomenon revolutionized our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrize possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Pre-requisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be very helpful.

References

1. Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).
2. Srednicki, M., *Quantum Field Theory*, CUP (2007).

3. Schwarz, M., *Quantum Field Theory and the Standard Model*, CUP (2014).

4. Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).

Additional support

There will be four problem sheets handed out during the course. Classes for each of these sheets will be arranged during Lent term (the 4th class will be scheduled for Easter term). There will also be a general revision class during Easter term.

The Standard Model (L24)

F. Quevedo

The Standard Model of particle physics is, by far, the most successful application of quantum field theory (QFT). At the time of writing, it accurately describes all experimental measurements involving strong, weak, and electromagnetic interactions. The course aims to demonstrate how this model, a QFT with gauge group $SU(3) \times SU(2) \times U(1)$ and fermion fields for the leptons and quarks, is realised in nature. It is intended to complement the more general Advanced QFT course. We begin by defining the Standard Model in terms of its local (gauge) and global symmetries and its elementary particle content (spin-half leptons and quarks, spin-one gauge bosons and spin-zero Higgs boson). The parity P, charge-conjugation C and time-reversal T transformation properties of the theory are investigated. These need not be symmetries manifest in nature; e.g. only left-handed particles feel the weak force and so it violates parity symmetry. Ideas of spontaneous symmetry breaking are applied to discuss Goldstone's theorem and the Higgs mechanism. We then describe how the weak and electromagnetic interactions arise from the spontaneous breaking of the $SU(2) \times U(1)$ gauge symmetry. We show how CP violation becomes possible in the electroweak sector when there are three generations of particles and describe its consequences. The topic of neutrino masses and oscillations is touched upon, an important window to physics beyond the Standard Model. We show how to obtain cross sections and decay rates, quantities which can be measured in experiments, from the matrix element of a process. Because the couplings are small, these can be computed for various scattering and decay processes in the electroweak sector using perturbation theory. The strong interaction is described by quantum chromodynamics (QCD), the non-abelian gauge theory of the (unbroken) SU(3) gauge symmetry. At low energies quarks are confined and form bound states called hadrons. The coupling constant decreases as the energy scale increases, to the point where perturbation theory can be used. As an example we consider electron-positron annihilation to final state hadrons at high energies. Time permitting, we may discuss nonperturbative approaches to QCD. Both very high-energy experiments and very precise experiments are currently striving to observe effects that cannot be described by the Standard Model alone. If time permits, we comment on how the Standard Model is treated as an effective field theory to accommodate (so far hypothetical) effects beyond the Standard Model and potential physics beyond the standard model.

Pre-requisites

It is necessary to have attended the Quantum Field Theory and the Symmetries, Fields and Particles courses, or to be familiar with the material covered in them. It would be advanta- geous to attend the Advanced QFT course during the same term as this course, or to study renormalisation and non-abelian gauge fixing.

Literature

1. M.E. Peskin and D.V. Schroeder, An Introduction to Quantum Field Theory, Addison- Wesley (1995).

2. F. Halzen and A.D. Martin, Quarks and Leptons: An Introductory Course in Modern Particle Physics, Wiley (1984).

3. I.J.R. Aitchison and A.J.G. Hey, Gauge Theories in Particle Physics, CRC Press (two volumes or earlier 1989 edition in one volume).

4. J.F. Donoghue, E. Golowich and B.R. Holstein, Dynamics of the Standard Model, Cam- bridge University Press (2014).

5. H. Georgi, Weak Interactions and Modern Particle Theory, Benjamin/Cummings (1984).

6. Cliff Burgess and Guy Moore, The Standard Model: A Primer, Cambridge University Press (2012).

7. M. Thomson, Modern Particle Physics, Cambridge University Press (2013).

Additional support

Four example sheets will be provided and four associated examples classes will be given. There will also be a revision class in Easter Term.

String Theory (L24)

R A Reid-Edwards

String theory is the quantum theory of interacting one-dimensional extended objects (strings). What makes the theory so apealing is that it is a quantum theory that necessarily contains gravitational interactions and therefore provides the first tentative steps towards a full quantum theory of gravity. It has become clear that string theory is also much more than this. It has become a framework in which to understand problems in quantum field theory, to ask meaningful questions about what we expect from a quantum theory of gravity, and as a crucible yielding new ideas in mathematics.

This course provides an introduction to String Theory. We begin by generalising the worldline of a relativistic particle to the two-dimensonal surface swept out by a string (the world*sheet*). The quantum theory of the embedding of these surfaces in spacetime is goverened by a two-dimensional quantum field theory and we shall study the simplest example - the bosonic string - in detail.

An introduction to relevant ideas in two-dimensional Conformal Field Theory (CFT), such as the operator product expansion and the Virasoro algebra, will be given. The quantisation of the string will be studied, its specturm obtained, and the relationship between physical states of the two dimensional CFT and fields in spacetime will be discussed. We will see the necessity of the critical dimension of spacetime.

The path integral approach to the theory will be discussed and the Fadeev-Popov and BRST methods will be introduced to deal with the redundanceies that appear in the theory. Vertex operators will be introduced and scattering amplitudes will be computed at tree level. Perturbation theory at higher loops and the role played by moduli space of Riemann surfaces will be sketched.

Time permitting, we will discuss open strings (worldsheets with boundaries) and D-branes. The physics of strings on circles and higher dimensional tori will be discussed and stringy phenomena such as duality and corrections to field theory may also be discussed.

Pre-requisites

Knowledge of the Quantum Field Theory course in Michaelmas term is assumed and some familiarity with General Relativity is helpful. Advanced Quantum Field Theory will complement this course but will not be assumed.

Literature

1. Polchinski, *String Theory: Vol. 1: An Introduction to the Bosonic String*, CUP 1998

2. Green, Schwarz and Witten, *Superstring Theory: Vol. 1:Introduction* CUP 1987.

3. Lust and Theisen, *Lecture Notes in Physics: Superstring Theory*, Springer 1989. (Note there is also a more recent expanded edition witten with Blumenhagen).
4. David Tong, *String Theory*, arXiv:0908.0333

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Supersymmetry (L16)

David Skinner

This course provides an introduction to the role of supersymmetry in quantum field theory, with the emphasis on mathematics rather than phenomenology. We study representations of the super Poincaré algebra in $d = 4$. We introduce superfields and construct supersymmetric actions for gauge and matter theories. The associated quantum theories are often easier to study than their non-supersymmetric cousins and some observables can even be computed exactly via localization. We also study Seiberg's non-renormalization theorems and phases of SYM theories.

Further topics may include the Witten index for SQM on a Riemannian manifold, and its relation to the Atiyah–Singer index theorem, $\mathcal{N} = 2$ theories in $d = 2$, their chiral rings and the associated A and B models, and Seiberg–Witten theory from extended supersymmetry in $d = 4$.

Pre-requisites

You will need to familiar with the material in both the QFT and General Relativity course from Michaelmas. In particular, we will assume knowledge of differential geometry to the level of Prof. Reall's notes, available at

`http://www.damtp.cam.ac.uk/user/hsr1000/lecturenotes_2012.pdf`

or the Lent term Part III course on Applications of Differential Geometry to Physics. It is also strongly recommended that you attend the Lent AQFT course in parallel with this one; the material on path integrals introduced in that course will be needed for this one.

Literature

1. K. Hori, S. Katz, C. Vafa *et al. Mirror Symmetry* Clay Math Monographs, AMS (2003).
2. P. Deligne, E. Witten *et al.*, *Quantum Fields and Strings: A Course for Mathematicians* vols. 1&2, AMS (1999).
3. J. Terning, *Modern Supersymmetry*, International Series of Monographs on Physics, OUP (2009).
4. E. Witten, *Supersymmetry and Morse Theory*, J. Diff. Geom. **17**, (1982) no. 4, 661-692. Also available at

   `https://projecteuclid.org/euclid.jdg/1214437492`
5. E. Witten, *Phases of $\mathcal{N} = 2$ theories in two dimensions*, Nucl. Phys. **B403** (1993) 159-222. Also available at

   `https://www.sciencedirect.com/science/article/pii/055032139390033L`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Classical and Quantum Solitons (E16)

N.S. Manton

Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. After quantisation, solitons give rise to new particle states in the underlying quantum field theory that are not seen in perturbation theory. We will focus mainly on kink solitons in one space dimension, vortices of the abelian Higgs model in two dimensions, and Skyrmions in three dimensions. Quantised Skyrmions give us a model for protons and neutrons and larger nuclei like the alpha particle, where the topological charge is the conserved baryon number.

Pre-requisites

This course assumes you have taken Quantum Field Theory and Symmetries, Fields and Particles. The small amount of topology that is needed will be developed during the course.

Literature

1. N. Manton and P. Sutcliffe, *Topological Solitons.* C.U.P., 2004 (Chapters 1,3,4,5,7,9).
2. E.J. Weinberg, *Classical Solutions in Quantum Field Theory.* C.U.P., 2012 (Chapters 1,2,3,4,8).
3. R. Rajaraman, *Solitons and Instantons.* North-Holland, 1987.

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Physics Beyond the Standard Model (E8)

Non-Examinable (Graduate Level)

Maria Ubiali

This graduate course gives a brief overview on the successes and theoretical problems of the Standard Model and discuss applications of Effective Field Theory (EFT) ideas and techniques to the study of particle physics beyond the Standard Model (SM).

After introducing the concept of EFTs and reviewing the Standard Model from an EFT perspective, the course will give examples of how precision analyses at the Large Hadron Colliders can be used as indirect probes of Physics Beyond the Standard Model, with a special emphasis on the SMEFT and HEFT frameworks. The course will give an overview of the most promising Ultra-Violet completions of the Standard Model.

Pre-requisites

Background knowledge of Standard Model and Quantum Field Theories is highly desirable.

Literature

1. S Weinberg, *The Quantum Theory of Fields*, Volume II, Cambridge University Press (1995)
2. A. V. Manohar, *Effective field theories*, hep-ph/9606222
3. D. B. Kaplan, *Five lectures on effective field theory*, nucl-th/0510023.

4. B. Gripaios, *Lectures on Physics Beyond the Standard Model*, arXiv:1503.0263 [hep-ph]

5. S. Willenbrock and C. Zhang, *Effective Field Theory Beyond the Standard Model* arXiv:1401.0470 [hep-ph]

6. I. Brivio, M. Trott, *The Standard Model as an Effective Field Theory*, arXiv:1706.08945 [hep-ph]

Relativity and Gravitation

These courses provide a thorough introduction to General Relativity and Cosmology. The Michaelmas term courses introduce these subjects, which are then developed in more detail in the Lent term courses on Black Holes and Advanced Cosmology. A non-examinable course explores the application of spinor techniques in General Relativity.

Desirable previous knowledge

Suffix notation, vector and tensor analysis. Variational principle and Lagrangian formulation of dynamics. Familiarity with Lorentz transformations and use of 4-vectors in special relativity, 4-momentum p^μ for a particle and energy-momentum conservation in 4-vector form. Relativistic formulation of electrodynamics using $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and Lagrangian density $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$.

Knowledge of basic mathematical methods, including Fourier transforms, normal modes, and δ-function (including 3-dimensions). Basic quantum mechanics, wave functions, amplitudes and probabilities. Familiarity with aspects of statistical physics and thermodynamics, including notions of thermal equilibrium, entropy, black body radiation, and Fermi-Dirac, Bose-Einstein and Boltzmann distributions.

The desirable previous knowledge needed to tackle the Relativity and Gravitation courses is covered by the following Cambridge undergraduate courses. Students starting Part III from outside might like to peruse the syllabuses on the WWW at

`https://www.maths.cam.ac.uk/undergrad/course`

Year		*Courses*
First	*Essential*:	Vectors & Matrices, Diff. Eq., Vector Calculus, Dynamics & Relativity.
Second	*Essential*:	Methods, Quantum Mechanics, Variational Principles.
	Helpful:	Electromagnetism, Geometry, Complex Methods.
Third	*Essential*:	Classical Dynamics.
	Very helpful:	General Relativity, Stat. Phys., Electrodynamics, Cosmology.
	Helpful:	Further Complex Methods, Asymptotic methods.

If you have not taken the courses equivalent to those denoted ‘essential’, then you should review the relevant material over the vacation.

General Relativity (M24)

David Tong

General relativity is the study of space and time and gravity. The theory is one of the great achievements of human civilisation, a single, elegant formula which, when unravelled, reveals many of the mysteries of the Universe and hints at what lies beyond. This formula, known as the Einstein field equation, tells us how apples fall, how planets orbit, how black holes evolve, how ripples of the spacetime continuum propagate, and how the entire Universe expands.

This is a second course on General Relativity, albeit one that could just about be followed without prior exposure to the subject. The first half of the course will give an introduction to Differential Geometry, the mathematics that underlies curved spacetime. The second half of the course will then turn to the physics of gravity.

Pre-requisites

Special Relativity (essential), General Relativity (highly desirable), Maxwell’s equations in relativistic form (also highly desirable)

Literature

1. S. Carroll, "Spacetime and Geometry: An Introduction to General Relativity", Pearson International Edition.
2. Robert M. Wald, "General Relativity", University of Chicago Press.
3. C.W. Misner, K.S. Thorne, and J.A. Wheeler, "Gravitation", Princeton University Press.
4. M. Nakahara, "Geometry, Topology and Physics", IOP Publishing

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Cosmology (M24)

Blake Sherwin

This course discusses what we know (and don't know) about the evolution of our universe, from inflationary quantum fluctuations in the first fraction of a second to the formation of galaxies and structures today. It also seeks to illustrate how cosmology can serve as a uniquely powerful laboratory for understanding fundamental physics.

In detail, the course will cover the following topics:

1. Geometry and dynamics of our Universe
2. Inflation
3. Thermal history
4. Cosmological perturbation theory
5. Structure formation
6. Cosmic microwave background basics
7. Initial conditions from inflation

Pre-requisites

Although the course is intended to be as self contained as possible, knowledge of relativity, quantum mechanics and statistical mechanics will be very useful. General relativity and quantum field theory courses may allow a deeper understanding of some of the material covered.

Literature

1. S. Dodelson, *Modern Cosmology*
2. E. Kolb and M. Turner, *The Early Universe*
3. S. Weinberg, *Cosmology*

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Black Holes (L24)

Harvey Reall

A black hole is a region of space-time that is causally disconnected from the rest of the Universe. The study of black holes reveals many surprising and beautiful properties, and has profound consequences for quantum theory. The following topics will be discussed:

1. Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
2. The initial value problem, strong cosmic censorship.
3. Causal structure, null geodesic congruences, Penrose singularity theorem.
4. Penrose diagrams, asymptotic flatness, weak cosmic censorship.
5. Reissner-Nordstrom and Kerr black holes.
6. Energy, angular momentum and charge in curved spacetime.
7. The laws of black hole mechanics. The analogy with laws of thermodynamics.
8. Quantum field theory in curved spacetime. The Hawking effect and its implications.

Pre-requisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

1. H. S. Reall, *Part 3 Black Holes*: lecture notes available at www.damtp.cam.ac.uk/user/hsr1000
2. R.M. Wald, *General relativity*, University of Chicago Press, 1984.
3. S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
4. V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.
5. N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. R.M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be distributed during the course. Four examples classes will be held to discuss these. A revision class will be held in the Easter term.

Field Theory in Cosmology (L24)

Enrico Pajer and Tobias Baldauf

This course discusses applications of classical, statistical and quantum field theory to cosmology. The course comprises three interconnected topics:

- Cosmological inflation and primordial quantum perturbations (QFT in curved spacetime)
- The matter and galaxy distribution in the Large Scale Structure of the Universe (statistical field theory)

- The physics of the Cosmic Microwave Background (classical and statistical field theory)

The goals of the course are: to discuss open problems in cosmology and describe their intimate relation to fundamental high energy physics; to provide the basic knowledge to understand modern research literature in cosmology; to explore how field theory provides a unifying formalism to describe disparate physical processes from the birth of the Universe to the highly non-linear cosmic web.

More specifically, after a general introduction to open current research and open problems in cosmology, we review inflation and introduce the Effective Field Theory of cosmological perturbations and its connection to field theories with non-canonical interactions. Then we present the so-called "in-in" or Schwinger-Keldysh formalism to compute cosmological correlators and discuss some simple examples, focusing on the leading non-Gaussian statistic, the bispectrum. After showing that cosmological perturbations become classical, we review some basic properties of stochastic fields and correlation functions. The equations determining the dynamics of Large Scale Structure are then introduced together with the concept of renormalization. As an application, we derive a prediction for the matter and galaxy power spectrum at next-to-leading order. Finally, we introduce the Boltzmann equations for the coupled photon-baryon "fluid" and use them to compute the observed temperature anisotropies in the Cosmic Microwave Background.

Pre-requisites

Some familiarity with introductory Quantum Field Theory and General Relativity, as for example provided by the respective Michaelmas courses, is highly recommended. Basic knowledge of introductory Cosmology is essential. Students who did not attend the Michaelmas course on Cosmology may still follow this course after reviewing the relevant course notes.

Literature

Lecture notes including references will be provided by the lecturers.

Additional support

Four example sheets will be provided and four associated example classes will be given. There will be a one-hour revision class in Easter Term.

Applications of Differential Geometry to Physics

Maciej Dunajski (L16)

This is a course designed to develop the Differential Geometry required to follow modern developments in Theoretical Physics. The following topics will be discussed.

- Geometry of Lie Groups
  1. Manifolds
  2. Vector fields and one–parameter groups of transformations
  3. Group action on manifolds
  4. Metrics on Lie Groups and Kaluza Klein theories.
- Classical mechanics
  1. Symplectic and Poisson structures
  2. Geodesic flow, Killing vectors, Killing Tensors.
  3. Null Kaluza–Klein reductions
  4. Integrable Systems

- Fibre bundles and instantons
  1. Principal bundles and vector bundles.
  2. Connection and Curvature
  3. Instantons

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

References

[1] Arnold. V. Mathematical Methods of Classical Mechanics. Springer.

[2] Dunajski, M. Solitons, Instantons, and Twistors, *Oxford Graduate Texts in Mathematics*, Oxford University Press, 2009.

[3] Eguchi, T., Gilkey, P. and Hanson. A. J. Physics Reports 66 (1980) 213-393

Gauge/Gravity Duality (E16)

Aron Wall

Gauge-Gravity duality (also known as AdS/CFT) is an amazing duality that relates theories of quantum gravity (with a negative cosmological constant) to certain quantum field theories living in a smaller dimensional spacetime. This is the most precise known realization of the holographic principle, the idea that all information in the universe is encoded somehow at the boundary of the universe. These lectures will describe in detail the "dictionary" used to relate observables on the bulk side to observables on the boundary side.

Topics covered: Anti-de Sitter spacetime; conformal field theory; wave equations in AdS, and their relationship to CFT operators and sources; the duality between black holes and thermal states; holographic entanglement entropy. If time permits: recent developments concerning bulk reconstruction, and the black hole information puzzle.

Pre-requisites

Required: General Relativity and Black Holes. Also, either Advanced Quantum Field Theory or Statistical Field Theory (specifically, partition functions and renormalization).

Helpful: Some basic aspects of quantum information theory and conformal symmetry will play an important role in this course, but the relevant aspects will be reviewed in a self-contained manner.

Not Required: String Theory, Supersymmetry. Although most of the specific known examples of AdS/CFT come from superstring theories, these aspects will not be emphasized in these lectures. (There will probably be one lecture on Juan Maldacena's original derivation of black hole entropy in string theory, but it will not be on the exam.)

Literature

Lecture notes for the non-examinable course last year, as well as some review articles, are located at:

`http://www.damtp.cam.ac.uk/user/aw846/AdSCFT.html`

(Note however that the topics covered this year may be slightly different, in particular Topic 5: Large N Gauge Theories and Examples of AdS/CFT will be mostly excluded.)

Additional support

Three examples sheets will be provided and three associated examples classes will be given.

Spinor Techniques in General Relativity (L24)

Non-Examinable (Graduate Level)

Irena Borzym (12 Lectures) and Peter O'Donnell (12 Lectures)

Spinor structures and techniques are an essential part of modern mathematical physics. This course provides a gentle introduction to spinor methods which are illustrated with reference to a simple 2-spinor formalism in four dimensions. Apart from their role in the description of fermions, spinors also often provide useful geometric insights and consequent algebraic simplifications of some calculations which are cumbersome in terms of spacetime tensors.

The first half of the course will include an introduction to spinors illustrated by 2-spinors. Topics covered will include the conformal group on Minkowski space and a discussion of conformal compactifications, geometry of scri, other simple simple geometric applications of spinor techniques, zero rest mass field equations, Petrov classification, the Plucker embedding and a comparison with Euclidean spacetime. More specific references will be provided during the course and there will be worked examples and handouts provided during the lectures.

The second half of the course will include: Newman-Penrose (NP) spin coefficient formalism, NP field equations, NP quantities under Lorentz transformations, Geroch-Held-Penrose (GHP) formalism, modified GHP formalism, Goldberg-Sachs theorem, Lanczos potential theory, Introduction to twistors. There will be no problem sets.

Pre-requisites

The Part 3 general relativity course is a prerequisite.

No prior knowledge of spinors will be assumed.

Literature

Introductory material.

1. L. P. Hughston and K. P. Tod, *Introduction to General Relativity.* Freeman, 1990.
2. C.W. Misner, K.S. Thorne and J.A. Wheeler, *Gravitation.* Freeman, 1973.

Best Course Reference Text for Lectures 1 to 12.

J.M. Stewart, *Advanced General Relativity.* CUP, 1993.

Best Course Reference Text for Lectures 13 to 24.

P O'Donnell, *Introduction to 2-spinors in general relativity.* World Scientific, 2003.

Reading to complement course material.

1. Penrose and Rindler, *Spinors and Spacetime Volume 1.* Cambridge Monographs on Mathematical Physics, 1987.
2. S. Ward and Raymond O. Wells, *Twistor Geometry and Field theory.* Cambridge Monographs on Mathematical Physics, 1991 .

3. Robert J. Baston, Michael G. Eastwood, *The Penrose Transform.* Clarendon Press, 1989.

4. S. A Huggett and P. Tod, *Introduction to Twistor Theory.* World Scientific, 2003.

5. R.M. Wald, *General Relativity.* World Chicago UP, 1984.

6. S.W. Hawking and G.F.R. Ellis, *The Large Scale Structure of Spacetime.* CUP, 1973.

Astrophysics

Introduction to Astrophysics courses

These courses provide a broad introduction to research in theoretical astrophysics; they are taken by students of both Part III Mathematics and Part III Astrophysics. The courses are mostly self-contained, building on knowledge that is common to undergraduate programmes in theoretical physics and applied mathematics. For specific pre-requisites please see the individual course descriptions.

Galaxy Formation (M24)

N Wyn Evans

This course describes our current state of knowledge of galaxy formation and evolution in a cold dark matter cosmology. We will start with structure formation in the non-linear regime, the formation and evolution of dark matter haloes and Press-Schechter theory. We will cover physical processes (shock heating, radiative cooling and star formation) as well as dynamical transformations (dynamical friction, tidal shocking, accretion and mergers) that are responsible for the shapes and properties of the galaxies we see today. We will end with a study of the formation and current day attributes of disk galaxies (Sersic profiles, thin and thick disks, stellar haloes) and elliptical galaxies (fast/slow rotators, major/minor mergers, Faber-Jackson relation). Recent discoveries on the structure of the Local Group and the Milky Way galaxy will be used as illustrative examples of formation processes throughout the course.

There is a complementary course by Prof. V. Belokurov in Lent Term on the present-day life and evolution of the galaxies.

Pre-requisites

This Part III course assumes that you have taken undergraduate courses in cosmology, relativity and dynamics.

Literature

1. J. Binney and S. Tremaine *Galactic Dynamics* 2nd edition, Princeton University Press, 2008
2. J. Bland-Hawthorn, K. Freeman *The Origin of the Galaxy and the Local Group*, Springer, 2014
3. A. Loeb *How Did the First Stars and Galaxies Form*, Princeton, 2010 (Background reading)
4. M. Longair, *Galaxy Formation* 2nd edition, Springer, 2008
5. H. Mo, F. van den Bosch and S. White, *Galaxy Formation and Evolution*, Cambridge Universitry Press, 2010
6. S. Phillips, *The Structure and Evolution of Galaxies*, Wiley, 2005
7. L. Sparke, J. Gallagher, *Galaxies in the Universe*, 2nd edition, Cambridge University Press, 2007 (Background reading)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Astrophysical Fluid Dynamics (M24)

Roman Rafikov

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as their stability and their response to tidal forcing.

Provisional synopsis

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- Conservation laws, symmetries and hyperbolic structure. Stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, shocks and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.
- Stellar oscillations. Introduction to asteroseismology and astrophysical tides.
- Local dispersion relation. Internal waves and instabilities in stratified rotating astrophysical bodies.

Pre-requisites

This course is suitable for both astrophysicists and fluid dynamicists. An elementary knowledge of vector calculus, fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas.* Cambridge University Press.
2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.
3. Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows.* Cambridge University Press. Available as an e-book from

   `http://ebooks.cambridge.org`

4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics.* University Science Books.
5. Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics.* Imperial College Press.
6. Ogilvie, G. I. (2016). *Lecture Notes: Astrophysical Fluid Dynamics.* J. Plasma Phys. **82**, 205820301.

Additional support

Four example sheets will be provided and four associated classes will be given. Extended notes supporting the lecture course are available from reference 6 in the list above. There will be a revision class in Easter Term.

Structure and Evolution of Stars (M24)

A.N.Żytkow

Our attempts at gaining insight into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. Such a description naturally follows the laws of conservation of mass, momentum and energy. The basic equations for spherical stars will be derived and boundary conditions described. These equations have to be supplemented by a descrition of the methods of energy transport, the equation of state, the physics of opacity and nuclear reactions, all of which will be discussed. Some familiarity with the principles of hydrodynamics, thermodynamics, quantum mechanics, atomic and nuclear physics will be assumed.

Approximate solutions of the equations will be described. Polytropic gas spheres, homology principles, the virial theorem will be presented. The evolution of a star will be discussed, starting from the main-sequence, following the stages in which various nuclear fuels are exhausted and leading to the final outcome as white dwarfs, neutron stars or black holes.

There will be a brief discussion of helioseismology, stellar rotation and mass loss from stars.

The only way in which we may test stellar structure and evolution theory is through the comparison of theoretical results to observations. Throughout the course, reference will be made to the observational properties of the stars, with particular reference to the Hertzsprung-Russell diagram, the mass-luminosity law and spectroscopic information.

Pre-requisites

At least a basic understanding of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics although a detailed knowledge of all of these is not expected.

Preliminary Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Literature

1. Eldridge, J.J. and Tout, C.A. *The Structure and Evolution of Stars*, World Scientific, 2019.
2. Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution, Second Edition*, Springer-Verlag, 2012.
3. Iben, I. *Stellar Evolution Physics, Vol. 1 and 2*, Cambridge University Press, 2013.
4. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
5. Padmanabhan, T. *Theoretical Astrophysics, Volume II: Stars and Stellar Systems*, CUP, 2001.

Additional support

There will be four example sheets each of which will be discussed during an examples class. There will be a one-hour revision class in the Easter Term.

Extrasolar Planets: Atmospheres and Interiors (L24)

Nikku Madhusudhan

The field of extrasolar planets (or 'exoplanets') is one of the most dynamic frontiers of modern astronomy. Exoplanets are planets orbiting stars beyond the solar system. Thousands of exoplanets are now known with a wide range of sizes, temperatures, and orbital parameters, covering all the categories of planets in the solar system (gas giants, ice giants, and rocky planets) and more. The field is now moving into a new era of Exoplanet Characterization, which involves understanding the atmospheres, interiors, and formation mechanisms of exoplanets, and ultimately finding potential biosignatures in the atmospheres of rocky exoplanets. These efforts are aided by both high-precision spectroscopic observations as well as detailed theoretical models of exoplanets.

The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and stratospheres, equilibrium/non-equilibrium chemistry, atmospheric dynamics, clouds/hazes, etc) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, as well as parametric models and retrieval techniques) (3) exoplanetary interiors (equations of state, mass-radius relations, and internal structures of giant planets, super-Earths, and rocky exoplanets), and (4) relating atmospheres and interiors to planet formation. Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres of all kinds. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed. The course will also include a discussion on detecting biosignatures in rocky exoplanets, the relevant theoretical constructs and expected observational prospects with future facilities.

Pre-requisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self-contained and basic concepts will be introduced as required.

Literature

1. Seager, S., Exoplanet Atmospheres: Physical Processes, Princeton Series in Astrophysics (2010).
2. Exoplanets, University of Arizona Press (2011), ed. S. Seager.
3. de Pater, I. and Lissauer J., Planetary Sciences, Cambridge University Press (2010).
4. Chapters on exoplanetary atmospheres and interiors in the book Protostars and Planets VI, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning. Available publicly on astro-ph arXiv (e.g., arXiv:1402.1169, arXiv:1401.4738).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

The Life and Death of Galaxies (L24)

Vasily Belokurov

This course will provide the observational perspective on the evolution of galaxies and will complement the theoretical Part III course "Galaxy Formation".

SEDs. Panchromatic view of galaxies. Dependance of the galactic structure on wavelength. Components of the galactic SED. Stars. Dust. Nebular emission. Stromgren sphere. Initial mass function. From SED to star-formation history, present star-formation, stellar and gas contents.

Star-formation. Star-formation rates. Metal-enrichment. Yield. Closed box, instantaneous mixing, one zone model. G-dwarf problem. Inflows and Outflows. Mass-metallicity relation. Star-Formation Law. Cloud collapse. Star formation on sub-galactic scales. Star-formation history of the Universe. Integrated light: Look back vs Archaeology. Halo assembly in the Cold Dark Matter Universe. Galaxy formation efficiency as a function of galaxy mass. SFH with resolved stellar populations.

Dynamics. Dynamics of elliptical galaxies. Distribution Functions. Collisionless Boltzman Equation (CBE). Integrals of CBE. Jeans Equations. Virial Theorem. Poisson Equation. Slow and Fast rotators. Dynamical Friction. Fundamental and Mass planes. Relaxation and phase mixing. Dynamics of spiral galaxies. Rotation curves. Dark matter. Navarro-Freak-White density profiles. Galactic halos. Mass measurements with gravitational lensing.

Structure and evolution. Structure of galaxies. Two types of galaxies: dead and alive. Light distribution. Sersic profile and its modifications. Detailed structure of elliptical galaxies. Shapes of ellipticals. Galaxy luminosity function. Schechter function. Connection between galaxy type/structure and environment. Mergers. Tides. Evolution of elliptical galaxies.

Pre-requisites

It is preferable (but not required) that you have attended "Galaxy Formation" course in M19.

Literature

1. Binney, J., and Tremaine, S. *Galactic Dynamics* Second Edition. Princeton University Press, 2008
2. Mo, H., van den Bosch, F., and White, S. *Galaxy Formation and Evolution* Cambridge University Press, 2010
3. Sparke, L., and Gallagher, J. *Galaxies in the Universe* Second Edition. Cambridge University Press, 2007
4. Longair, M. *Galaxy Formation* Second Edition. Springer, 2008

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Dynamics of Astrophysical Discs (L16)

Gordon Ogilvie

A disc of matter in orbital motion around a massive central body is found in numerous situations in astrophysics. For example, Saturn's rings consist of trillions of metre-sized iceballs that undergo gentle collisions as they orbit the planet and behave collectively like a fluid. Protostellar or protoplanetary discs are the dusty gaseous nebulae that surround young stars for their first few million years; they accommodate the angular momentum of the collapsing cloud from which the star forms, and are the sites of planet formation. Plasma accretion discs are found around black holes in interacting binary star systems and in the centres of active galaxies, where they can reveal the properties of the compact central objects and produce some of the most luminous sources in the Universe. These diverse systems have much in common dynamically.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). The evolution of an accretion disc is governed by the conservation of mass and angular momentum and is regulated by the efficiency of angular momentum transport. An astrophysical disc is a rotating shear flow whose local behaviour can be analysed in a convenient model known as the shearing sheet. Various instabilities can occur and give rise to sustained angular momentum transport. The resonant gravitational interaction of a planet or other satellite with the disc within which it orbits generates waves that carry angular momentum and energy. This process leads to orbital evolution of the satellite and is one of the factors shaping the observed distribution of exoplanets.

Provisional synopsis:

Occurrence of discs in various astronomical systems, basic physical and observational properties.

Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.

Evolution of an accretion disc.

Vertical disc structure, scaling relations and timescales, thin-disc approximations, thermal and viscous stability.

Shearing sheet, symmetries, shearing waves.

Incompressible dynamics: hydrodynamic stability, vortices and dust dynamics.

Compressible dynamics: density waves, gravitational instability and turbulence.

Satellite-disc interaction: tidal potential, resonant torques, migration and gap opening.

Magnetorotational instability and turbulence.

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some previous knowledge of basic magnetohydrodynamics is helpful for the magnetorotational instability, but self-contained notes on this topic will be available.

Literature

Much information on the astrophysical background is contained in [1]. Some of the basic theory of accretion discs is described in review articles [2,3].

1. Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.
2. Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.
3. Papaloizou, J. C. B. & Lin, D. N. C. (1995), Annu. Rev. Astron. Astrophys. 33, 505.

Additional support

Three example sheets will be provided and three associated example classes will be given. There will be a revision class in Easter Term.

Binary Stars (L16)

Christopher Tout

A binary star is a gravitationally bound system of two component stars. Such systems are common in our Galaxy and a substantial fraction interact in ways that can significantly alter the evolution of the individual stellar components. Many of the interaction processes lend themselves to useful mathematical modelling when coupled with an understanding of the evolution of single stars.

In this course we begin by exploring the observable properties of binary stars and recall the basic dynamical properties of orbits by way of introduction. This is followed by an analysis of tides, which represent the simplest way in which the two stars can interact. From there we consider the extreme case in which tides become strong enough that mass can flow from one star to the other. We investigate the stability of such mass transfer and its effects on the orbital elements and the evolution of the individual stars. As a prototypical example we examine Algol-like systems in some detail. Mass transfer leads to the concept of stellar rejuvenation and blue stragglers. As a second example we look at the Cataclysmic Variables in which the accreting component is a white dwarf. These introduce us to novae and dwarf novae as well as a need for angular momentum loss by gravitational radiation or magnetic braking. Their formation requires an understanding of significant orbital shrinkage in what is known as common envelope evolution. Finally we apply what we have learnt to a number of exotic binary stars, such as progenitors of type Ia supernovae, X-ray binaries and millisecond pulsars.

Pre-requisites

The Michaelmas term course on Structure and Evolution of Stars is very useful but not absolutely essential. Knowledge of elementary Dynamics and Fluids will be assumed.

Literature

1. Pringle J. E. and Wade R. A., *Interacting Binary Stars.* CUP.

Reading to complement course material

1. Eggleton P. P., *Evolutionary Processes in Binary and Multiple Stars.* CUP.

Additional support

Three examples sheets will be provided and three associated two-hour examples classes will be given. There will be a two-hour revision class in the Easter Term.

Quantum Computation, Information and Foundations

Quantum Computation (M16)

Richard Jozsa

Quantum mechanical processes can be exploited to provide new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

This course will be a 'second' course in the subject, following the Part II course Quantum Information and Computation (see below in prerequisites) that was introduced in the year 2017-2018.

In this course we will aim to cover the following topics:
• The hidden subgroup problem and quantum Fourier transform on a group;
• The quantum phase estimation algorithm and applications;
• Amplitude amplification and applications;
• Quantum simulation for local hamiltonians;
• The Harrow-Hassidim-Lloyd quantum algorithm for systems of linear equations.
If time permits we may also discuss (or substitute) further topics such as: Introduction to Clifford operations; Classical simulation properties of Clifford circuits (Gottesman-Knill theorem); Measurement based quantum computing; The Pauli based model of quantum computing (Bravyi, Smith and Smolin 2016).

Pre-requisites

This course will assume a prior basic acquaintance with quantum computing, to the extent presented in the course notes for the Cambridge Part II course Quantum Information and Computation available at `http://www.qi.damtp.cam.ac.uk/node/272`
In particular you should be familiar with Dirac notation and principles of quantum mechanics, as presented in the course notes sections 2.1, 2.2 and 2.3. You should also have a basic acquaintance with quantum computation to the extent of the second half of the course notes, pages 47 to 86 (Chapters 6-11). *It would be desirable for you to look through this material before the start of the course.*

Literature

Further useful literature includes the following.

1. Nielsen, M. and Chuang, I., *Quantum Computation and Quantum Information.* CUP, 2000.

2. John Preskill *Lecture Notes on Quantum Information Theory* (especially Chapter 6) available at `http://www.theory.caltech.edu/people/preskill/ph219/`

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

Quantum Information Theory (L24)

Sergii Strelchuk

Quantum Information Theory (QIT) lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the mathematical theory of acquisition, storage, transmission and processing of information.

QIT is the study of how these tasks can be accomplished, using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical counterparts. These new features can be exploited, not only to improve the performance of certain information-processing tasks but also to accomplish tasks which are impossible or intractable in the classical realm.

The course will start with a short introduction to some of the basic concepts and tools of Classical Information Theory, which will prove useful in the study of QIT. Topics in this part of the course will include a brief discussion of data compression, transmission of data through noisy channels, Shannon's theorems, entropy and channel capacity. The quantum part of the course will commence with a study of open systems and a discussion of how they necessitate a generalization of the basic postulates of quantum mechanics. Topics will include quantum states, quantum operations, generalized measurements, POVMs, the Kraus Representation Theorem, the Choi-Jamilkowski isomorphism, quantum data compression limit, and random coding arguments.

We will further focus on data compression, reliable transmission of information over noisy communication channels, and introduce accessible information and coherent information. In particular, we will discuss the Holevo bound on the accessible information, the Holevo-Schumacher-Westmoreland (HSW) Theorem, and key properties of coherent information leading to surprising superadditivity effects for quantum channel capacities.

Pre-requisites

Familiarity with the Part II course *Quantum Information and Computation* or equivalent is essential.

Knowledge of basic quantum mechanics will be assumed.

Elementary knowledge of Probability Theory, Vector Spaces and Linear Algebra will be useful.

Literature

1. M. A. Nielsen and I. L. Chuang *Quantum Computation and Quantum Information.* Cambridge University Press, 2002.

2. M. M. Wilde *From Classical to Quantum Shannon Theory* Cambridge University Press, 2013.

3. J. Preskill, *Lecture notes on Quantum Information Theory*, Acta Applicandae, **56**, 1-98 (1999). Also available at `http://www.theory.caltech.edu/~preskill/ph229/#lecture`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Philosophy of Physics

The courses in Philosophy of Physics are open to all students doing Part III, but are formally listed as graduate courses. This means there is no exam at the end of May for any such course; but a Part III student can get credit for them by doing their submitted Part III essay in association with one of the courses. More generally, the Philosophy of Physics courses are intended as a refreshing and reflective companion to the other Part III courses, especially the courses in theoretical physics.

Philosophical Aspects of Classical and Quantum Mechanics (M8)

Non-Examinable (Part III Level)

J. Butterfield and B. Roberts

This course surveys some philosophical aspects of classical and quantum mechanics. Since philosophy of physics is an inter-disciplinary subject (and the course is not examinable!), we will let the content be influenced by the interests of those attending. But we will begin with elements of (a) the modern formulation of Lagrangian and Hamiltonian mechanics in terms of tangent and cotangent bundles, and (b) the quantum measurement problem (including density matrices, mixtures and decoherence). Then we will continue with such topics as: (i) the treatment of time, and time observables, in both classical and quantum mechanics; (ii) the geometric formulation of quantum mechanics as discussed by Kibble, Gibbons, Ashtekar and Schilling; (iii) symmetry and symmetry principles in physics, including time reversal symmetry; and (iv) uncertainty relations, including analogues in classical mechanics.

Pre-requisites

There are no formal prerequisites. Previous familiarity with the frameworks of classical and quantum mechanics will be essential; but the technicalities of each topic will be developed as needed in the lectures.

Preliminary Reading

Among the many good books about the course's topics, one might pick out as especially inspiring:

1. Weyl, H. *Philosophy of Mathematics and Natural Science.* Princeton University Press, 1949, 2009.
2. Bell, J. *Speakable and Unspeakable in Quantum Mechanics.* Cambridge University Press, 1987, 2004.

Literature

For topics (a) and (b), we expect to use parts of the following. The last item is freely downloadable, and an invaluable resource for the whole course.

1. Geroch, R. "Geometrical quantum mechanics", 1974, `http://strangebeautiful.com/other-texts/geroch-geom-qm.pdf`
2. Marsden, A. and Ratiu, T. *Introduction to Mechanics and Symmetry*, 2nd Edition, Springer 2002, `https://www.fis.unam.mx/~max/mecanica/b_Marsden.pdf`
3. Butterfield, J. *On Symplectic Reduction in Classical Mechanics*, in J. Earman and J. Butterfield (eds.) *The Handbook of Philosophy of Physics*, 2 volumes, Elsevier; pp. 1 - 131. Available at: physics/0507194 and at `http://philsci-archive.pitt.edu/archive/00002373/`
4. Isham, C. *Modern Differential Geometry for Physicists.* World Scientific. 1999.

5. Landsman, N. *Between classical and quantum.* In Butterfield, J. and Earman, J. (eds.) *Handbook of the Philosophy of Physics*, 2 volumes, Elsevier. Available at: http://arxiv.org/abs/quant-ph/0506082, and at: http://philsci-archive.pitt.edu/archive/00002328/

6. Landsman, N. *Foundations of Quantum Theory.* Springer 2017: especially Chapters 1-5 and Chapter 11. Open access: downloadable at: https://link.springer.com/book/10.1007/978-3-319-51777-3

For topics (i)-(iv), we expect to use parts of the following.

(i) Time and time observables

1. Hilgevoord, J., *Time in quantum mechanics*, American Journal of Physics **70**, 301-306 (2002).

2. Roberts, B. *A general perspective on time observables.* Studies in History and Philosophy of Physics, **47**,50-54 (2014), http://philsci-archive.pitt.edu/10600/.

3. Roberts, B. *Observables, Disassembled*, Studies in History and Philosophy of Modern Physics, **63**, 150-162 (2018), http://philsci-archive.pitt.edu/1444.

(ii) Geometric quantum mechanics

1. Gibbons, G.W. *Typical states and density matrices*, Journal of Geometry and Physics, **8**(1-4), 147-162 (1992).

2. Ashtekar, A. and Schilling, T.A. *Geometrical formulation of quantum mechanics.* In Alex Harvey, editor, *On Einstein's path: essays in honor of Engelbert Schücking*, pages 23-65. Springer-Verlag New York, Inc., (1999). https://arxiv.org/abs/gr-qc/9706069

3. Brody, D.C. and Hughston, L.P. *Geometric quantum mechanics*, Journal of Geometry and Physics, **38**, 19-53 (2001), https://arxiv.org/abs/quant-ph/9906086

(iii) Symmetry

1. Roberts, B. *Three myths about time reversal in quantum mechanics*, Philosophy of Science, **84**, 1-20 (2017), http://personal.lse.ac.uk/robert49/pdf/Roberts2017a.pdf.

2. Roberts, B. *Three merry roads to T-violation.* Studies in History and Philosophy of Modern Physics, **52**, 8-15 (2015), http://philsci-archive.pitt.edu/9851/.

3. Roberts, B. *The simple failure of Curie's principle*, Philosophy of Science, **80**, 579-592 (2013), http://philsci-archive.pitt.edu/9862/.

(iv) Uncertainty relations

1. Hilgevoord, J. and Uffink J. (2006), *The uncertainty principle*, Stanford Encyclopedia of Philosophy: http://www.seop.leeds.ac.uk/entries/qt-uncertainty/

2. Uffink, J. (1990), *Measures of Uncertainty and the Uncertainty Principle*, Utrecht University PhD. Available at: http://www.projects.science.uu.nl/igg/jos/

Additional support

One or two Part III essays will be offered in conjunction with this course.

Philosophical Aspects of Quantum Field Theory (L8)

Non-Examinable (Part III Level)

J. Butterfield and B. Roberts

Quantum field theory has for many decades been the framework for several basic and outstandingly successful physical theories. Nowadays, it is being addressed by philosophy of physics (which has traditionally concentrated on conceptual questions raised by non-relativistic quantum mechanics and relativity). This course will introduce this literature. The content will be moulded by students' interests. But we hope: (i) to emphasize quantization theory, and algebraic methods; (ii) to mostly use the books by Folland, and by de Faria and de Melo; and (iii) to lead up to the Unruh effect.

We also expect, in the first half of the course, to review: (a) the mathematical structure of quantum theories in general, at the level of the books by Hannabuss, Jordan and Prugovecki; (b) some foundational issues, using the books by Araki, Clifton and Landsman (which is Open Access); (c) ideas of operator algebras, using the books by Emch, Haag and Ruetsche.

Pre-requisites

There are no formal prerequisites. Previous familiarity with quantum field theory, such as provided by the Part III courses, will be helpful.

Preliminary Reading

This list of reading gives an overview of the course's topics, and is approximately in order of increasing difficulty.

1. S. Weinberg (1997), 'What is Quantum Field Theory, and What Did We Think It Is?'. Available online at: http://arxiv.org/abs/hep-th/9702027; and in T. Cao, (ed.) *The Conceptual Foundations of Quantum Field Theory*. Cambridge University Press, 1999.

2. D. Wallace (2006), 'In defense of naiveté: The conceptual status of Lagrangian quantum field theory', *Synthese*, **151** (1):33-80, 2006. Available online at: http://arxiv.org/pdf/quant-ph/0112148v1

3. D. Wallace (2001), 'Emergence of particles from bosonic quantum field theory'. Available online at: http://arxiv.org/abs/quant-ph/0112149

4. L. Ruetsche, *Interpreting Quantum Theories*: especially up to Chapter 9. Oxford University Press, 2011.

5. R. Clifton and H. Halvorson (2001), 'Are Rindler quanta real? Inequivalent particle concepts in quantum field theory', *British Journal for Philosophy of Science*, **52**, pp 417-470. Sections 1, 2.1, 2.2, 3.1, 3.2. Available online at: http://arxiv.org/abs/quant-ph/0008030. Reprinted as Chapter 9 in R. Clifton *Quantum Entanglements*, ed. J. Butterfield and H. Halvorson, Oxford University Press 2004.

Literature

The main resource will be the books by Folland, and by de Faria and de Melo. For mathematical background, we will draw on the books by Hannabuss, Jordan and Prugovecki. For foundational issues, we will draw on the books by Araki, Clifton and Landsman (the last being freely downloadable, and an invaluable resource for the whole course). For operator algebras, we will also use the books by Emch, Haag and Ruetsche. An overall reference is Ticciati (1999); a recent advanced monograph is Rejzner (2016).

1. E. de Faria and W. de Melo. *Mathematical Aspects of Quantum Field Theory*: up to Chapter 6. Cambridge University Press, 2010.

2. G. Folland. *Quantum Field Theory: a tourist guide for mathematicians*: up to Chapter 6. American Mathematical Society, 2008.

3. K. Hannabuss. *An Introduction to Quantum Theory*: up to Chapter 11. Oxford University Press, 1997.

4. T. Jordan. *Linear Operators for Quantum Mechanics*: especially Chapters 3 to 5. John Wiley 1969; Dover 2006.

5. E.Prugovecki. *Quantum Mechanics in Hilbert Space*: especially Parts III, IV. Academic Press 1981; Dover 2006.

6. H. Araki. *Mathematical Theory of Quantum Fields*: up to Chapter 4. Oxford University Press, 1999.

7. R. Clifton. *Quantum Entanglements*, edited by J. Butterfield and H. Halvorson: Chapters 6 to 9. Oxford University Press 2004.

8. N. Landsman. *Foundations of Quantum Theory*. Springer 2017: especially Chapters 5, 6,7,9,10. Open access: downloadable at: https://link.springer.com/book/10.1007/978-3-319-51777-3

9. G. Emch. *Algebraic Methods in Statistical Mechanics and Quantum Field Theory*: especially Chapter 1. John Wiley 1972; Dover 2009.

10. R. Haag. *Local Quantum Physics: fields, particles, algebras*: Chapters I, II, III and V.4: Springer 1992.

11. R. Ticciati. *Quantum Field Theory for Mathematicians*: up to Chapter 6. Cambridge University Press, 1999.

12. K. Rejzner. *Perturbative Algebraic Quantum Field Theory: an introduction for mathematicians*: up to Chapter 5. Springer 2016.

Additional support

One or two Part III essays—one of them probably about the Unruh effect—will be offered in conjunction with this course.

Applied and Computational Analysis

Inverse Problems (M24)

Hanne Kekkonen & Yury Korolev

Inverse problems arise whenever there is a need to infer quantities of interest from indirectly measured data. Inverse problems are ubiquitous in science; they arise in physics, biology, medicine, engineering, finance and computer science (e.g., in machine learning and computer vision). Many imaging problems, such as reconstruction of medical images (computer tomography, magnetic resonance imaging, positron-emission tomography) and deblurring or denoising of microscopy and astronomy images, are also instances of inverse problems. Inverse problems typically share a feature that makes them challenging to solve in practice; they lack continuous dependence on the data and, therefore, small errors in the measurements can lead to large errors in naive reconstructions, making them useless. To deal with this issue, special *regularisation* and *Bayesian* techniques have been developed to overcome the instability by using additional a priori information about the unknown, such as smoothness or sparsity in some basis.

In this course we will present the mathematical theory for solving inverse problems using regularisation and Bayesian methods, from the classical foundations to modern state-of-the-art methods. We will apply the theory to some problems in imaging and introduce efficient numerical algorithms.

Pre-requisites

This course assumes basic knowledge in linear algebra, analysis and probability theory (e.g. linear analysis or analysis of functions and Probability and Measure). Additional knowledge in convex analysis is beneficial, but not mandatory.

Literature

1. H. W. Engl, M. Hanke and A. Neubauer. *Regularization of Inverse Problems.* Vol. 375, Springer Science & Business Media, 1996, ISBN: 9780792341574

2. O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier and F. Lenzen. *Variational Methods in Imaging.* Applied Mathematical Sciences, Springer New York, 2008, ISBN: 9780387309316

3. M. Dashti and A.M. Stuart, *The Bayesian approach to inverse problems, Handbook of Uncertainty Quantification.* Springer, 2017.

4. A.M. Stuart, *Inverse problems: a Bayesian perspective.* Acta Numerica, 2010.

5. M. Benning, M. Burger *Modern Regularization Methods for Inverse Problems*, Acta Numerica, **27**, 1-111 (2018). Also available at

   `https://www.cambridge.org/core/journals/acta-numerica/article/modern-regularization-methods-for-inverse-problems/1C84F0E91BF20EC36D8E846EF8CCB830`

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Distribution Theory and Applications (M16)

Dr. A. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we will look at operations such as differentiation, translation, convolution and the Fourier transform. We will breifly look at Sobolev spaces in $\mathbf{R}^n$ and their description in terms of the Fourier transform of tempered distrbutions. The material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. The final section of the course will be concerned with Hörmander's oscillatory integrals.

Pre-requisites

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB Complex Methods or Complex Analysis).

Preliminary Reading

1. F.G. Friedlander & M.S. Joshi, *Introduction to the Theory of Distributions*, Cambridge Univ Pr, 1998.
2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, Cambridge Univ Pr, 1958.
3. G.B. Folland, *Introdution to Partial Differential Equations*, Princeton Univ Pr, 1995.

Literature

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols* I-II, Springer Verlag, 1985.
2. M. Reed & B. Simon, *Methods of Modern Mathematical Physics: Vols* I-II, Academic Press, 1979.
3. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be made available. There will be a revision class in the Easter Term.

Topics in Convex Optimisation (M16)

Hamza Fawzi

Mathematical optimisation problems arise in many areas of science and engineering, including statistics, machine learning, robotics, signal/image processing, and others. This course will cover some techniques known as *convex relaxations*, to deal with optimisation problems involving polynomials, which are in general intractable. The emphasis of the course will be on semidefinite programming which is a far-reaching generalization of linear programming. A tentative list of topics that we will cover include:

- From linear programming to conic programming. Duality theory.
- Semidefinite optimisation and convex relaxations. Sums-of-squares and moment problems.
- Applications: binary quadratic optimisation and rounding methods (e.g., Goemans-Williamson rounding), stability of dynamical systems, matrix completion/low-rank matrix recovery, etc.

Pre-requisites

This course assumes basic knowledge in linear algebra and analysis. Some knowledge of convex analysis will be useful.

Literature

1. A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, 2001 (`http://dx.doi.org/10.1137/1.9780898718829`).
2. G. Blekherman, P. Parrilo, R. Thomas, *Semidefinite optimization and convex algebraic geometry*, SIAM 2013 (`http://dx.doi.org/10.1137/1.9781611972290`).
3. S. Boyd, L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004 (`http://web.stanford.edu/~boyd/cvxbook/`).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Numerical Solution of Differential Equations (L24)

Arieh Iserles

The course will address modern algorithms for the solution of ordinary and partial differential equations, inclusive of finite difference and finite element methods, with an emphasis on broad mathematical principles underlying their construction and analysis.

Pre-requisites

Although prior knowledge of *some* numerical analysis and of abstract function spaces is advantageous, it will not be taken for granted. Reasonable understanding of basic concepts of analysis (complex analysis and analytic functions, basic existence and uniqueness theorems for ODEs and PDEs, elementary facts about PDEs) and of linear algebra is a prerequisite.

Literature

1. U. Ascher, *Numerical Methods for Evolutionary Differential Equations*, SIAM, 2008.
2. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations* (2nd edition), Cambridge University Press, 2006.

Additional support

An extensive printed handout, covering the entire material of the course, will be provided in the first week. There will be weekly examples' classes, starting from the third week, as well as a revision supervision in the Easter Term.

Hybrid Photonics Computing (L16)

Natalia Berloff

Recently Coherent Networks emerged as a promising alternative to universal classical or quantum computing and to quantum simulators/annealers. Their physical implementation is based on coherent dynamics of so-called coherent centers in the network of lasers, optical parametric oscillators, cold atomic gases, exciton-polariton condensates, memristors, VO2 oscillators, ring oscillators, multicore fibers, etc. They are expected to serve as fast and accurate accelerators for modern digital computers in the specialized tasks for NP-hard integer and continuous optimization problems in vastly different areas such as vehicle routing and scheduling problems, dynamic analysis of neural networks and financial markets, prediction of new chemical materials and machine learning. The theoretical framework of coherent networks was proposed as heuristic algorithms for NP-hard optimization problems and efficient simulators for many-body systems. This course covers fundamental principles, algorithms, and applications, as well as the physical implementation of coherent network computing.

Pre-requisites

Undergraduate level degree in physics or applied mathematics and basic knowledge of scientific computing is expected.

Literature

1. P. R. Prucnal, B. J. Shastri *Neuromorphic Photonics.* 1st edition. CRC Press, 2017.
2. M. Newman *Networks.* 2nd edition, Oxford University Press, 2018.
3. K. Staliunas and VJ Sanches-Morcillo *Transverse Patterns in Nonlinear Optical Resonators*, Springer, 2003.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Mathematical Analysis of the Incompressible Navier-Stokes Equations (L24)

Non-Examinable (Graduate Level)

Edriss S. Titi

In these lectures I will introduce the basics of the mathematical analytical theory of the Navier-Stokes equations of viscous incompressible fluids. These equations appear in a wide range of physical and biological applications, varying from oceanic and atmospheric dynamics, to combustion theory and body-fluid transport. On the one hand, from the mathematical analysis point of view these equations have been recognized to be among the most challenging problems in Applied Analysis. On the other hand, from the computational point of view they are prohibitively expensive to simulate for very large Reynolds numbers, and out of reach even for most powerful state-of-the-art computers.

Topics to be covered:

1. Introducing the Navier-Stokes and Euler equations of incompressible fluids.
2. Sobolev spaces, Sobelev inequalities, interpolation inequalities; and the relevant Functional Spaces.

3. Steady state solutions to the incompressible Navier-Stokes equations and their regularity.
4. Time dependent Leray-Hopf weak solutions to the incompressible Navier-Stokes equations.
5. Global regularity of strong solutions for the two-dimensional incompressible Navier-Stokes equations
6. Short time existence of strong solutions in the three-dimensional incompressible Navier-Stokes equations.
7. Weak-strong uniqueness and the role of the energy inequality in the Leray-Hopf weak solutions.

Pre-requisites

Students who are interested in attending the course are expected to have a background in Real Analysis and some basics of Functional Analysis. The rest of the course will be, to a large extend, self-contained.

Literature

1. P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, 1988.
2. R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holand. New print published by the AMS 2001.
3. R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF Regional Conference Series in Applied Math 66, SIAM, 2nd Ed, 1995.
4. R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd Ed, Applied Math Sci. 68, Springer-Verlag, 1997.
5. H. Sohr, *The Navier–Stokes Equations, An Elementary Functional Analytic Approach*, Birkhäuser Verlag, Basel, 2001.

Additional References

1. A. Chorin and J. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag.
2. C. Doering and J. Gibbon, *Applied Analysis of the Navier-Stokes Equations*, Cambridge University Press.
3. C. Foias, O. Manley, R. Rosa and R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001.
4. A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, 2002.
5. J. C. Robinson, *Infinite-dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics.

Continuum Mechanics

The four courses in the Michaelmas Term are intended to provide a broad educational background for any student preparing to start a PhD in fluid dynamics. The courses in the Lent Term are more specialized and in some cases (see the course descriptions) build on the Michaelmas Term material.

Desirable previous knowledge

For all the fluid dynamics courses, previous attendance at an introductory course in fluid dynamics will be assumed. In practice, familiarity with the continuum assumption, the material derivative, the stress tensor and the Navier-Stokes equation will be assumed, as will basic ideas concerning incompressible, inviscid fluid mechanics (e.g. Bernoulli's Theorem, vorticity, potential flow). Some knowledge of basic viscous flow, such as Stokes flow, lubrication theory and elementary boundary-layer theory, is highly desirable. Previous attendance at a course on wave theory covering concepts such as wave energy and group velocity, is desirable for some courses. No previous knowledge of solid mechanics, Earth Sciences, or biology is required.

In summary, knowledge of Chapters 1-8 of 'Elementary Fluid Dynamics' (D.J. Acheson, Oxford), plus Chapter 3 of 'Waves in Fluids' (J. Lighthill, Cambridge)(which deals with dispersive waves) would give a student an excellent grounding.

Familiarity with basic vector calculus (including Cartesian tensors), differential equations, complex variable techniques (e.g. Fourier Transforms) and techniques for solution of elementary PDEs, such as Laplace's equation, Poisson's equation, the diffusion equation and the simple wave equation, will be assumed. Knowledge of elementary asymptotic techniques would be helpful.

A Cambridge student taking continuum courses in Part III would be expected to have attended the following undergraduate courses

Year	*Courses*
First	Differential Equations, Dynamics and Relativity, Vector Calculus, Vectors & Matrices.
Second	Methods, Complex Methods, Fluid Dynamics.
Third	Fluid Dynamics, Waves, Asymptotic Methods.

Students starting Part III from outside Cambridge might like to peruse the syllabuses for the above courses on WWW with URL:

`http://www.maths.cam.ac.uk/undergrad/schedules/`

Slow Viscous Flow (M24)

J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Pre-requisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the summation convention is particularly useful for the first half of the course.

Preliminary Reading

1. D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7
2. G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.
3. L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 & 5.

Literature

1. J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).
2. S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)
3. C. Pozrikidis. *Boundary Integral and Singularity Methods for Linearized Viscous Flow.* CUP (1992).
4. O.M. Phillips. *Flow and Reactions in Permeable Rocks.* CUP (1991).

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Fluid dynamics of the solid Earth (M24)

Jerome A. Neufeld & M. Grae Worster

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth's core is formed by the solidification of a mixture of molten iron and various lighter elements, a process which drives predominantly compositional convection in the liquid outer core, thus producing the geodynamo responsible for the Earth's magnetic field. At very much longer time scales, radiogenic heating of the solid mantle drives solid-state convection resulting in plume-like features possibly responsible for features such as the Hawaiian sea mounts. Nearer the surface, convection drives the motion of brittle plates which are responsible for the Earth's topography as can be felt and imaged through the seismic record. Upwelling mantle material also drives partial melting of mantle rocks resulting in compaction, and ultimately in the propagation of viscous melt through the elastic crust. On the Earth's surface, and at very much faster rates, the same physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of sea ice to the flow of glacial ice.

This course will use the wealth of observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through porous materials.

Pre-requisites

A basic understanding of viscous fluid dynamics. Mathematical methods, particularly the solution of ordinary and partial differential equations.

Literature

1. M.G. Worster. *Solidification of Fluids.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)

2. H.E. Huppert. *Geological fluid mechanics.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)

3. D.L. Turcotte, G. Schubert. Geodynamics, second edition. CUP (2002)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Perturbation Methods (M16)

S.J. Cowley

This course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate in the problem assumes large or small values. Many problems of physical interest are covered by such asymptotic limits. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases providing accurate predictions when the parameter or coordinate has only moderately large or small values.

Some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called 'exponential asymptotics'), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to 'asymptotics beyond all orders' in which exponentially small corrections are extracted from the tails of asymptotic series. The advantage of uniformly valid expansions for comparison with experiment and numerical solutions will be covered. [7]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. Boundary-layer theory in fluid mechanics was the subject in which the method was first developed, but it has since been greatly extended and applied to many fields. Further examples will be given of asymptotics beyond all orders. This section will include a brief introduction to the summation of [divergent] series, e.g. covering Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks' transformations, Richardson extrapolation, and Domb-Sykes plots. [6]

- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKB[JLG]' method can be viewed as a special case). It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied, potentially including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium). [3]

Pre-requisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward differential equations and partial differential equations and evaluate simple integrals.

Literature

Relevant Textbooks

1. Bender, C.M. & Orszag, S., *Advanced Mathematical Methods for Scientists and Engineers*, McGraw-Hill (1978). *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to 'Stokes' lines as 'anti-Stokes' lines, and vice versa. The course will use Stokes' convention.*
2. Hinch, E.J., *Perturbation Methods*, Cambridge University Press (1991). *This is the book of the course; some view it as somewhat terse.*
3. Van Dyke, M.D., *Perturbation Methods in Fluid Mechanics*, Parabolic Press, Stanford (1975). *This is the original book on perturbation methods; somewhat dated, but still a useful read.*

Reading to Complement Course Material

1. Berry, M.V., *Waves near Stokes lines*, Proc. R. Soc. Lond. A, **427**, 265–280 (1990).
2. Boyd, J.P., *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, **56**, 1-98 (1999). Also available at

   `http://hdl.handle.net/2027.42/41670` and
   `http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf`.

3. Kevorkian, J. & Cole, J.D., *Perturbation Methods in Applied Mathematics*, Springer (1981).

Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.

Non-Newtonian Fluid Mechanics (M16)

Eric Lauga

Standard courses in fluid mechanics are concerned with the dynamics of Newtonian flows. In the Newtonian limit, viscous stresses only depend linearly on the instantaneous deformation rate of the fluid. However, in many instances relevant to industry as well as natural and physical sciences, a wide variety of fluids display non-Newtonian behaviour. In fact, we are all familiar with these fluids in our daily life. For example, in the kitchen, while water and olive oil are Newtonian, mayonnaise and ketchup are non-Newtonian fluids. Similarly, in the bathroom, toothpaste and shaving cream are materials which can be made to flow like liquids but also share many properties with elastic solids. In this course, we give an introduction to the mathematical modelling of Non-Newtonian fluids.

After introducing the experimental phenomenology of non-Newtonian flows, we will detail the mathematical modelling approach to tackle (i) Generalised Newtonian fluids with instantaneous but nonlinear responses, (ii) linear viscoelastic fluids with memory of their deformation and (iii) nonlinear viscoelastic fluids displaying normal stress differences and resistance to extension. We will then highlight how non-Newtonian constitutive relations can give rise to novel flow instabilities. We will conclude by considering yield stress (visco-plastic) fluids which can only deform if stresses exceed critical values. Throughout the course, mathematical modelling will be motivated and compared with experiments. At the end of the course, students will be equipped with the necessary skills to carry out independent research in complex fluids and rheology relevant to a wide range of industrial and scientific problems.

Pre-requisites

Undergraduate fluid dynamics, vector calculus and mathematical methods.

Literature

1. National Committee for Fluid Mechanics Film on "Rheological Behavior of Fluids" at: `http://web.mit.edu/hml/nc`
2. Morrison (2001) Understanding Rheology, Oxford University Press.
3. Bird, Curtiss, Armstrong, and Hassager (1987) Dynamics of Polymeric Liquids, Vol. 1: Fluid Mechanics, 2nd ed, Wiley.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Fluid Dynamics of Climate (L24)

P.H. Haynes & J.R. Taylor

Understanding the Earth's climate and predicting its future evolution is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the climate system, helping to make the planet hospitable for life. The dynamical complexity of this fluid motion and the wide range of space and time scales involved is one of the most difficult aspects of climate prediction.

This course, focusing on the large-scale behaviour of stratified and rotating flows, provides an introduction to the fluid dynamics necessary to build mathematical models of the climate system. The course begins by considering flows which evolve on a timescale which is long compared with a day, where the Earth's rotation plays an important role. The rotation is felt through the Coriolis force (a fictitious force arising from use of a frame of reference rotating with the Earth) which causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics, particularly in combination with stable density stratification. Canonical models are introduced and studied to illustrate phenomena such as adjustment to a state of geostrophic balance, where Coriolis force balances pressure gradient, new wave modes that can communicate dynamical information on both regional and global scales, and new hydrodynamic instabilities that lead to atmospheric weather systems and ocean eddies.

The course then moves on apply these basic ideas to important aspects of the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the structure and hence the effects of eddies and weather systems, the dynamics of ocean gyres and boundary currents like the Gulf Stream, the dynamics of the meridional (north/south) circulation in the ocean and atmosphere and the associated transport of heat and of chemical and biological tracers and special dynamics of tropical regions which give rise to phenomena such as El Nino.

Desirable Previous Knowledge

Undergraduate fluid dynamics

Reading to complement course material

1. Vallis, G.K. Atmospheric and Oceanic Fluid Dynamics (2nd edition). Cambridge University Press. (2017).
2. Gill, A.E., Atmosphere-Ocean Dynamics. Academic Press (1982).
3. Marshall, J. and R.A. Plumb. Atmosphere, Ocean, and Climate Dynamics. Academic Press. (2008).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Theoretical Physics of Soft Condensed Matter (L24)

Mike Cates, Ronojoy Adhikari, Rob Jack

Soft Condensed Matter refers to liquid crystals, emulsions, molten polymers and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, generally with noise so that the equations of motion are stochastic PDEs. The form of these equations is helpfully constrained by the requirement that the Boltzmann distribution is regained in the steady state (when this indeed holds, i.e. for systems in contact with a heat bath but not subject to forcing). Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a 'smectic liquid crystal' (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field (and several steps short of the renormalization group, whose application to classical field theories is discussed in other courses but not this one).

Important models of soft matter include diffusive ϕ^4 field theory ('Model B'), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates 'Model H', a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth. We will also explore the continuum theory of nematic liquid crystals, which spontaneously break rotational but not translational symmetry, focusing on topological defects and their associated mathematical structure such as homotopy classes.

A section of the course will present the mechanical equations for low-dimensional soft materials informed by concepts of topology and differential geometry. We will first identify kinematic variables suitable for the description of one-dimensional materials like filaments and two-dimensional materials like membranes, and then consider dynamical conservation laws, emphasising their topological character. We will move on to material-specific relations that close the conservation laws, emphasising their geometric character. These general principles will be illustrated by examples of specific one- and two-dimensional materials.

Finally, the course will cover some recent extensions of the same general approach to systems whose microscopic dynamics does not have time-reversal symmetry, such as self-propelled colloidal swimmers. We will discuss how the absence of time-reversal symmetry leads to qualitative changes in dynamical behaviour, both for averaged quantities and for fluctuations. This part of the course will describe some general results, particularly fluctuation theorems, and their consequences for the observation of rare events. The implications of these results in specific soft-matter systems may also be discussed.

Note on lectures

This course has evolved from a 16 lecture course of the same title given for several years by Prof. Cates alone. This is now extended by two sections, approximately corresponding to the last two paragraphs above, lectured by Dr Adhikari and Dr Jack respectively. The format will be roughly 16+4+4 but this may be adjusted as the course proceeds.

Pre-requisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. This course complements in part the following Michaelmas Term courses although none are prerequisites: Statistical Field Theory; Active Biological Fluids; Slow Viscous Flow; Quantum Field Theory.

In previous years the audience has included a mix of students whose main specialism is either fluid dynamics or field theory. People with these differing backgrounds may find different parts of the course easier or harder, but the intention is to create a roughly level playing field.

Preliminary Reading

1. D. Tong *Lectures on Statistical Physics*

   `http://www.damtp.cam.ac.uk/user/tong/statphys.html`

   Before embarking on this course you do need to understand the equation $F = -k_B T \ln Z$ and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Familiarity with the Landau theory of phase transitions is desirable. We will not need much abstract thermodynamics (e.g. Maxwell relations) but you do need to know the zeroth, first and second laws. The above lecture notes are an excellent resource for revising and reviewing the key material.

2. M. E. Cates and E. Tjhung *Theories of binary fluid mixtures: from phase-separation kinetics to active emulsions.* J. Fluid Mech. (2018), **836**, pp1-66.

   `https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/theories-of-binary-fluid-mixtures-from-phaseseparation-kinetics-to-active-emulsions/5BD133CB20D89F47E724D77C296FEF80/share/106fd30f307db12134745de39fd568fbbaa3f9d2`

   This JFM perspectives article has significant overlap (perhaps 40%) with the course but takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics when needed. It gives a good flavour of the types of problem we will address and some of the methodologies involved. However we will not have time to cover much of the material it contains on active systems.

Literature

So far there are no books that treat this material at the right level. But it may be worth looking at:

1. P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics.* Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, topological defects all feature in the description of condensed matter systems at $\hbar = 0$. More for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will also be a two-hour revision class in the Easter Term.

Aeroacoustics (L16)

Nigel Peake

An understanding of noise generation and propagation is important in a wide range of areas, from aero-engine and wind turbine design to musical acoustics and noninvasive surgery, and in this course we describe the analytical techniques which form the common framework on which all these problems can be treated. Typically, sound waves possess only a small amplitude and can be modelled very satisfactorily in many situations using linear theory. Nonlinear effects do become important, however, particularly when the sound propagates over large distances.

The content of the course, with approximate number of lectures in brackets, will be as follows:

- the generation of sound by unsteady fluid motion, and by the motion of solid bodies, with particular reference to the aeroacoustics of high-speed jets and propellers [4];
- the interaction between sound waves and solid boundaries, including the theory of diffraction and use of the Wiener-Hopf technique [6];
- the (linear) propagation of sound waves through nonuniform media [2].
- nonlinear acoustics, and the solution of relevant model equations using a blend of exact and asymptotic techniques. Shock formation and propagation [4].

Pre-requisites

Some familiarity with complex variable theory and elementary fluid mechanics is important, but no other Part III courses are prerequisite. The emphasis throughout will be on obtaining physical insight from exact and asymptotic mathematical solutions.

Literature

1. D.G. Crighton et al. 1992 *Modern Methods in Analytical Acoustics*, Springer-Verlag.
2. M.E. Goldstein 1976 *Aeroacoustics.* McGraw-Hill.
3. D.S. Jones 1986 Acoustic and Electromagnetic Waves. Oxford.
4. B. Noble 1958 Methods Based on the Wiener-Hopf Technique. Chelsea.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Hydrodynamic Stability (L16)

Rich Kerswell

Developing an understanding of how "small" perturbations grow, saturate and modify fluid flows is central to addressing many challenges of interest in fluid mechanics. Furthermore, many applied mathematical tools of much broader relevance have been developed to solve hydrodynamic stability problems, and hydrodynamic stability theory remains an exceptionally active area of research, with several exciting new developments being reported over the last few years.

In this course, an overview of some of these recent developments will be presented. After a brief introduction to the general concepts of flow instability, presenting a range of examples, the major content of this course will be focussed on the broad class of flow instabilities where velocity "shear" and fluid inertia play key dynamical roles. Such flows, typically characterised by sufficiently"high" Reynolds number Ud/ν, where U and d are characteristic velocity and length scales of the flow, and ν is the kinematic viscosity of the fluid, are central to modelling flows in the environment and industry.

A hierarchy of mathematical approaches will be discussed to address a range of stability problems, from more classical concepts of normal mode growth on laminar parallel shear flows and their subsequent weakly nonlinear behaviour, to transient but significant growth of infinitesimal perturbations and finite amplitude instabilities.

Pre-requisites

Undergraduate fluid mechanics, linear algebra, complex analysis and asymptotic methods.

Literature

1. F. Charru *Hydrodynamic Instabilities* CUP 2011.
2. P. G. Drazin & W. H. Reid *Hydrodynamic Stability* 2nd edition. CUP 2004.
3. P. J. Schmid & D. S. Henningson, *Stability and transition in shear flows.* Springer, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be an hour and a half revision class in the Easter Term.

Demonstrations in Fluid Mechanics (L8)

Non-Examinable (Part III Level)

Dr. J.A. Neufeld

While the equations governing most fluid flows are well known, they are often very difficult to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- sedimentation and resuspension;
- avalanches and granular flows;
- porous media and carbon sequestration;
- fluid flow and elastic deformation;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low Reynolds' number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account.

Pre-requisites

Undergraduate Fluid Dynamics.

Literature

1. M. Van Dyke. An Album of Fluid Motion. Parabolic Press.
2. G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.
3. M. Samimy, K. Breuer, P. Steen, & L. G. Leal. A Gallery of Fluid Motion. CUP.