Forcing and the Continuum Hypothesis (L24)

Dr R. Matthews

An early result in the history of set theory is Cantor's famous theorem that the real numbers cannot be put into one-to-one correspondence with the natural numbers. This naturally leads to the question of whether these are the only two possible sizes for infinite sets of reals, with the Continuum Hypothesis being the assertion that this is indeed the case.

(CH) Every infinite set of reals is either equinumerous with the set of natural numbers or equinumerous with the set of all real numbers. Equivalently, $2^{\aleph_0} = \aleph_1$.

This question was considered so important to the foundations of mathematics that David Hilbert posed it as the first of his twenty-three problems for the 20th Century in 1900. In 1938, Kurt Gödel invented the *Constructible Universe* and used this to show that CH cannot be disproved in ZFC. Then, in 1963, Paul Cohen invented the *method of forcing* to show that CH cannot be proved in ZFC. From these two results it follows that CH is independent from the standard axioms of set theory.

In this course we shall study the concept of independence results in set theory. The course will cover a selection of topics including:

- **Models of Set Theory.** Transitive models. Absoluteness. Reflection Principles.
- **Inner Models.** Definability. The Constructible Universe. Condensation. Gödel's proof of the consistency of CH.
- **Forcing.** Generic extensions. The Forcing Theorems. Adding reals. Cohen's proof of the consistency of $\neg$CH.

Prerequisites

The Part II course *Logic & Set Theory* or an equivalent course is essential.

The Part III course *Model Theory & Non-Classical Logic* (Michaelmas 2023) is not directly related to this course but will contain various mathematical topics that are useful in understanding the material in this course.

This course will run concurrently with the Part III course *Large Cardinals* (Lent 2024). The two courses are self-contained and can be taken separately, but given the close relationship between the underlying topics, it is strongly suggested that students consider taking both courses.

Literature

1. Kenneth Kunen *Set Theory An Introduction to Independence Proofs.* Elsevier, 1980.
2. Thomas Jech *Set Theory.* The Third Millenium Edition, revised and expanded, Springer, 2003.
3. Jon Barwise *Admissible sets and structures.* Cambridge University Press, 2017.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.