

# Perturbation Methods (M16)

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This course will deal with the asymptotic solution to problems in applied mathematics in general when some parameter or coordinate assumes large or small values. Many problems of physical interest are covered by such asymptotic limits; indeed, it plays a crucial role in the field of mathematical modelling. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases can provide accurate predictions when the parameter or coordinate has only moderately large or small values.

Some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, but most of the lectures will be devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions, and so-called ‘exponential asymptotics’), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the ‘divide & conquer’ strategy, Laplace’s method, stationary phase and steepest descents. This will be followed by a discussion of Stokes lines and an introduction to ‘asymptotics beyond all orders’ in which exponentially small corrections are extracted from the tails of asymptotic series. The advantage of uniformly valid expansions for comparison with experiment and numerical solutions will be covered. [7]
- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. The method was first developed in fluid mechanics, where it is referred to as boundary-layer theory, but it has since been greatly extended and applied to many fields. Further examples will be given of asymptotics beyond all orders. This section will include a brief introduction to the summation of [divergent] series, e.g. covering Cesàro, Euler and Borel sums, Padé approximants, continued fractions, Shanks’ transformations, Richardson extrapolation, and Domb-Sykes plots. [6]
- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the ‘WKB[JLG]’ method can be viewed as a special case.) It is a systematic method, capable of extension in many ways, and includes such ideas as those of ‘averaging’ and ‘time scale distortion’ in a natural way. A number of applications will be studied, as time allows, potentially including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium.) [3]

## Prerequisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no specific knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward ordinary & partial differential equations and evaluate simple integrals.

## Literature

### *Relevant Textbooks*

1. C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, 1978. This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to ‘Stokes’ lines as ‘anti-Stokes’ lines, and vice versa. The course will use Stokes’ convention.
2. E.J. Hinch. *Perturbation Methods*. Cambridge University Press, 1991. The lecture course was originally based on this book; some may view it as somewhat concise.
3. M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics*. Parabolic Press, Stanford, 1975. This is the original book on perturbation methods; somewhat dated, but still an excellent read.

### *Reading to Complement Course Material*

1. M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. A, **427**, 265–280 (1990).
2. J.P. Boyd. *The Devil’s invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, 56, 1-98 (1999). Also available at  
<http://hdl.handle.net/2027.42/41670> and  
<http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf>.
3. J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer, 1981.

## Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.