Lie algebras and their representations (M24)

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This course is an introduction to the theory of semisimple Lie algebras usually over an algebraically closed field of characteristic 0, with an emphasis on their representations.

Lie algebras are 'infinitesimal symmetries', which one might think of as linearisations of Lie groups. They are ubiquitous in many branches of mathematics: in topology, in arithmetic and algebraic geometry and in theoretical physics (string theory, exactly solvable models in statistical mechanics,...). One reason for their importance is that the finite-dimensional complex representations of the simple Lie algebras are exactly the same as those of the corresponding groups. So instead of needing to study the topology and geometry of the simple Lie groups, or the algebraic geometry of the simple algebraic groups, we can use only linear algebra and still manage to describe completely these representations.

While our algebras play a role in many diverse areas of mathematics, the material in this course is inherently attractive in its own right, combining as it does a certain amount of depth and a satisfying degree of completeness in its basic results. The main purpose of the course is to gain familiarity with these objects so that you can work with them in the context of your own interests. Because of this, we will spend time on examples. However, because of the amount of material we will need to cover to give you a solid background, we will sometimes skip or only sketch proofs that can be found in the references below, especially references [8] and [3] and [4].

Of the following I will cover (most of) the first four topics and, depending on time, may sketch the fifth.

- Definitions, motivations, and basic structure theory.
- Root systems, Weyl groups, the finite simple Lie algebras.
- Theorems of Engel and Lie (on solvability), Cartan (on Killing forms), Weyl (on complete reducibility).
- Classification of finite-dimensional representations, Verma modules, the combinatorics of characters.
- (if time) Crystal bases.

Prerequisites

Linear algebra up to the Jordan canonical form; basic abstract algebra (groups, rings, modules and fields); the rudiments of representation theory (Part II or equivalent course, see [4]). A little multilinear algebra and familiarity with Lie group theory (the circle group S^1, SU(2), SO(3), etc.) are nice to have but not dealbreakers. Material in the Part III Commutative Algebra course (tensor products of vector spaces) is also tangentially useful.

Literature

There is no shortage of books on the theory of Lie algebras: the treatment in [8] cannot be bettered.

1. Carter, R. W., *Simple groups of Lie type.* Wiley Classics Library, John Wiley & Sons, Inc., New York, 1989 reprint.

2. Carter, R. W., *Lie algebras of finite and affine type.* Cambridge Studies in Advanced Mathematics, 96, Cambridge University Press, Cambridge, 2005.

3. Erdmann, K. and Wildon, M., *Introduction to Lie algebras.* Springer, 2006

4. Fulton, W. and Harris, J.. *Representation theory, a first course.* Graduate Texts in Mathematics, 129, Springer-Verlag, New York, 1991.

5. Hall, B. C., *Lie groups, Lie algebras and representations.* Springer GTM 222, Springer-Verlag 2004.

6. Henderson, A., *Representations of Lie algebras: an introduction through* gl_n (Australian Mathematical Society Lecture Series, No 22), 2012.

7. Humphreys, J. E., *Linear Algebraic Groups,* vol 21 of GTM, Springer, New York, 1975.

8. Humphreys, J. E., *Introduction to Lie algebras and representation theory,* vol. 9 of GTM Springer, New York, 1978.

9. Jacobson, N., *Lie algebras.* Dover reprint, 2003.

10. Kac, V., *Infinite dimensional Lie algebras* (3rd edn), Cambridge University Press, Cambridge, 1990.

11. Serre, J.-P., *Complex semisimple Lie algebras.* transl. G.A. Jones. Springer monographs in mathematics, Springer-Verlag, New York, 2001.

12. Kostant, B., *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group,* American Journal of Mathematics, Vol. 81, No. 4 (Oct., 1959), pp. 973-1032.

13. Slodowy, P., *Simple singularities and simple algebraic groups,* Springer Lecture Notes 815, Springer 1980.

14. Stubhaug A., *The mathematician Sophus Lie: it was the audacity of my thinking.* Springer, Berlin, 2002.

Additional support

Exercises (and hints) will be provided in lectures and four sheets of examples will be given. There will be a one-hour revision class in the Easter Term 2024.