Infinite-dimensional Spectral Computations (L16)

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The spectral theory of operators is a fundamental area of mathematics with profound implications across diverse fields. It plays a pivotal role in quantum mechanics, where the spectrum of an operator corresponds to potential measurement outcomes. Furthermore, in areas such as signal processing, control theory, data-driven discovery in dynamical systems, and partial differential equations (PDEs), spectral analysis methods are essential for solving complex problems.

This course aims to offer a comprehensive understanding of operator theory on separable Hilbert spaces, with a particular emphasis on the foundations of computing spectra of operators. The course is ideally suited for graduate students in mathematics or physics who have a keen interest in advanced functional analysis concepts. A significant aspect of the course involves learning how to manage the error and ensure convergence when reducing an infinite-dimensional problem to a series of finite-dimensional ones.

The course will cover a selection of topics including:

- Frameworks for establishing the foundations of computation and classifying problem difficulty. For example, we will explore the concept of the Solvability Complexity Index and discuss developing canonical embedding problems for problem classification.
- Computing spectra of operators via computing the resolvent operator and addressing the fundamental issue of spectral pollution (spurious eigenvalues).
- Spectral measures and continuous spectra of self-adjoint operators, convolution methods for their computation, and contour projection methods.
- Contemporary applications in data-driven study of so-called Koopman operators and transfer operators for nonlinear dynamical systems.
- Time permitting, we will discuss the broader implications of these techniques beyond pure spectral theory, including areas such as computer-assisted proofs and PDEs.

Prerequisites

This course presumes basic knowledge in linear algebra, analysis, and linear analysis. Other useful Part III courses include *Functional Analysis* and *Numerical Solution of Differential Equations*. However, no formal background in numerical analysis courses is required.

Literature

1. W. Rudin, *Functional analysis* 2nd edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.
2. M. Reed, B. Simon, *Methods of modern mathematical physics. I*, Academic Press, Inc., New York, 1980.
3. J.B. Conway, *A course in functional analysi*s, Vol. 96. Springer, 2019.
4. E.B. Davies, *Linear operators and their spectra*, Vol. 106. Cambridge University Press, 2007.

5. M. Colbrook, A. Horning, A. Townsend, *Computing spectral measures of self-adjoint operators*, SIAM review 63.3 (2021): 489-524.

6. M. Colbrook, *On the computation of geometric features of spectra of linear operators on Hilbert spaces*, Foundations of Computational Mathematics (2022): 1-82.

Additional support

Three examples sheets will be provided, accompanied by three associated examples classes. A one-hour revision class will also be offered in the Easter Term.