Noisy Mechanics (L24)

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When modelling a mechanical system of any complexity, one soon abandons the idea of solving the equations of motion in complete detail. For example, in a fluid, we replace Newton's laws for the molecules with the Navier Stokes equation for the fluid velocity, which approximates the mean molecular velocity in each neighbourhood. Coarse-graining further, for say a solid particle pushed through a viscous fluid, one can replace the main effect of the surrounding medium with a frictional drag force proportional to the particle's velocity.

The resulting coarse-grained descriptions can be very powerful but are inherently incomplete. There are underlying microscopic degrees of freedom that remain unresolved, whose effects cannot all be captured at deterministic level. Fortunately, in very many cases, their influence can be accurately included by adding noise to the more macroscopic description.

Here 'noise' means a stochastic term or terms in the equations of motion. For instance, a solid colloidal particle in a fluid undergoes Brownian motion (diffusion) and this can be accurately described either by adding noise to the equation for the particle alongside the frictional drag (giving a noisy ODE), or, more fundamentally, by adding noise to the Navier Stokes equation itself (giving a noisy PDE).

Similar remarks apply to models used in other scientific areas ranging through population dynamics, climate science, materials science, ecology, and human behaviour. (We exclude quantum physics from this course, although the same can apply there.) The noise can either be internally generated from unobserved degrees of freedom, or caused by external perturbations that are not monitored in detail. For systems close to thermal equilibrium we can say a lot about the statistics of the noise, because it must allow the Boltzmann distribution to emerge as the stationary state. More generally though, the noise terms need to be derived from explicit coarse-graining or identified phenomenologically.

In some cases, the main effect of noise is to cause small fluctuations about whatever state the dynamics would reach without it. These fluctuations can be calculated – and measured in many cases – and we will spend some time understanding them. Often however, the noise causes quite new things to happen, and we will study these as well. Some of them are unintuitive. Examples include formation of an oscillatory state or pattern where none previously existed; a slow drift towards extinction of a population that would otherwise be stable; conversion of a continuous dependence of the stationary state upon parameters into a discontinuous one; and rare but dramatic transitions between different basins of attraction in systems that, without noise, would be trapped forever in the same one.

This course on Noisy Mechanics is intended to give a pragmatic introduction to noise effects, focusing on how to predict the emergent behaviour in applications drawn from the various fields mentioned above, both near and far from thermal equilibrium, and with emphasis on phase separation and emergence of spatial patterns.

Unlike most courses and textbooks on stochastic differential equations, we will not present (let alone assume) a detailed and rigorous foundation of the underlying probabilistic concepts. Instead we will introduce the tools we need as we go along and resolve ambiguities as and when we encounter them.

This course was new 2025/6 and is still under development. The synopsis below remains provisional; we do not guarantee coverage of all the listed topics.

Part 1 - Introduction to noisy mechanics using Brownian motion of a particle. Langevin approach. Additive vs multiplicative noise. Thermal noise, Fluctuation-Dissipation Theorem, detailed balance principle and Boltzmann distribution. Ornstein-Uhlenbeck process, calculation of correlations. Models with several degrees of freedom (coupled stochastic ODEs). Examples of noise-induced phenomena: El Nino southern oscillation, cyclic population dynamics of lizards. The Fokker-Planck approach: derivation and use. First look at barrier crossing. Stochastic resonance and the statistics of ice ages. Potential conditions (absence of stationary currents) in higher dimensional systems. Statistics of Trajectories. Use of stochastic action to address barrier crossing, entropy production, and detailed balance condition.

Part 2 - Noisy continuum mechanics (stochastic PDEs). Definition of functional Langevin equations via Fourier series. Linearized dynamics: density fluctuations in population biology and diffusive particle gas. Hydrodynamic modes in thermal fluids. Noisy Navier-Stokes equation. Nonlinear theory – the Swift Hohenberg equation. Emergence of detailed balance / gradient flow structure; solution via free-energy minimization. Mean field theory: second order phase transition, Self-consistent treatment of nonlinearities. Fluctuation-induced first order transition. Non-conserved (Model A) and conserved (Model B) dynamics of a scalar field: fluctuations at rest and under shear. Barrier crossing in infinite dimensions: Classical Nucleation Theory for Models A and B.

Part 3 - Noise-induced transitions between basins of attraction: an introduction to large deviations using Bernoulli trials. Cramer's theorem for large deviations. Wiener process and its large deviations. Ito process and its large deviations. The Freidlin-Wentzell action. The small noise limit of the Fokker-Planck equation and singular perturbation theory. General expressions for escape rate, pre-factors and the quasipotential. Applications to gradient and non-gradient problems, including low-dimensional models of fluids, weather, and epidemics.

Prerequisites

The course assumes you already know how to handle standard (rather than stochastic) ODEs and PDEs. Otherwise, it aims to be self contained and accessible to people from many different backgrounds. Applications will be drawn from fluid mechanics, statistical field theory, and population biology; while familiarity with one or more of these is helpful, it is never essential.

Literature

Most textbooks on stochastic dynamics lay great emphasis on rigorous foundations which can obscure the pragmatic approach taken in this course. However, with effort, useful material can be excavated from the following: C. Gardiner, *Stochastic Methods* (Springer); W. Coffey and Y. Kalmykov, The Langevin Equation (World Scientific); H. Risken, *The Fokker-Planck Equation* (Springer); N. van Kampen *Stochastic Processes in Physics and Chemistry* (North Holland). The same applies to R. Kubo, M. Toda, N. Hashitzume, *Statistical Physics II, Nonequilibrium Statistical Physics* (Springer); P. Chaikin and T. Lubensky, *Principles of Condensed Matter Physics* (Cambridge). These two physics texts are less formal but assume a higher level of background knowledge in statistical mechanics.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.