Noisy Mechanics (L24)

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When modelling a mechanical system of any complexity, one soon abandons the idea of solving the equations of motion in complete detail. For example, in a fluid, we replace Newton's laws for the molecules with the Navier Stokes equation for the fluid velocity, which approximates the mean molecular velocity in each neighbourhood. Coarse-graining further, for a larger-than-molecular solid particle pushed through a viscous fluid, one can replace the main effect of the surrounding medium with a frictional drag force proportional to the particle's velocity.

The resulting coarse-grained descriptions can be very powerful but are inherently incomplete. There are underlying microscopic degrees of freedom that remain unresolved, whose effects cannot all be captured at deterministic level. Fortunately, in very many cases, their influence can be accurately included by adding noise to the more macroscopic description.

Here 'noise' means a stochastic term or terms in the equations of motion. For instance, a solid colloidal particle in a fluid undergoes Brownian motion (diffusion) and this can be accurately described either by adding noise to the equation for the particle alongside the frictional drag (giving a noisy ODE), or, more fundamentally, by adding noise to the Navier Stokes equation itself (giving a noisy PDE).

Similar remarks apply to models used in other scientific areas ranging through population dynamics, climate science, materials science, ecology, and human behaviour. (We exclude quantum physics from this course, although the same can apply there.) The noise can either be internally generated from unobserved degrees of freedom, or caused by external perturbations that are likewise not monitored in detail. For systems close to thermal equilibrium we can say a lot about the statistics of the noise, because it must allow the Boltzmann distribution to emerge as the stationary state. More generally though, the noise terms need to be derived from explicit coarse-graining or identified phenomenologically.

In some cases, the main effect of noise is to cause small fluctuations about whatever state the dynamics would reach without it. These fluctuations can be calculated – and measured in many cases – and we will spend some time understanding them. Often however, the noise causes quite new things to happen, and we will study these as well. Some of them are unintuitive. Examples include formation of an oscillatory state or pattern where none previously existed; a slow drift towards extinction of a population that would otherwise be stable; conversion of a continuous dependence of the stationary state upon parameters into a discontinuous one; and rare but dramatic transitions between different basins of attraction in systems that, without noise, would be trapped forever in the same one. For instance we will study the condensation of a supersaturated vapour into a liquid by nucleation of liquid droplets that must exceed a critical mass before they can grow further. (This happens every time it rains.)

This course on Noisy Mechanics is intended to give a pragmatic introduction to noise effects, focusing on how to predict the emergent behaviour in applications drawn from the various fields mentioned above, both near and far from thermal equilibrium, and with emphasis on phase separation and emergence of spatial patterns.

Unlike most courses and textbooks on stochastic differential equations, we will not present (let alone assume) a detailed and rigorous foundation of the underlying probabilistic concepts. Instead we will introduce the tools we need as we go along and resolve ambiguities as and when we encounter them.

This is a new course for 2025/6. The synopsis below is therefore provisional; we do not guarantee coverage of all the listed topics.

Part 1 - Introduction to noisy mechanics using Brownian motion of a particle. Langevin and Fokker-Planck descriptions. Discussion of additive and multiplicative noise. Thermal noise, Fluctuation-Dissipation Theorem, detailed balance principle and Boltzmann distribution. Ornstein-Uhlenbeck process, calculation of correlations. A first look at barrier crossing. The case of a nonconservative force. Models with several degrees of freedom (coupled stochastic ODEs) including inertial Brownian particles and active particle models but also models from population dynamics etc.. Survey of various emergent phenomena: stationary fluxes, stochastic resonance, stability replaced by extinction, et al..

Part 2 - Noisy continuum mechanics. Definition of functional Langevin equations and path weights via Fourier series. Thermal fluctuations in a fluid of interacting Brownian particles: noise terms found via detailed balance. Momentum-conserving fluids: the noisy Navier Stokes equation. Velocity fluctuations and their correlation functions. Thermal fluctuations and phase separation in binary fluids (Model B). Gradient flow on the free energy landscape, phase transitions, noise effects on these. Emergence of spatial structure and spatiotemporal patterning in continuum models from various fields of study, including cases without detailed balance. Outline (using Model B) of classical nucleation theory via free energy landscape.

Part 3 - Noise-induced transitions between basins of attraction. Motivation of the general question by analysis of the barrier crossing of an inertial particle. The limit of small noise and the relation to singular perturbation theory. Boundary layer analysis of the Fokker-Planck equation using the JWKB approximation. Instantons and the Freidlin-Wentzell action functional. General expressions for escape rate, pre-factors and the quasipotential. Applications to gradient and non-gradient problems, including low-dimensional models of fluids, weather, epidemics. Numerical approaches to computing large deviations via Freidlin-Wentzell action functionals.

Prerequisites

The course assumes you already know how to handle standard (rather than stochastic) ODEs and PDEs. Otherwise, it aims to be self contained and accessible to people from many different backgrounds. However, some familiarity with basic aspects of fluid mechanics and statistical thermodynamics will certainly be helpful for a number of the applications areas discussed.

Literature

Most textbooks on stochastic dynamics lay great emphasis on rigorous foundations which can obscure the pragmatic approach taken in this course. However, with effort, useful material can be excavated from the following: C. Gardiner, *Stochastic Methods* (Springer); W. Coffey and Y. Kalmykov, The Langevin Equation (World Scientific); H. Risken, *The Fokker-Planck Equation* (Springer); N. van Kampen *Stochastic Processes in Physics and Chemistry* (North Holland). The same applies to R. Kubo, M. Toda, N. Hashitzume, *Statistical Physics II, Nonequilibrium Statistical Physics* (Springer); P. Chaikin and T. Lubensky, *Principles of Condensed Matter Physics* (Cambridge). These two physics texts are less formal but assume a higher level of background knowledge in statistical mechanics.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.