# Random structures in finite-dimensional space (L24)

**Wendelin Werner**

As opposed to the cases where randomness is naturally structured/created and read/collected along a one-dimensional time-line (such as for Markov chains, martingales, or stochastic calculus), the present course will deal with random structures defined in $d$-dimensional space (discrete, such as $\mathbf{Z}^d$ or other lattices, or in the continuum such as $\mathbf{R}^d$) for $d \geq 2$. In the discrete cases, these will typically be random functions defined on subsets of $\mathbf{Z}^d$, with values in $\{0, 1\}$ or $\mathbf{R}$.

Examples include percolation, the Ising model, uniform spanning trees and forests, the Gaussian Free Field and others. Most of these models are motivated from physics. They are all simple to define, yet they often have very rich behaviour, and a number of fundamental and easy-to-state questions remain open.

This course will present a selection of ideas and results on this topic. More specifically, we first plan to focus on discrete models defined on lattices and discuss:

- The phase transition for percolation and for the Ising model.

- The asymptotic conformal invariance of critical percolation on the triangular planar lattice ("Smirnov's Theorem").

- Wilson's algorithm to construct uniform spanning trees.

- Some properties of the discrete Gaussian Free Field.

In the final part of the course, we will discuss some features of random structures defined in the continuum:

- The definition of the Gaussian Free Field as a random generalized function and some of its properties.

- How conformal invariance enables to characterize and construct natural random planar fractal structures.

### Prerequisites

This course assumes familiarity with probability, measure theory, and analysis at Part II level, as well as the content of the Advanced Probability course. It is probably useful but not absolutely necessary to follow the stochastic calculus course in parallel.

### Literature

Lecture notes will be provided. Some additional helpful references are:


1. H. Duminil-Copin, *Introduction to percolation theory.* (available here: `https://www.unige.ch/~duminil/publi/2017percolation.pdf`)

2. G. Grimmett, *Probability on Graphs*, Cambridge University Press (available here: `http://www.statslab.cam.ac.uk/~grg/books/pgs.html`)

3. W. Werner and E. Powell, Lecture notes on the Gaussian Free Field, Cours Spécialisés S.M.F. (available here: `https://arxiv.org/abs/2004.04720`)

**Additional support**

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term..