The Ising model (M16)

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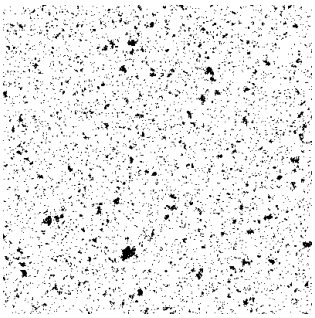
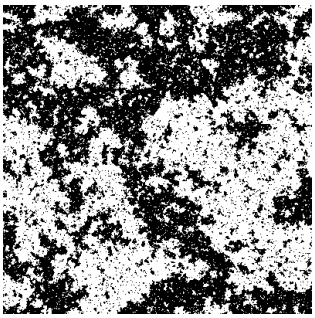
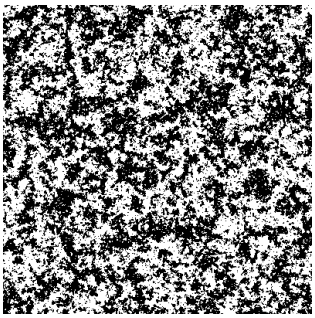
The Ising model is one of the simplest models of phase transitions. It has inspired many developments in statistical mechanics as well as probability. The model describes a system that is very correlated at low temperatures and at high temperature has exponentially decaying correlations. There is exactly one crossover point: the *critical* temperature.



(a) A typical low-temperature configuration.



(b) A typical configuration at the critical temperature.



(c) A typical high-temperature configuration.

Figure 1: Typical configurations of the Ising model on a 1000×1000 square grid for different temperatures. Properties of the phase diagram will be established rigorously.

The Ising model is a sandbox for understanding more complicated models in probability as well as statistical physics. Mastering it gives one a language that is useful in many different contexts. Despite the simplicity of the model many easily stated, exciting open problems remain.

The course will include the following content.

- The Curie-Weiss model: Ising model on the complete graph, where exact calculations are possible.
- Graphical representations (random-cluster and random current models): The Ising model can be understood through percolation (in these correlated random graphs).
- Phase diagram on the hypercubic lattice $\mathbb{Z}^d$: Existence of a phase transition, sharpness and continuity.
- Glimpse of conformal invariance: On the square lattice $\mathbb{Z}^2$ at the critical point the system acquires additional symmetries, which are subject to intensive research.

Prerequisites

None, but an elementary course in either probability or statistical physics could be useful.

Literature

1. S. Friedli and Y. Velenik, *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*, Cambridge University Press, 2017. Chapter 2: The Curie–Weiss Model. https://www.unige.ch/math/folks/velenik/smbook/
2. H. Duminil-Copin, *Lectures on the Ising and Potts models on the hypercubic lattice*. https://arxiv.org/abs/1707.00520