Diophantine Analysis (L24)

Professor P. P. Varjú

This course will discuss two classical methods in Diophantine analysis. The first one can be traced back to Thue, who proved the following result in Diophantine approximation. Let α be an algebraic number of degree d. Then for any

$$\kappa > \frac{d}{2} + 1,$$

there is a constant c such that

$$\left|\alpha - \frac{r}{s}\right| > \frac{c}{s^\kappa} \qquad (\dagger)$$

for all $r, s \in \mathbf{Z}$. This result is a significant improvement of Liouville's bound, in which the exponent is $\kappa = d$. Thue used his result to prove that certain Diophantine equations have only finitely many solutions. Thue's result has been subsequently improved by Siegel, Dyson and finally Roth, who achieved the optimal exponent $\kappa = 2+\varepsilon$. Schmidt generalized Roth's result to the setting of a system of inequalities for linear forms, which is known as the subspace theorem.

The second method that will be discussed in the course goes back to the work of Baker. Gelfond and Schneider independently proved that α^β is transcendental whenever $\alpha \neq \{0, 1\}$ is algebraic and β is an irrational algebraic number, which was Hilbert's seventh problem. This can be reformulated as

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0$$

for any non-zero algebraic numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$, provided $\log \alpha_1$ and $\log \alpha_2$ are linearly independent over the rationals. Baker generalized this result to linear forms in arbitrarily many logarithms, and, moreover, he gave lower bounds for the absolute value of such a form. These estimates have been revisited and improved on by many authors.

Both methods have been utilized by many authors for a wide range of applications in number theory and beyond. In the course, we will discuss both methods, and we will sample from their applications.

Some highlight of the course will be the following.

- Proof of $(\dagger)$ for any $\kappa > \sqrt{2d}$.
- Proof of the Gelfond-Scheinder theorem.
- A version of Siegel's theorem on integral points on curves using the subspace theorem.
- Effective improvement of Liouville's theorem using linear forms in logarithms.

Prerequisites

Some knowledge of Galois theory, number fields and complex analysis will be assumed.

Literature

The course will not follow any particular source, but there are many excellent books that discuss some of the course material including the following.

1. A. Baker, *Transcendental number theory.* Cambridge University Press, Cambridge, 1990.
2. E. Bombieri and W. Gubler, *Heights in Diophantine geometry.* Cambridge University Press, Cambridge, 2006.
3. Y. Bugeaud, *Linear forms in logarithms and applications.* European Mathematical Society (EMS), Zürich, 2018.
4. J. W. S. Cassels, *An introduction to Diophantine approximation.* Hafner Publishing Co., New York, 1972.
5. D. Masser, *Auxiliary polynomials in number theory.* Cambridge University Press, Cambridge, 2016.
6. M. Waldschmidt, *Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables.* Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 326. Springer-Verlag, Berlin, 2000.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.