Forcing & the Continuum Hypothesis (L16)

Professor Benedikt Löwe

The method of forcing is one of the most important model constructions in set theory and its versatility is the reason for the plethora of independence results in set theory. It was developed to solve one of the most important foundational problems of the 20th century: determining the cardinality of the set of real numbers. Cantor's *Continuum Hypothesis* asserts that the cardinality of the set of real numbers has the smallest possible value:

> Every infinite set of reals is either equinumerous with the set of natural numbers or equinumerous with the set of all real numbers. Equivalently, $2^{\aleph_0} = \aleph_1$. (CH)

When he presented his list of twenty-three problems for the 20th century at the *International Congress of Mathematicians* in Paris in 1900, David Hilbert listed the the question whether the Continuum Hypothesis is true as the very first problem on his list. It turned out that this question cannot be solved on the basis of ZFC: in 1938, Kurt Gödel showed that CH cannot be disproved in ZFC; in 1963, Paul Cohen invented the method of forcing to show that CH cannot be proved in ZFC.

In this course we shall study the basics of the method of forcing in order to present Cohen's proof. The course will discuss:

1. models of set theory (in particular, transitive models, the Lévy reflection theorem, and the theory of absoluteness);
2. the construction of the generic extension of a countable transitive model (in particular, the syntactic and semantic forcing relation, the forcing theorem, and the generic model theorem);
3. showing relative consistency proofs with the method of forcing;
4. applications to cardinal arithmetic (in particular, preservation results and the consistency of $\neg$CH and related results).

Prerequisites

The Part II course *Logic & Set Theory* or an equivalent course is essential. This course uses some content from the Part III course *Model Theory* (Lent 2024). It is recommended (but not strictly necessary) to take *Model Theory* to supplement the discussion of models of set theory in this course.

Literature

Kenneth Kunen. *Set Theory. An Introduction to Independence Proofs.* Elsevier 1980 [Studies in Logic and the Foundations of Mathematics, Vol. 102].

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.