Logic and Computability (L24)

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This course builds upon the foundations laid out by the Part II courses *Logic and Set Theory* and *Automata and Formal Languages*, introducing a new notion of computability and a non-classical logic, as well as discussing the relationship between these subjects.

In Part II, students meet two general models of computation: register machines and recursive functions. There is, however, another alternative which has certain practical advantages and is at the core of functional programming languages like Haskell: the (untyped) λ-calculus. We will explore its properties and see that a version of this calculus has a strong connection to intuitionistic logical proofs.

Intuitionistic logic departs from classical logic by adopting a new interpretation for the logical connectives and quantifiers that makes it constructive and, therefore, carry computational content. We will discuss its syntax and semantics, the accompanying completeness theorems, and how it compares to classical logic.

Finally, there are interesting points in which logic and computability intersect. Examples include the incompleteness phenomenon and the arithmetic hierarchy (which generalises the characterisation of recursively enumerable sets as Σ_1-definable sets). We will explore some of these ideas and see what computability theory has to say about models of arithmetic.

The intent is to cover the following topics:

Computability

Recursive functions and computability; Recursive and recursively enumerable sets; The untyped λ-calculus; Reduction and the Church-Rosser Theorem; Church numerals; Fixed point combinators and λ-computability.

Logic

Nonstandard models of arithmetic; Natural deduction and intuitionistic logic; Diaconescu's Theorem; Completeness of the Heyting and Kripke semantics; Modalities; Negative translations.

Computability in logic

Craig's Theorem; Gödel numbering and Incompleteness, Tennenbaum's Theorem; The simply typed λ-calculus and the Curry-Howard isomorphism; Weak and strong normalisation for the simply typed λ-calculus.

The course will appeal to those wishing to pursue research in the Foundations of Mathematics, Computer Science (particularly in the semantics of programming languages), and Proof Formalisation. Those interested in topos theory will find the material on intuitionistic logic useful.

Prerequisites

Familiarity with the contents of the Part II courses Logic and Set Theory and Automata and Formal Languages (or equivalent) is essential. Some of the relevant facts and definitions pertaining to computability, however, will be revised at a fast pace.

Literature

A primary source for intuitionistic logic and its relation to the λ-calculus is '*Proofs and Types*' [1]. It doesn't address the Heyting and Kripke semantics, however, and thus has to be complemented in other ways. A categorically minded student will enjoy Ghilardi's '*A First Introduction to the Algebra of Sentences*' [2], but everyone can appreciate Palmgren's notes '*Semantics of intuitionistic propositional logic*' [3] — they were both made available online. This is also the case for the very good and straightforward notes on the untyped λ-calculus by Barendregt and Barendsen '[4]'.

Kaye's '*Models of Peano Arithmetic*' [5] is the go-to reference on anything related to the formal theory of arithmetic.

References

[1] J. Girard, Y. Lafont, P. Taylor. *Proofs and Types*, Cambridge University Press, 1989.

[2] S. Ghilardi. *A First Introduction to the Algebra of Sentences*, Lecture Notes, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, 2000, http://users.mat.unimi.it/users/ghilardi/allegati/dispcesena.pdf.

[3] E. Palmgren. *Semantics of intuitionistic propositional logic*, Lecture Notes, Department of Mathematics, Uppsala University, 2009, http://www2.math.uu.se/~palmgren/tillog/heyting3.pdf.

[4] H. Barendregt, E. Barendsen. *Introduction to Lambda Calculus*, Revised edition, 2000, https://www.cse.chalmers.se/research/group/logic/TypesSS05/Extra/geuvers.pdf.

[5] R. Kaye. *Models Of Peano Arithmetic*, Oxford Logic Guides **15**, Oxford University Press, 1991.

[6] A. S. Troelstra, D. van Dalen. *Constructivism in Mathematics: an introduction*, Vol. 1, Elsevier, 1988.

[7] H. Rogers Jr. *Theory of Recursive Functions and Effective Computability*, Higher Mathematics Series, McGraw-Hill, 1967.

Additional support

Four example sheets and four associated examples classes will be given. There will be a one-hour revision class in Easter Term.