Logic and Computability (M24)

J. Siqueira

This course builds upon the foundations laid out by the Part II courses *Logic and Set Theory* and *Automata and Formal Languages*, introducing a new notion of computability and a non-classical logic, as well as discussing the relationship between these subjects.

In Part II, there was an exposition of two general models of computation: register machines and recursive functions. There is, however, another alternative which has certain practical advantages and is at the core of functional programming languages like Haskell: the (untyped) λ-calculus. We will explore its properties and see that a version of this calculus has a strong connection to intuitionistic logical proofs.

Intuitionistic logic departs from classical logic by adopting a new interpretation for the logical connectives and quantifiers that makes it constructive and, therefore, carry computational content. We will discuss its syntax and semantics, the accompanying completeness theorems, and how it compares to classical logic.

Finally, there are interesting points in which logic and computability intersect. Examples include the incompleteness phenomenon and the arithmetic hierarchy (which generalises the characterisation of recursively enumerable sets as Σ_1-definable sets). We will explore some of these ideas and see what computability theory has to say about models of arithmetic.

The intent is to cover the following topics:

Computability

Recursive functions and computability; Recursive and recursively enumerable sets; The untyped λ-calculus; Reduction and the Church-Rosser Theorem; Church numerals; Fixed point combinators and λ-computability.

Logic

Nonstandard models of arithmetic; Natural deduction and intuitionistic logic; Diaconescu's Theorem; Completeness of the Heyting and Kripke semantics; Negative translations.

Computability in logic

Craig's Theorem; Gödel numberings and Incompleteness, Tennenbaum's Theorem; The simply typed λ-calculus and the Curry-Howard isomorphism; Weak and strong normalisation for the simply typed λ-calculus.

If time permits, we may also discuss either the Friedberg-Muchnik Theorem or Priestley Duality.

The course will appeal to those wishing to pursue research in the Foundations of Mathematics, Computer Science (particularly in semantics of programming languages), and Proof Formalisation. Those interested in topos theory will find the material on intuitionistic logic particularly useful.

Prerequisites

Familiarity with the contents of the Part II courses Logic and Set Theory and Automata and Formal Languages (or equivalent) is essential. Some of the relevant facts and definitions pertaining to computability, however, will be revised at a fast pace.

Literature

A primary source for intuitionistic logic and its relation to the λ-calculus is '*Proofs and Types*' [1]. It doesn't address the Heyting and Kripke semantics, however, and thus has to be complemented in other ways. A categorically minded student will enjoy Ghilardi's '*A First Introduction to the Algebra of Sentences*' [2], but everyone can appreciate Palmgren's notes '*Semantics of intuitionistic propositional logic*' [3] — they were both made available online. This is also the case for the very good and straightforward notes on the untyped λ-calculus by Barendregt and Barendsen '[4]'.

Kaye's '*Models of Peano Arithmetic*' [5] is the go-to reference on anything related to the formal theory of arithmetic.

If we end up pursuing the extra topics, Soare's book [6] has the best exposition of Turing degrees and finite priority injury methods I am aware of. Discussions on duality will benefit from Priestley's own book [7], as well as the more modern exposition by Gehrke and van Gool [9].

References

[1] J. Girard, Y. Lafont, P. Taylor. *Proofs and Types*, Cambridge University Press, 1989.

[2] S. Ghilardi. *A First Introduction to the Algebra of Sentences*, Lecture Notes, Dipartimento di Scienze dell'Informazione, Università degli Studi di Milano, 2000, http://users.mat.unimi.it/users/ghilardi/allegati/dispcesena.pdf.

[3] E. Palmgren. *Semantics of intuitionistic propositional logic*, Lecture Notes, Department of Mathematics, Uppsala University, 2009, http://www2.math.uu.se/~palmgren/tillog/heyting3.pdf.

[4] H. Barendregt, E. Barendsen. *Introduction to Lambda Calculus*, Revised edition, 2000, https://www.cse.chalmers.se/research/group/logic/TypesSS05/Extra/geuvers.pdf.

[5] R. Kaye. *Models Of Peano Arithmetic*, Oxford Logic Guides **15**, Oxford University Press, 1991.

[6] R. I. Soare. *Turing Computability: Theory and Applications*, Theory and Applications of Computability Series, Springer, 2016.

[7] B. A. Davey, H. A. Priestley. *Introduction to Lattices and Order*, 2nd Edition, Cambridge University Press, 2002.

[8] A. S. Troelstra, D. van Dalen. *Constructivism in Mathematics: an introduction*, Vol. 1, Elsevier, 1988.

[9] M. Gehrke, S. van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*, Cambridge University Press (Cambridge Tracts in Theoretical Computer Science), 2024.

[10] H. Rogers Jr. *Theory of Recursive Functions and Effective Computability*, Higher Mathematics Series, McGraw-Hill, 1967.

Additional support

Four Example Sheets (and accompanying Example Classes) are planned for this course. There will also be a one-hour revision class in Easter Term.