Large Cardinals (L16)

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The following definitions and facts should be familiar from an introductory course on set theory: a cardinal κ is called *regular* if every unbounded subset of κ has cardinality κ; successor cardinals, i.e., cardinals of the form $\aleph_{\alpha+1}$, are always regular; the usual limit cardinals, e.g., $\aleph_\omega$, $\aleph_{\omega+\omega}$, or $\aleph_{\omega_1}$, are not.

Thus, the following is a natural question:

"Are there any uncountable regular limit cardinals?".

If they exist, they must be very large, in particular, much larger than any of the mentioned limit cardinals.

It turns out that this question is intricately connected with the incompleteness phenomenon in set theory: if there is an uncountable regular limit cardinal, then there is a model of ZFC; therefore, ZFC is consistent, and hence (by Gödel's Second Incompleteness Theorem) ZFC cannot prove the existence of these cardinals (unless, of course, it is inconsistent).

Regular limit cardinals (a.k.a. *weakly inaccessible cardinals*) are the smallest examples of set-theoretic notions called *large cardinals*: cardinals with properties that imply that they must be very big and whose existence cannot be proved in ZFC.

In this course, we shall get to know a number of these large cardinals, study their behaviour, observe consequences of their existence for set theory, and develop techniques to determine the logical strength of large cardinals (the so-called *consistency strength hierarchy*). In modern set theory, large cardinals are used as the standard way to calibrate logical strength of extensions of ZFC.

Prerequisites

The Part II course *Logic & Set Theory* or an equivalent course is essential.

This course shares some content with the Part III courses *Model Theory & Non-classical Logic* (Michaelmas 2023) and *Forcing & the Continuum Hypothesis* (Lent 2024) that will be proved in the other courses and used in this course. Following these two courses is not necessary, but recommended.

Literature

1. Thomas Jech, *Set Theory*, The Third Millenium Edition, revised and expanded, Springer, 2003 [Springer Monographs in Mathematics]

2. Akihiro Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings.* Springer, 2003 [Springer Monographs in Mathematics]

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.