Perturbation Methods (M16)

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This course will deal with the asymptotic solution to problems in applied mathematics when some parameter or coordinate assumes large or small values. Many problems of physical interest are covered by such asymptotic limits; indeed, they play a crucial role in the field of mathematical modelling. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases can provide accurate predictions even when the parameter or coordinate has only moderately large or small values.

Some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, with the remaining lectures devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the 'divide & conquer' strategy, Laplace's method, stationary phase and steepest descents. [6]
- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKB[JLG]' method can be viewed as a special case.) It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied, as time allows, potentially including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium.) [5]
- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. The method was first developed in fluid mechanics, where it is referred to as boundary-layer theory, but it has since been greatly extended and applied to many fields. [5]

Prerequisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no (or only elementary) knowledge of these fields will be assumed. The only prerequisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward ordinary & partial differential equations and evaluate reasonably simple integrals.

Literature

Relevant Textbooks (not required for the course, but which may assist)

1. C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill, 1978. *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to 'Stokes' lines as 'anti-Stokes' lines, and vice versa. The course will use Stokes' convention.*

2. E.J. Hinch. *Perturbation Methods.* Cambridge University Press, 1991. *The lecture course was originally based on this book; some may view it as somewhat concise.*

3. M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics.* Parabolic Press, Stanford, 1975. *This is the original book on perturbation methods; somewhat dated, but still an excellent read.*

4. J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer, 1981. *A useful and accessible book, covering a range of asymptotic methods, and applications.*

Additional Reading on Topics Beyond the Syllabus

1. M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. A, **427**, 265–280 (1990).

2. J.P. Boyd. *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, 56, 1-98 (1999). Also available at `http://hdl.handle.net/2027.42/41670` and `http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf`.

Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.