Mathematical Tripos Part III Lecture Courses in 2023-2024

Department of Pure Mathematics and Mathematical Statistics

Department of Applied Mathematics and Theoretical Physics

1 Algebra

Commutative Algebra (M24)

Dr O. Becker

This course will provide an introduction to the theory of commutative rings and modules over these rings. It should be viewed as a foundational course for Algebraic Geometry and Algebraic Number Theory.

The course will cover a selection of topics including:

- Unital commutative rings, ideals, Hilbert's basis theorem
- Tensor products, flatness, localization
- Nakayama's lemma, integral and finite extensions, going up and going down theorems
- Noether's normalization theorem and Hilbert's nullstellensatz, primary decomposition
- Direct and inverse limits, a brief introduction to completions, the Artin-Rees lemma
- Dimension theory and Hilbert functions
- Discrete valuation rings and Dedekind domains

Prerequisites

You will have attended a first course in ring theory, such as the IB course Groups, Rings and Modules. Experience of more advanced material such as

Part II courses Galois Theory, Representation Theory, Algebraic Geometry or Number Fields is desirable (contributing to your level of algebraic maturity) but not essential.

Literature

1. M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
2. J.S. Milne, *A Primer of Commutative Algebra, https://www.jmilne.org/math/xnotes/ca.html*
3. *D. Eisenbud,* Commutative Algebra with a view toward Algebraic Geometry*, Spring-Verlag, 1995.*
4. *H. Matsumura,* Commutative Algebra*, second edition, The Benjamin/Cummings publishing co., 1980.*
5. *H. Matsumura,* Commutative Ring Theory*, Cambridge Studies 8, CUP, 1989.*

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Group Cohomology (L16)

Dr C.J.B. Brookes

Group cohomology uses methods motivated by algebraic topology to study groups and their representations. Applications appear throughout mathematics, in representation theory, topology, geometry and number theory (Galois cohomology).

I hope to cover all/most of the following:

- Definitions and resolutions.
- Low degree cohomology, derivations and group extensions.
- Group homology, the Schur multiplier (or multiplicator) and central extensions.
- Crossed products, division algebras and Brauer groups of fields.
- Cup products and cohomology rings.

- Lyndon-Hochschild-Serre spectral sequence, inflation-restriction exact sequence.
- Cohomological dimension, finiteness conditions F_n and FP_n.

The course will be similar, but not identical, to the one given last year, with a little more emphasis on spectral sequences.

Prerequisites

My approach will be purely algebraic but it will be useful to have some acquaintance with (co-)homology from other contexts. Many of the examples will be of groups arising in geometry and topology.

You should know your way about groups, rings and modules. For this, the III course on Commutative Algebra will provide good experience. It would also be useful to have met the group algebra, the associative algebra underlying group representation theory.

I shall assume acquaintance with tensor products (of vector spaces/modules and algebras), and, for the section on Brauer groups of fields, I shall also assume that you have met Galois groups (though you won't in fact need much more than the definition).

Literature

1. D.J. Benson, *Representations and Cohomdlogy, volume II, Cohomology of Groups and Modules.* Cambridge University Press, 1991.
2. K.S. Brown, *Cohomology of Groups.* Springer, 1982.
3. L. Evens. *The Cohomology of Groups.* Oxford University Press, 1991.
4. R.S. Pierce, *Associative Algebras.* Springer, 1982.
5. D.J.S. Robinson, *A Course in the Theory of Groups.* Springer, 1982.
6. J.J. Rotman, *An Introduction to Homological Algebra.* Springer, 2009.
7. C.A. Weibel, *An Introduction to Homological Algebra.* Cambridge University Press, 1994.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Lie algebras and their representations (24 lectures, MT23)

Prof. S. Martin

This course is an introduction to the theory of semisimple Lie algebras usually over an algebraically closed field of characteristic 0, with an emphasis on their representations.

Lie algebras are 'infinitesimal symmetries', which one might think of as linearisations of Lie groups. They are ubiquitous in many branches of mathematics: in topology, in arithmetic and algebraic geometry and in theoretical physics (string theory, exactly solvable models in statistical mechanics,...). One reason for their importance is that the finite-dimensional complex representations of the simple Lie algebras are exactly the same as those of the corresponding groups. So instead of needing to study the topology and geometry of the simple Lie groups, or the algebraic geometry of the simple algebraic groups, we can use only linear algebra and still manage to describe completely these representations.

While our algebras play a role in many diverse areas of mathematics, the material in this course is inherently attractive in its own right, combining as it does a certain amount of depth and a satisfying degree of completeness in its basic results. The main purpose of the course is to gain familiarity with these objects so that you can work with them in the context of your own interests. Because of this, we will spend time on examples. However, because of the amount of material we will need to cover to give you a solid background, we will sometimes skip or only sketch proofs that can be found in the references below, especially references [8] and [3] and [4].

Of the following I will cover (most of) the first four topics and, depending on time, may sketch the fifth.

- Definitions, motivations, and basic structure theory.
- Root systems, Weyl groups, the finite simple Lie algebras.
- Theorems of Engel and Lie (on solvability), Cartan (on Killing forms), Weyl (on complete reducibility).
- Classification of finite-dimensional representations, Verma modules, the combinatorics of characters.
- (if time) Crystal bases.

Prerequisites

Linear algebra up to the Jordan canonical form; basic abstract algebra (groups, rings, modules and fields); the rudiments of representation theory (Part II or equivalent course, see [4]). A little multilinear algebra and familiarity with Lie group theory (the circle group S^1, SU(2), SO(3), etc.) are nice to have but not dealbreakers. Material in the Part III Commutative Algebra course (tensor products of vector spaces) is also tangentially useful.

Literature

There is no shortage of books on the theory of Lie algebras: the treatment in [8] cannot be bettered.

1. Carter, R. W., Simple groups of Lie type, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1989 reprint.
2. Carter, R. W., Lie algebras of finite and affine type, Cambridge Studies in Advanced Mathematics, 96, Cambridge University Press, Cambridge, 2005.
3. Erdmann, K. and Wildon, M., Introduction to Lie algebras, Springer, 2006
4. Fulton, W. and Harris, J.. Representation theory, a first course, Graduate Texts in Mathematics, 129, Springer-Verlag, New York, 1991.
5. Hall, B. C., Lie groups, Lie algebras and representations. Springer GTM 222, Springer-Verlag 2004.
6. Henderson, A., Representations of Lie algebras: an introduction through gl_n (Australian Mathematical Society Lecture Series, No 22), 2012.
7. Humphreys, J. E., Linear Algebraic Groups, vol 21 of GTM, Springer, New York, 1975.
8. Humphreys, J. E., Introduction to Lie algebras and representation theory, vol. 9 of GTM Springer, New York, 1978.
9. Jacobson, N., Lie algebras, Dover reprint, 2003.
10. Kac, V., Infinite dimensional Lie algebras (3rd edn), Cambridge University Press, Cambridge, 1990.
11. Serre, J.-P., Complex semisimple Lie algebras. transl. G.A. Jones. Springer monographs in mathematics, Springer-Verlag, New York, 2001.
12. Kostant, B., The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group, American Journal of Mathematics, Vol. 81, No. 4 (Oct., 1959), pp. 973-1032.
13. Slodowy, P., Simple singularities and simple algebraic groups, Springer Lecture Notes 815, Springer 1980.
14. Stubhaug A., The mathematician Sophus Lie: it was the audacity of my thinking, Springer, Berlin, 2002.

Additional support

Exercises (and hints) will be provided in lectures and four sheets of examples will be given. There will be a one-hour revision class in the Easter Term 2024.

2 Algebraic Geometry

Algebraic Geometry (M24)

Dr D. Ranganathan

The course will be an introduction to modern algebraic geometry. The first part of the course will develop the basic theory of sheaves, and use this to give a treatment of schemes, morphisms between them, and fibre products. We will then explore morphisms the basic properties of morphisms, including separatedness and properness. After this, we turn to the study of quasi-coherent sheaves, with a particular focus on line bundles, the Picard group, and other proximate notions. The final few lectures in the course will give a rapid introduction to sheaf cohomology "for the working mathematician".

The course will develop the basic toolkit used by practicing algebraic geometers, and will also be appropriate for anyone with an interest in acquiring a working knowledge of algebraic geometry, including those in nearby fields such as differential geometry, topology, and number theory.

Prerequisites

The basic theory of rings and modules, and elementary topology will all be assumed. It is recommended that students have had a previous course in Commutative Algebra, take one in parallel, or familiarize themselves with the concepts from Atiyah–MacDonald's text on the subject. In the latter case, the crucial content is in Chapters 1–3 and 5–7. The key ideas from algebra that we will need are ideals, modules, tensor products, and localization. Basic intuition is provided from topology and the theory of manifolds, and this may be helpful but is not essential.

A prior course in the algebraic geometry of affine and projective varieties is not logically necessary for the course, i.e. the contents of the Part II Algebraic Geometry course. However, students should be aware that certain key examples in scheme theory do come from classical algebraic geometry. Students with no prior exposure to algebraic geometry may occasionally need to look up such constructions. However, this will be the exception rather than the rule, and ample references will be provided.

Literature

The following books complement the course material. The first and second are roughly interchangeable. The final text provides valuable intuition for the many unfamiliar concepts in scheme theory.

1. R. Hartshorne, *Algebraic Geometry*, Springer, 1977.
2. R. Vakil, *The rising sea. Foundations of algebraic geometry.* Available online.
3. D. Eisenbud and J. Harris, *Geometry of schemes*, Springer, 2000.

Additional support

Four examples sheets will be provided and four associated examples classes will be given; these will be essential. Additional problems and examples may be provided to clarify more difficult concepts. There will be a revision class in the Easter Term.

Abelian Varieties (L24)

Professor A. J. Scholl

An abelian variety is an irreducible projective variety over a field which is also a group. The simplest example is an elliptic curve. Other examples include the Jacobian variety of a (smooth projective) curve, whose points parametrise divisor classes of degree zero. There are beautiful analytic and algebraic theories of such objects. They also crop up in other areas of geometry and in number theory.

Although elliptic curves can be studied using fairly elementary algebraic geometry and explicit equations, for abelian varieties of higher dimension scheme theory is pretty much essential, and gives a much better understanding (as well as a good illustration of how powerful scheme theory is). This course will therefore be in 2 parts.

The first 3–4 weeks will be a continuation of the Algebraic Geometry course from Michaelmas. We will cover various topics in abstract algebraic geometry. One important topic will be the study of how cohomology varies in a family (cohomology and basechange).

In the remainder of the lectures, we will introduce group schemes, and develop some of the algebraic theory of abelian varieties, approximately covering Chapter 2 of Mumford's book. If time permits, we may also give the construction of the Jacobian of a curve.

Prerequisites

Algebraic Geometry (MT) and Commutative Algebra (MT) or equivalent are essential.

Literature

1. David Mumford: *Abelian varieties.* Second edition. Tata Institute of Fundamental Research, 1974.
2. Edixhoven, Moonen and Van Der Geer: draft book, available at http://van-der-geer.nl/ gerard/AV.pdf or https://www.math.ru.nl/ bmoonen/research.html#bookabvar
3. Robin Harsthorne: *Algebraic geometry.* Springer, corrected edition. (For the first half of the course)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Toric Varieties (L16)

Dr R. Picciotto

This examinable Part III course will provide an introduction to Toric Varieties. These varieties are accessible through combinatorial methods, and provide a large class of examples in Algebraic Geometry for which explicit computations are often possible. I will cover the basic constructions of toric varieties from fans and polytopes, as well as toric morphism. Further topics will include toric bundles and divisors and an introduction to GIT theory.

The course will cover a selection of topics including:

- Toric varieties from fans and polytopes;
- Toric morphisms and their properties;
- Quotients and coordinate rings;
- Toric divisors and line bundles;
- Toric surfaces and resolutions of singularities.

Prerequisites

A course in Algebraic Geometry, at the level of the Part II or Part III course.

Literature

1. *D. Cox, J. Little, H. Schenck*, Toric Varieties. Graduate Studies in Mathematics 124, 2011.
2. *W. Fulton*, Introduction to Toric Varieties. AM-131, Princeton University Press, 1993.
3. Lecture notes by David Cox, 'Lectures on toric varieties'.
4. Lecture notes by Frank Sottile, 'Ibadan lectures on toric varieties'.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be one revision class in the Easter Term.

3 Analysis and PDEs

Functional Analysis (L24)

András Zsák

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. We will cover the following topics:

Hahn–Banach Theorems on the extension of linear functionals. Locally convex spaces.

Duals of the spaces $L_p(\mu)$ and $C(K)$. The Radon–Nikodym Theorem and the Riesz Representation Theorem.

Weak and weak-* topologies. Theorems of Mazur, Goldstine, Banach–Alaoglu. Reflexivity and local reflexivity.

Hahn–Banach Theorems on separation of convex sets. Extreme points and the Krein–Milman theorem. Partial converse and the Banach–Stone Theorem.

Banach algebras, elementary spectral theory. Commutative Banach algebras and the Gelfand representation theorem. Holomorphic functional calculus.

Hilbert space operators, C^*-algebras. The Gelfand–Naimark theorem. Spectral theorem for commutative C^*-algebras. Spectral theorem and Borel functional calculus for normal operators.

Some additional topics time permitting. For example, uniform convexity and smoothness, ultraproducts, the Fréchet–Kolmogorov Theorem, weakly compact subsets of $L_1(\mu)$, the Eberlein–Šmulian and the Krein–Šmulian theorems, the Gelfand–Naimark–Segal construction.

Prerequisites

Thorough grounding in basic topology and analysis. Some knowledge of basic functional analysis and basic measure theory (much of which will be recalled either in lectures or via handouts). In Spectral Theory we will make use of basic complex analysis. For example, Cauchy's Theorem, Cauchy's Integral Formula and the Maximum Modulus Principle.

Literature

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales).* Oxford University Press, 2011.
2. Bollobás, Béla *Linear analysis: an introductory course.* Cambridge University Press, 1990.
3. Murphy, Gerard J. *C^*-Algebras and Operator Theory.* Academic Press, Inc., 1990.
4. Rudin, Walter *Real & Complex Analysis.* McGraw-Hill, 1987.
5. Rudin, Walter *Functional Analysis.* McGraw-Hill, 1990.
6. Taylor, S. J. *Introduction to measure and integration.* Cambridge University Press 1973.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be some material as well as examples sheets and announcements available at `www.dpmms.cam.ac.uk/~az10000/`

Analysis of Partial Differential Equations (M24)

Dr Z. Wyatt

This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is presently a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on developing the theory and methods of the modern approach to PDE theory. Emphasis will be given to functional analytic techniques, relying on a priori estimates rather than explicit solutions. The course will primarily focus on approaches to linear, elliptic and evolutionary problems through energy estimates, with the prototypical examples being Laplace's equation, the heat equation, the wave equation and Schrödinger's equation.

The following concepts will be studied: well-posedness; the Cauchy problem for general (non-linear) PDE; characteristics; Sobolev spaces; elliptic boundary value problems (solvability and regularity); evolutionary problems (hyperbolic, parabolic and dispersive PDE).

Prerequisites

There are no specific pre-requisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly self-contained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Preliminary Reading

Students are strongly encouraged to read through Claude Warnick's notes for the Part II Analysis of Functions course. These will be made available online. The following article also gives an overview of the field of PDEs

1. Klainerman, S., *Partial Differential Equations*, Princeton Companion to Mathematics (editor T. Gowers), Princeton University Press, 2008.

Literature

1. Some lecture notes from a previous lecturer of the course are available online at:

   http://cmouhot.wordpress.com/teachings/

The following textbooks are excellent references:

2. Evans, L. C., *Partial Differential Equations*, Springer, 2010
3. Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer,2010.
4. John, F., *Partial Differential Equations*, Springer, 1991

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will be one office hour a week.

Elliptic Partial Differential Equations (L24)

Neshan Wickramasekera & Greg Taujanskas

This course will provide an introduction to the theory of linear second order elliptic partial differential equations. Second order elliptic equations play a fundamental role in many areas of mathematics including Calculus of Variations, Riemannian Geometry and Mathematical Physics. Linear elliptic theory provides the foundation for studying a number of non-linear problems arising in these fields, such as minimal submanifolds, harmonic maps and evolution problems in Geometry and Mathematical Physics.

The course will provide a rigorous treatment, based on a priori estimates, of both classical and weak solutions to linear elliptic equations, focusing on the question of existence and uniqueness of solutions to the Dirichlet problem and the question of regularity of solutions. Specific topics include harmonic functions, maximum principles for general second order equations, Schauder estimates (via L. Simon's scaling argument), the continuity method for existence of solutions, solvability of the Dirichlet problem in balls, Perron's method, divergence form operators, De Giorgi–Nash–Moser estimates and the Harnack theory and, as time permits, a brief discussion of the quasilinear theory centred around the prototypical example of the Minimal Surface Equation.

Prerequisites

Part III *Analysis of PDEs*.

Literature

1. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order.*
2. L. Simon, *Schauder estimates by scaling.* Calc. Var. & PDE, **5**, (1997), 391–407.
3. P. Minter, Lecture notes for the courses *Elliptic PDEs* and *The Minimal Surface Equation and Related Topics*, https://minterscompactness.wordpress.com/lecture-notes/

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

4 Applied and Computational Analysis

Distribution Theory and Applications (L16)

Dr. A. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we look at operations such as differentiation, translation, convolution and the Fourier transform. We will introduce the Sobolev spaces $H^s(\mathbf{R}^n)$ and $H^s_{\mathrm{loc}}(X)$ and describe them in terms of Fourier transforms for tempered distributions. The material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution with compact support?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. The final section of the course will be concerned with Hörmander's theory of oscillatory integrals.

Prerequisites

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB complex methods or analysis). Knowledge of measure theory and functional analysis will *not* be needed, but might be complementary.

Preliminary Reading

1. F.G. Friedlander & M.S. Joshi, *Introduction to the Theory of Distributions*, C.U.P, 1998.
2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, C.U.P, 1958.
3. G.B. Folland, *Introdution to Partial Differential Equations*, Princeton Univ Pr, 1995.

Literature

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols* I-II, Springer, 1985.
2. M. Reed & B. Simon, *Methods of Modern Mathematical Physics: Vols* I-II, Academic Press, 1979.
3. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be made available. There will be a revision class in the Easter Term.

Approximation Theory (M16)

Professor A. Shadrin

The course will give an overview of basic concepts of Approximation Theory, i.e., best and good approximation of a large family of functions by a smaller set (usually finitely generated, linear or nonlinear) in certain normed spaces (such as C and L_p), construction of good approximants (in various settings), finding approximation order for smooth functions (say, order n^{-k} for approximation by polynomials of degree n).

The course consists of three parts.

- We start with the classical approximation by polynomials, which includes the Weierstrass theorem, positive linear operators, and direct and inverse theorems for trigonometric approximation.
- We move then the emphasis to univariate splines which are piecewise polynomial functions. Here we study representation through the B-spline basis, spline interpolation theory and norm-minimization property of splines via orthogonal spline projector.
- Finally, we make a tour into wavelets which will cover the multiresolution analysis and Daubechies orthogonal wavelets with a compact support.

Prerequisites

The course is mostly self-contained, but it assumes a standard mathematical analysis background (say, normed linear spaces, inner products, Fourier series) and some linear algebra (solution of linear systems of equations). Some functional analysis tools (e.g., Hahn-Banach theorem) appear in comments and advanced problems, so they are helpful but not required.

Literature

To a large extent, the course follows the Lecture Notes by C. de Boor where one can find much more details on each topic. In the first part, it is also based on Cheney's book.

1. C. de Boor, Notes on Approximation Theory
   https://pages.cs.wisc.edu/ deboor/ma887.html
2. E. W. Cheney, *Approximation theory*, MgGrow-Hill, New-York, 1966.
3. R. A. DeVore, G. G. Lorents, *Constructive Approximation,* Springer-Verlag, Berlin, 1993.
4. I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

Additional suppport

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter term.

Numerical solution of differential equations (M24)

Professor A. Iserles

The goal of this lecture course is to present and analyse efficient numerical methods for ordinary and partial differential equations. The exposition is based on few basic ideas from approximation theory, complex analysis, theory of differential equations and linear algebra, leading in a natural progression to a wide range of numerical methods and computational strategies. The emphasis is on algorithms and their mathematical analysis, rather than on applications.

The course consists of three parts:

- methods for **ordinary differential equations** (with an emphasis on initial-value problems and a thorough treatment of stability issues and stiff equations)
- numerical schemes for **partial differential equations** (both boundary and initial-boundary value problems, featuring finite differences and finite elements)
- time allowing, **numerical algebra of sparse systems** (inclusive of fast Poisson solvers, sparse Gaussian elimination and iterative methods).

We start from the very basics, analysing approximation of differential operators in a finite-dimensional framework, and proceed to the design of state-of-the-art numerical algorithms.

Desirable Previous Knowledge

Good preparation for this course assumes relatively little in numerical mathematics *per se,* except for basic understanding of elementary computational techniques in linear algebra and approximation theory. Prior knowledge of numerical methods for differential equations will neither be assumed nor is necessarily an advantage. Experience with programming and application of computational techniques will obviously aid comprehension but is not a prerequisite.

Fluency in linear algebra and decent understanding of mathematical analysis are a must. Thus, linear spaces (inner products, norms, basic theory of function spaces and differential operators), complex analysis (analytic functions, complex integrals, the Cauchy formula), Fourier series, basic facts about dynamical systems and, needless to say, elements from the theory of differential equations.

There are several undergraduate textbooks on numerical analysis. The following present material at a reasonable level of sophistication. Often they present material well in excess of the requirements for the course in computational differential equations, yet their contents (even the bits that have nothing to do with the course) will help you acquire valuable background in numerical techniques:

Introductory Reading

1. S. Conte & C. de Boor, *Elementary Numerical Analysis,* McGraw–Hill, New York, 1980.

2. G.H. Golub & C.F. van Loan, *Matrix Computations,* 3rd edition. Johns Hopkins Press 1996.

3. M.J.D. Powell, *Approximation Theory and Methods,* Cambridge University Press, Cambridge, 1981.

4. G. Strang, *Introduction to Linear Algebra,* Wellesley-Cambridge Press, Cambridge (Mass.), 3rd ed. 2003..

Reading to complement course material

1. U. M. Ascher, *Numerical Methods for Evolutionary Differential Equations,* SIAM, Philadelphia, 2008.

2. E. Hairer, S. P. Nørsett and G. Wanner, *Solving Ordinary Differential Equations I: Nonstiff Problems*, Springer-Verlag, Berlin, 2nd ed. 1993.

3. E. Hairer and G. Wanner, *Solving Ordinary Differential Equations II: Stiff and Differential Algebraic Problems,* Springer-Verlag, Berlin, 2nd ed. 1996.

4. A. Iserles, *A First Course in the Numerical Analysis of Differential Equations*, Cambridge University Press, Cambridge, 2nd ed. 2008.

5. K.W. Morton & D.F. Mayers, *Numerical Solution of Partial Differential Equations: An Introduction,* Cambridge University Press, Cambridge, 2005.

6. G. Strang and G. Fix, *An Analysis of the Finite Element Method*, Wellesley–Cambridge Press, Cambridge (Mass.), 2nd ed. 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Infinite-dimensional Spectral Computations (L16)

Matthew Colbrook

The spectral theory of operators is a fundamental area of mathematics with profound implications across diverse fields. It plays a pivotal role in quantum mechanics, where the spectrum of an operator corresponds to potential measurement outcomes. Furthermore, in areas such as signal processing, control theory, data-driven discovery in dynamical systems, and partial differential equations (PDEs), spectral analysis methods are essential for solving complex problems.

This course aims to offer a comprehensive understanding of operator theory on separable Hilbert spaces, with a particular emphasis on the foundations of computing spectra of operators. The course is ideally suited for graduate students in mathematics or physics who have a keen interest in advanced functional analysis concepts. A significant aspect of the course involves learning how to manage the error and ensure convergence when reducing an infinite-dimensional problem to a series of finite-dimensional ones.

The course will cover a selection of topics including:

- Frameworks for establishing the foundations of computation and classifying problem difficulty. For example, we will explore the concept of the Solvability Complexity Index and discuss developing canonical embedding problems for problem classification.
- Computing spectra of operators via computing the resolvent operator and addressing the fundamental issue of spectral pollution (spurious eigenvalues).
- Spectral measures and continuous spectra of self-adjoint operators, convolution methods for their computation, and contour projection methods.
- Contemporary applications in data-driven study of so-called Koopman operators and transfer operators for nonlinear dynamical systems.
- Time permitting, we will discuss the broader implications of these techniques beyond pure spectral theory, including areas such as computer-assisted proofs and PDEs.

Prerequisites

This course presumes basic knowledge in linear algebra, analysis, and linear analysis. Other useful Part III courses include *Functional Analysis* and *Numerical Solution of Differential Equations*. However, no formal background in numerical analysis courses is required.

Literature

1. W. Rudin, *Functional analysis* 2nd edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.

2. M. Reed, B. Simon, *Methods of modern mathematical physics. I*, Academic Press, Inc., New York, 1980.

3. J.B. Conway, *A course in functional analysi*s, Vol. 96. Springer, 2019.

4. E.B. Davies, *Linear operators and their spectra*, Vol. 106. Cambridge University Press, 2007.

5. M. Colbrook, A. Horning, A. Townsend, *Computing spectral measures of self-adjoint operators*, SIAM review 63.3 (2021): 489-524.

6. M. Colbrook, *On the computation of geometric features of spectra of linear operators on Hilbert spaces*, Foundations of Computational Mathematics (2022): 1-82.

Additional support

Three examples sheets will be provided, accompanied by three associated examples classes. A one-hour revision class will also be offered in the Easter Term.

Topics in Mathematics for Deep Learning (L16)

Non-Examinable (Part III Level)

Dr C. Esteve Yagüe

This non-examinable course is intended to be an introduction to the mathematical theory of Deep Learning. The course is aimed at students with a mathematical background, with the goal of attracting their attention to this new research field which is nowadays experiencing a huge development. The course can also be interesting for computer scientists, with experience (or not) in deep learning, who want to explore the topic from a mathematical viewpoint.

The great success of the application of deep learning techniques to a wide range of real-life problems has raised a number of mathematical questions, some of them being still unanswered, or partially answered. We will start by introducing the notation used to describe artificial neural networks and formulate the underlying learning problems. We will review the most important theoretical results concerning the approximation and generalization properties of deep neural networks in the overparametrized regime (infinite width and depth limit), as well as the behaviour of the optimization algorithms used during training (typically formulated as a non-convex optimization problem). Then, we will discuss the use of different NN architectures, which guarantee desirable properties of the learned model. Finally, we will present some of the applications of deep learning techniques, such as inverse problems, image analysis, and the numerical approximation of partial differential equations.

The course will cover a selection of topics including:

- **Introduction to Deep Learning:** formulation of the learning problem, neural network architectures and types of error.
- **Expressivity of Neural Networks:** universal approximation of overparametrized neural networks in different functional spaces.
- **Optimization of Deep Neural Networks:** convergence of stochastic gradient descent, loss landscape analysis, implicit regularization, etc.
- **Generalization of Neural Networks:** connection to kernel methods, Gaussian processes and the kernel regime.
- **Special architectures:** residual NNs, convolutional NNs, scattering transform, equivariant NNs, etc.
- **Applications of deep learning:** Physics Informed NNs, Deep Learning for inverse problems, etc.

Prerequisites

No previous knowledge on deep learning or machine learning is required for this course, which is mainly addressed to students with a background in mathematics. Concerning the mathematical knowledge, it is desirable to have some fundamental background on functional analysis, measure theory and optimization.

Preliminary Reading

1. Julius Berner, Philipp Gtohs, Gitta Kutyniok, Philipp Petersen, *The Modern Mathematics of Deep Learning*, 2022. Book chapter in *Theory of Deep Learning*, Cambridge University Press. https://arxiv.org/abs/2105.04026

Literature

1. Francis Bach, *Learning theory from first principles*, 2021. Online version https://www.di.ens.fr/ fbach/ltfp$_b$*ook.pdf F.Bach*, *Breaking the curse of dimensionality with convex neur* 629–681, 2017.*Linktojournalhttps : //jmlr.org/papers/volume*18/14 − 546/14 − 546.*pdf*

2. J. E. Gerken, J. Aronsson, O. Carlsson, H. Linander, F. Ohlsson, C. Petersson, and D. Persson, *Geometric deep learning and equivariant neural networks*, Artificial Intelligence Review, pages 1–58, 2023. Link to journal https://link.springer.com/content/pdf/10.1007/s1046 023-10502-7.pdf

3. A. Jacot, F. Gabriel, and C. Hongler, *Neural tangent kernel: Convergence and generalization in neural networks*, Advances in neural information processing systems, 31, 2018. Link to journal https://proceedings.neurips.cc/paper$_f iles/paper/2018/file/5a4be1fa34e62b$ $Paper.pdf A.G.deG.Matthews, J.Hron, M.Rowland, R.E.Turner, andZ.Ghahramani,$ *Gaussian process behav* $//arxiv.org/abs/$1804.11271

4. S. Mukherjee, A. Hauptmann, O. Öktem, M. Pereyra, and C.-B. Schönlieb. *Learned reconstruction methods with convergence guarantees: A survey of concepts and applicattions.* IEEE Signal Processing Magazine, 40(1):164–182, 2023. Link to journal https://ieeexplore.ieee.org/abstract/document/10004773

Additional support

Lecture notes will be provided and accompanying office hours will be offered.

Topics in Convex Optimisation (L16)

Dr H. Fawzi

Mathematical optimisation problems arise in many areas of science and engineering, including statistics, machine learning, robotics, signal/image processing, and others. This course will cover algorithms for convex optimization problems, i.e., algorithms to solve problems of the form

$$\min_{x \in X} f(x)$$

Prerequisites

This course assumes basic knowledge in linear algebra and analysis. Some knowledge of convex analysis will be useful.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

5 Astrophysics

Astrophysical Fluid Dynamics (M24)

Professor Roman Rafikov

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of jets and accretion discs around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is 'frozen in' to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be discussed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs.

The course will cover a selection of topics including:

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD, with examples of basic phenomena.
- MHD equilibria, Grad-Shafranov equation.
- Conservation laws, stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, astrophysical shocks (supernova explosions) and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: astrophysical jets.

Prerequisites

This course is suitable for both astrophysicists and fluid dynamicists. Knowledge of key elements of vector calculus, fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas.* Cambridge University Press.

2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.

3. Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows.* Cambridge University Press. Available as an e-book from

   http://ebooks.cambridge.org

4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics.* University Science Books.

5. Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics.* Imperial College Press.

6. Ogilvie, G. I. (2016). *Lecture Notes: Astrophysical Fluid Dynamics.* J. Plasma Phys. **82**, 205820301.

Additional support

Four example sheets will be provided and four associated classes will be given. Extended notes supporting the lecture course are available from reference 6 in the list above. There will be a revision class in Easter Term.

Dynamics of Astrophysical Discs (L16)

Henrik Latter

Discs are ubiquitous in astrophysics and participate in some of its most important processes. Most, but not all, feed a central mass: by facilitating the transfer of angular momentum, they permit the accretion of material that would otherwise remain in orbit. As a consequence, discs are essential to star, planet, and satellite formation. They also regulate the growth of supermassive black holes and thus indirectly influence galactic structure and the intracluster medium. Although astrophysical discs can vary by ten orders of magnitude in size and differ hugely in composition, all share the same basic dynamics and many physical phenomena.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). In fact, an astrophysical disc is a rotating shear flow that can sustain various instabilities, waves, and turbulence, which allow the necessary accretion. On the other hand, the resonant gravitational interaction of a planet within the disc generates waves that leads to orbital evolution of the planet and is one of the processes shaping the observed distribution of extrasolar planets.

Provisional synopsis:

- Occurrence of discs in various astronomical systems, basic physical and observational properties.
- Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.
- Viscous evolution of an accretion disc.
- Vertical disc structure: thin-disc approximations, thermal instability in cataclysmic variables.
- The shearing sheet, symmetries, shearing waves.
- Incompressible dynamics: hydrodynamic stability, vortices and dust in protoplanetary discs.
- Compressible dynamics: density waves, gravitational instability and 'gravitoturbulence' in planetary rings and protoplanetary discs.
- Satellite-disc interaction, impulse approximation, gap opening by embedded planets.
- Magnetorotational instability, 'dead zones' and outbursts in protoplanetary discs, magnetised outflows and jets

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is helpful for the topic of magnetorotational instability.

Literature

Astrophysical background can be found in reference [1] below, while the classical theory is summarised in the two review articles [2,3].

1. Frank, J., King, A. & Raine, D. (2002), Accretion Power in Astrophysics, 3rd edn, CUP.
2. Pringle, J. E. (1981), Annu. Rev. Astron. Astrophys. 19, 137.
3. Papaloizou, J. C. B. & Lin, D. N. C. (1995), Annu. Rev. Astron. Astrophys. 33, 505.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Advanced Stellar Evolution (L16)

Non-Examinable (Part III Level)

Dr A. N. Żytkow

We will proceed with the investigation of further aspects of stellar structure and evolution following on from the course "Structure and Evolution of Stars" in the Michaelmas Term.

The only way we know how to form heavy elements is through stellar processes. Thus stars are central in building the world around us. Insights into the structure and evolution of stars rely on a mathematical description of the physical processes which determine the nature of stars. The only way in which we may test stellar structure and evolution theory is through the comparison of theoretical results to observations.

The main subjects to be covered will be an introduction to asteroseismology, stellar rotation, as well as further details regarding mass loss from stars and some examples of stellar models. All these are essential if we endeavour to understand, interpret and study current observational data.

Prerequisites

Some understanding of hydrodynamics, electromagnetic theory, thermodynamics, nuclear physics, quantum mechanics as well as theory of structure and evolution of stars, although a detailed knowledge of all of these is not necessary.

Literature

1. Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution*, Second Edition, Springer- Verlag, 2012.
2. Cox, J. P. and Giuli, R. T. *Principles of Stellar Structure*, Gordon and Breach, 1968.
3. Padmanabhan, T. *Theoretical Astrophysics*, Volume II: Stars and Stellar Systems, Cambridge University Press, 2001

Astrophysical black holes (L16)

Professor D. Sijacki

Black holes are one of the most fascinating objects lying at the interface of mathematics, physics and astronomy. From the astrophysical stand point they give rise to extremely rich and complex phenomena occurring from sub-parsec to Mega-parsec scales and covering the full electromagnetic spectrum. A large body of state-of-the-art current and upcoming observational facilities and theoretical models is aimed at investigating black hole properties and their link with the larger scale environment, making this an exciting and fast paced research field. With the recent gravitational wave detections of merging black hole binaries the field has experienced further stimulus, as black holes have become unique multi-messengers to explore cosmology, gravity in the strong regime, high energy phenomena and complex (magneto)hydrodynamic flows.

This course will cover a range of concepts pertinent to astrophysical (supermassive) black holes highlighting both observational and theoretical advances in the field. The aim of the course is to give an overview of the possible formation and growth channels of these objects and to discuss various mechanisms through which black holes interact with their surroundings, with the synopsis as follows:

- Basic concepts; observational evidence for dormant and non-dormant objects (SgrA*)
- AGN properties and classification
- Formation pathways for supermassive black holes
- Black hole growth overview
- Fuelling mechanisms from kpc to sub-pc scales
- Bondi-Hoyle solution and limitations
- Accretion disk models: thin, slim and thick discs
- Outflows: basic concepts; collimated wind and jet phenomena
- Energy-, momentum- and radiation pressure-driven outflow solutions
- Impact of outflows on host properties

Prerequisites

Good knowledge of material covered in Part II courses: Astrophysical fluid dynamics, stellar dynamics and structure of galaxies, structure and evolution of stars is required. Some knowledge of (thermo)dynamics, electromagnetism and galaxy formation is advantageous but not strictly necessary.

Literature

1. Frank, J., King, A. R., & Raine, D., *Accretion Power in Astrophysics*, Cambridge University Press, 2002.

2. Netzer, H., *The Physics and Evolution of Active Galactic Nuclei*, Cambridge University Press, 2013.

3. Misner C. W., Thorne K. S., & Wheeler J. A., *Gravitation*, W. H. Freeman and company, 1973.

4. Pringle, J. E., & King, A. R., *Astrophysical Flows*, Cambridge University Press, 2007.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Planetary System Dynamics (M24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of 1000s of extrasolar planets and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems.

Specific topics to be covered include:

1. Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation

2. Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits

3. Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift
4. Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits
5. Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide
6. Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces
7. Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments
8. Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems
9. Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap

Pre-requisites

This course is self-contained.

Literature

1. Murray C. D. and Dermott S. F.,*Solar System Dynamics.* Cambridge University Press, 1999.
2. Tremaine S., *Dynamics of Planetary Systems.* Princeton University Press, 2023.
3. Armitage P. J., *Astrophysics of Planet Formation.* Cambridge University Press, 2010.
4. de Pater I. and Lissauer J. J., *Planetary Sciences.* Cambridge University Press, 2010.
5. Valtonen M. and Karttunen H., *The Three-Body Problem.* Cambridge University Press, 2006.

6. Seager S., *Exoplanets.* University of Arizona Press, 2011.

7. Perryman M., *The Exoplanet Handbook.* Cambridge University Press, 2011.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Radiative processes in astrophysical plasma

Non-Examinable (Part III Level)

Dr. G. Del Zanna

The main aim of this non-examinable 16-lecture Part-III course is to provide a broad overview of the physics underlying the main radiative processes occurring in astrophysical plasma. We will focus first on emission line diagnostics, which mostly covers the lower-energies, then discuss the continuum emission. We provide general background theory but devote a good fraction of the lectures to applications, to provide an insight on current research in astrophysics based on high-resolution spectroscopy

The course will cover the following topics:

- Introduction to Radiation quantities and examples of spectra of astrophysical plasma, from the X-rays to the near infrared. An introduction to the background theory and the plasma diagnostics using line and continuum emission.

- Radiation and matter in thermodynamical equilibrium: black-body radiation, thermal plasma, Maxwell, Boltzmann, Saha distributions.

- Basics of radiative transfer. The interaction between atoms and radiation: spontaneous emission and absorption, stimulated emission, Einstein's coefficients and their relation with the absorption and emission coefficients. Spectral line profiles and line broadening effects. Optical depth and opacity of a spectral line.

- Basics of atomic structure for complex atoms. Non-relativistic Breit-Pauli vs. relativistic (Dirac) approach. Central field approximation, coupling schemes, good quantum numbers. Transition probabilities, dipole approximation, selection rules, forbidden lines. The formation of the periodic table.

- Atomic data (wavelengths, A-values, identifications) and their uncertainties. Examples.

- Fundamental processes affecting the level populations within an ion. Collisional excitation due to impact by electrons and protons and reverse processes. Basics of scattering calculations. Cross-sections and rates.

- Statistical equilibrium equations. LTE vs. NLTE. Level population within an atom. Two-level atom. Metastable levels and the principle of measuring electron densities from line ratios.

- Photoexcitation, resonant scattering, Doppler dimming in moving plasma.

- Fundamental processes affecting the ion charge state distributions in plasmas. Collisional ionization by electrons and protons, photoionization. Dielectronic and radiative recombination. Cross-sections and reverse processes. Rates. Ion charge state distributions.

- Density-dependent effects on the ion charge state distributions. Non-equilibrium effects: ionising and recombining plasma, non-Maxwellian electron distributions.

- Measuring electron densities and temperatures in low-density plasma using emission lines e.g. in planetary nebulae, galaxy cluster hot plasma, stellar coronae.

- Techniques for measuring temperature distributions and chemical abundances using emission lines. Examples of solar/stellar spectra.

- Theory of the X-ray satellite lines for He-like ions, formed by inner-shell excitation and dielectronic recombination. Diagnostic applications: how to directly measure electron temperatures, the ionization state of the plasma and non-thermal electrons. The Perseus cluster observed by Hitomi.

- Continuum radiation. Free-free, free-bound. Measuring electron temperatures from continuum emission.

- Cyclotron and synchrotron emission and absorption. Examples of synchrotron emission in astrophysical plasma. Thompson scattering, Compton scattering, inverse Compton scattering.

- Active Galactic Nuclei: accretion disks, broad and narrow emission line spectra, Compton reflection, X-ray emission and the 6 keV iron feature.

Prerequisites

his course assumes you are familiar with Principles of Quantum Mechanics. It would be useful to have knowledge of Atomic structure (at least the Hydrogen atom), and of Elements of quantum field theory.

Literature

1. Del Zanna, G. and Mason, H.E., *Solar UV and X-Ray Spectral Diagnostics*, Living Reviews in Solar Physics,2018, 15,5, DOI: 10.1007/s41116-018-0015-3
https://link.springer.com/article/10.1007/s41116-018-0015-3

2. Landi Degl' Innocenti, E., *Atomic Spectroscopy and Radiative Processes*, 2014, Springer.

3. Rybicki, G. B. and Lightman, A. P., *Radiative processes in astrophysics.*, 1979, New York, Wiley-Interscience.

4. Ghisellini, G., *Radiative processes in high energy astrophysics*, 2013.
http://adsabs.harvard.edu/abs/2013LNP...873.....G

Structure and Evolution of Stars (M24)

Professor C. A. Tout

The structure of a star can be mathematically described by certain differential equations which can be derived from the principles of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics. Some familiarity with these theories will be assumed.

The basic equations of a spherical star will be derived in detail and the mode of energy transport, the equation of state, and the physics of the opacity sources and the nuclear reactions will be discussed.

Approximate solutions of the equations for stellar structure will be given. Attention will be given to the virial theorem, polytropic gas spheres and homology principles. The procedure for numerical solution of the equations will be mentioned briefly.

The evolution of a star will be discussed with reference to its main-sequence evolution, the exhaustion of various nuclear fuels and the end points of evolution such as white dwarfs, neutron stars and black holes.

Close binary stars are two stars in a gravitationally bound system in which the ratio of orbital separation to stellar radius is sufficiently small for the stars to interact through tides and mass transfer. The concept of Roche lobe overflow and its stability will be discussed, as well as the effects on the stellar evolution. Algols and Cataclysmic Variables will be considered as observed examples.

Throughout the course, reference will be made to the observational properties of stars and these will be discussed at appropriate times with particular reference to the Hertzsprung–Russell diagram, the mass-luminosity law and spectroscopic information.

There will be four examples sheets each of which will be discussed during an examples.

Prerequisites

A basic understanding of the principles of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics.

Preliminary Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Literature

1. J. J. Eldridge and C. A. Tout, *The Strusture and Evolution of Stars* World Scientific, 2019.
2. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
3. J. E. Pringle and R. A. Wade *Interacting Binary Stars*, CUP, 1985.

Additional support

There will be four examples sheets each of which will be discussed during an examples class. There will be a revision class in the Easter term.

The Life and Death of Galaxies (L24)

Professor V. Belokurov

This course will provide the observational perspective on the evolution of galaxies. A large variety of the currently available data will be discussed as well as the theoretical and numerical tools necessary to interpret it.

The course will cover a selection of topics including:

- **Dynamics.** Dynamics of elliptical galaxies. Distribution Functions. Collisionless Boltzman Equation (CBE). Integrals of CBE. Jeans Equations. Mass-anisotropy-density degeneracy. Virial Theorem. Poisson Equation.

Slow and Fast rotators. Dynamical Friction. Fundamental and Mass planes. Relaxation and phase mixing. Violent relaxation. Dynamics of spiral galaxies. Rotation curves. Dark matter. Navarro-Freak-White density profiles. Galactic halos. Dark matter structure and sub-structure in the Local Group. Dwarf galaxies as dark matter laboratories.

- **Structure and evolution.** Structure of galaxies. Two types of galaxies: dead and alive. Light distribution. Density de-projection. Sersic profile and its modifications. Detailed structure of elliptical galaxies. Cusp scouring. 3D shapes of ellipticals. Galaxy luminosity function. Schechter function. Connection between galaxy type/structure and environment. Mergers. Tides. Stellar streams. Accretion signatures in the Local Group. Formation and evolution of elliptical galaxies.

- **Spectral Energy Distribution.** Panchromatic view of galaxies. Dependance of the galactic structure on wavelength. Components of the galactic SED. Stars. Dust. Nebular emission. Stromgren sphere. Spectral lines. Initial mass function: its shape and observational constraints. Links between SEDs (continuum, absorption and emission lines) and star-formation history, the current (recent) star-formation, stellar and gas contents. Age-metallicity degeneracy.

- **Star-formation.** Star-formation rates. Metal-enrichment. Galactic chemical evolution modes. Yields. Closed box, instantaneous one-zone mixing. G-dwarf problem. Inflows and Outflows. Mass-metallicity relation. Star-Formation Law. Cloud collapse and fragmentation. Star formation on sub-galactic scales. Quenching. Star-formation history of the Universe. Integrated light: Look back vs Archaeology. Halo and galaxy assembly in the Cold Dark Matter Universe. Galaxy formation efficiency as a function of galaxy mass. SFH with resolved stellar populations. Stellar Feedback. Dwarf galaxies and early star formation.

- **Active Galactic Nuclei and Quasars.** Evidence for (S)MBHs in the centres of galaxies. Energy production. BH growth. Co-evolution of central BHs and galaxies. M-sigma relation. AGN feedback.

Prerequisites

Knowledge of Galactic Dynamics (e.g. Part II Astrophysics Course "Stellar dynamics and Structure of Galaxies") would come in handy. Some facts from Part II's "Structure and Evolution of Stars" will also be used.

Literature

1. Binney, J., and Tremaine, S. *Galactic Dynamics* Second Edition. Princeton University Press, 2008.

2. Mo, H., van den Bosch, F., and White, S. *Galaxy Formation and Evolution* Cambridge University Press, 2010.

3. Sparke, L., and Gallagher, J. *Galaxies in the Universe* Second Edition. Cambridge University Press, 2007.

4. Longair, M. *Galaxy Formation* Second Edition. Springer, 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

6 Combinatorics

Combinatorics (M16)

Professor B. Bollobás

I hope to cover the topics below, in the order indicated. If time remains, which is most unlikely, there are plenty of further exciting topics to present.

Part I.

- Basic Results
- Sperner families and the MBL–Inequality.
- The Erdős–Ko–Rado Theorem.
- The Sauer–Shelah–Perles Inequality.
- The Bollobás Inequality and its extension by Frankl.
- The Kruskal–Katona Inequality and its simplified version, due to Lovász.
- The Box Theorem of Bollobás and Thomason and its applications.
- The Cauchy–Davenport Inequality and its extension, and the Erdős–Ginzburg–Zif Theorem.

Part II.

- The Polynomial Method

- Alon's Combinatorial Nullstellensatz and its applications, including Chevalley's Theorem and the Cauchy-Davenport Theorem.
- The Erdős-Heilbronn Conjecture, proved by the Dias da Silva and Hamidoune. The simple proof of Alon, Nathanson and Ruzsa.
- The conjecture of Kemnitz and Rónyai's theorem.
- The Croot–Lev–Pach lemma and the Ellenberg–Gijswijt theorem.

Pre-requisites

Mathematical maturity and love of combinatorial arguments would be welcome. Familiarity with the Part II Graph Theory course is an asset, but not necessary.

Literature

For the 'classical' results in extremal combinatorics, we shall use my old book below, which was written for Part III courses. This short book is soon to be updated and enlarged with Imre Leader. For the more recent results in the course, most of the original papers are listed below.

1. Alon, N., Combinatorial Nullstellensatz, in *Recent Trends in Combinatorics* (Mátraháza, 1995), *Combin. Probab. Comput.* **8** (1999), 7–29.

2. Bollobás, B., On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16** (1965) 447–452.

   Bollobás, B., *Combinatorics – Set Systems, Hypergraphs, Families of Vectors and Combinatorial Probability*, Cambridge University Press, Cambridge, 1986. xii+177 pp.

3. Bollobás, B. and A Thomason, Projections of bodies and hereditary properties of hypergraphs, *Bull. London Math. Soc.* **27** (1995) 417–424.

4. Croot, E., V.F. Lev, and P.P. Pach, Progression-free sets in $\mathbf{Z}_n^4$ are exponentially small, *Ann. of Math.* **185** (2017) 331–337.

5. Ellenberg, J.S., and D. Gijswijt, On large subsets of $\mathbf{F}_q^n$ with no three-term arithmetic progression, *Ann. of Math.* (2) **185** (2017) 339–343.

6. Erdős, P., Chao Ko and R. Rado, Intersection theorem for system of finite sets, *Quart. J. Math. Oxford* **12** (1961) 313–318.

7. Frankl, P., An extremal problem for two families of sets, *European J. Combinatorics* **3** 125–127.

8. Lubell, D., A short proof of Sperner's lemma, *J. Combinatorial Theory* **1** (1966), 299.

9. G.O.H. Katona, A Simple Proof of the Erdijs-Chao Ko-Rado Theorem, *Journal of Combinatorial Theory (B)* **13** (1972) 183-184.

10. Sauer, N., On the density of families of sets, *J. Combinatorial Theory Ser. A* **13** (1972) 145–147.

11. Shelah, S., A combinatorial problem; stability and order for models and theories in infinitary languages, *Pacific J. Math.* **41** (1972), 247–261.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

Introduction to computational complexity (L16)

Professor W. T. Gowers

Computational complexity is the study of the resources needed to perform computational tasks. Two resources of obvious importance are the time needed, which corresponds to the number of steps an algorithm needs to take, and the amount of storage needed. A less obvious one is the "amount of randomness" needed by a randomized algorithm. Over the last 50 years, the subject has developed in many important directions, so in one course it is not possible to cover more than a small fraction of it, even at an introductory level. Thus, while this course has the same title as a course given last year, its content will be substantially different: last year the focus of the course was on completeness results and on lower bounds for circuit complexity, while this year there will be a strong emphasis on the role played in the theory by randomness and pseudorandomness.

The following list of topics is a rough guide to what will be covered (but some topics listed here may not end up in the course and others not listed may).

- Some of the basic complexity classes, including P, NP, PSPACE, AC, NC, P/poly, RP, and BPP, and relationships between them.
- Arithmetic complexity.

- The power of randomness.
- Derandomization.
- Pseudorandom generators.
- Interactive proof systems.

Prerequisites

Familiarity with the basics of (discrete) probability and graph theory will be assumed. Otherwise, there are few prerequisites.

Literature

1. Sanjeev Arora and Boaz Barak, *Computational complexity: a modern approach*, CUP, 2009. Also available in draft form: https://users.cs.duke.edu/ reif/courses/complectures/bo
2. Oded Goldreich, *Computational complexity: a conceptual perspective* CUP, 2008. Also available online in draft form: https://www.wisdom.weizmann.ac.il/ oded/cc-drafts.html

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Introduction to Additive Combinatorics (L16)

Professor J. Wolf

This course provides an introduction to additive combinatorics, a fast-moving area of mathematics motivated by long-standing questions about additive structure in sets of integers. It will attempt to reflect both the elementary appeal of the key problems in the area as well as the breadth of techniques (analytic, combinatorial, algebraic, probabilistic) used to tackle them. By the end of the course, students should be able to appreciate various recent breakthroughs in the area.

The course will be organised along the following lines.

- Part I: Fourier-analytic techniques
  Bogolyubov's lemma; Roth's theorem on 3-term arithmetic progressions

- Part II: Combinatorial methods
  Plünnecke's inequality for sum sets; the Freiman-Ruzsa theorem
- Part III: Probabilistic methods
  the Balog-Szemerédi-Gowers theorem; Croot-Sisask almost periodicity and applications
- Part IV: Further topics (if time permits)
  algebraic constructions and obstructions; beyond 3-term progressions

Prerequisites

This course assumes basic familiarity with finite abelian groups, linear algebra, discrete probability and graph theory, at a level found in most undergraduate mathematics degrees. As such, it should be accessible to a broad range of students.

Literature

The following books give a flavour of the course, but should not be regarded as compulsory or definitive reading.

1. T. Tao and V. Vu. *Additive Combinatorics* (Cambridge Studies in Advanced Mathematics). Cambridge University Press, 2006.
2. Y. Zhao. *Graph Theory and Additive Combinatorics: Exploring Structure and Randomness.* Cambridge University Press, 2023.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Ramsey theory on graphs (M16)

Dr J. Sahasrabudhe

This course will introduce several of the fundamental methods in modern combinatorics by focusing on the Ramsey numbers $R(\ell, k)$ and related objects. In this course I aim to cover most of the following topics:

- **Upper bounds for classical ramsey numbers** Erdős-Szekeres upper bound on $R(\ell, k)$; The Ajtai-Komlos-Szemeredi theorem when ℓ is bounded; recent progress on $R(k)$.

- **Lower bounds for classical ramsey numbers** Erdős' lower bound on $R(k)$; The Lovász Local lemma and $R(3,k)$; The method of Graph containers and the Alon-Rődl technique for multicolour Ramsey numbers; Recent lower bounds on $R(4,k)$.

- **Hypergraph ramsey numbers** Erdős-Hajnal upper bounds for diagonal hypergraph ramsey; the "stepping up" method.

- **Szemerédi's regularity lemma**
  Basic statement and philosophy; triangle removal lemma; Roth's theorem; Ramsey numbers of bounded degree graphs.

- **Dependent random choice**
  Extremal numbers of bipartite graphs; Ramsey-Turán; Ramsey numbers of hypercubes.

- **Related colouring topics**
  Graphs with high girth and chromatic number; hypergraphs with property B.

Prerequisites

Familiarity with the topics and flavor of Part II graph theory (or an equivalent course) is useful and recommended although not strictly required.

Literature

This course draws from various sources, and there is no one good resource for course. However, many of the topics are standard and one can find lecture notes online. The first few lectures will see quite a bit of overlap with David Conlon's course on the same topic. Then the latter part of the course is also covered in R. Morris and R.I. Oliveira's lecture notes. The The probabilistic method is a good source for more probabilistic examples we are using.

1. N. Alon and J. Spenser *The Probabilistic Method*, Wiley (any edition).

2. D. Conlon *Graph Ramsey Theory*, Lecture notes available at http://www.its.caltech.edu/ dconlon/Ramsey course.html.

3. R. Morris and R.I. Oliveira *Extremal and Probabilistic Combinatorics*, available at https://impa.br/wp-content/uploads/2017/04/28CBM$_0$4.*pdf*.1993.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extremal Graph Theory (E12)

Non-Examinable (Part III Level)

Dr O. Janzer

Extremal Graph Theory studies questions of the following kind: at most how many edges can a graph on n vertices have if it satisfies a certain property? One of the most important instances of this problem is the question of how many edges an n-vertex graph can have without containing some given graph H as a subgraph (called the *extremal number* of H). Turán's theorem answers this question exactly for complete graphs, and the Erdős–Stone theorem determines the asymptotics whenever H is not bipartite. The bipartite case remains open in general, but there are many important and interesting bipartite graphs H for which the above question has been answered.

The course will cover a selection of topics including:

- The Erdős–Stone theorem
- Stability and supersaturation
- Extremal number of complete bipartite graphs
- The Erdős–Simonovits regularization lemma
- Extremal number of cycles
- Dependent random choice and Füredi's theorem on bipartite graphs with bounded degree on one side
- Extremal number of subdivisions of complete graphs and a connection to Ramsey numbers

Prerequisites

Familiarity with basic concepts and techniques in Graph Theory will be assumed.

Literature

1. B. Bollobás, *Extremal Graph Theory.* Dover Books on Mathematics.
2. Z. Füredi and M. Simonovits, *The history of degenerate (bipartite) extremal graph problems*, Erdős centennial, 2013. Also available at https://arxiv.org/abs/1306.5167.

Additional support

Office hours will be offered.

7 Continuum Mechanics

Direct and Inverse Scattering of Waves (L16)

Dr O. Rath Spivack

The study of wave scattering is concerned with how the propagation of waves is affected by ob-jects, and has a variety of applications in many fields, from environmental science to seismology, medicine, telecommunications, materials science, and many others. If we know the nature of the objects and we want to find how an incident wave is scattered, we call this a 'direct scattering problem' and practical applications will include for example underwater sound propagation, light transmission through the atmosphere, or the effect of noise in built-up areas. If we measure and know the scattered field produced by an incident wave, but we do not know the nature of the objects that have scattered it, we call this an 'inverse scattering problem' and applications will include for example non-destructive testing of materials, remote sensing with radar or lidar, or medical imaging.

This course will provide the basic theory of wave propagation and scattering and an overview of the main mathematical methods and approximations, with particular emphasis on inhomo-geneous and random media, and on the regularisation of inverse scattering problems. It is intended as a kind of bridge between the Applied and Computational Analysis courses and the Continuum Mechanics courses and should appeal to students interested in waves as well as in inverse problems.

The course will cover a selection of topics including:

- Boundary value problems and the integral form of the wave equation.
- The parabolic equation and Born and Rytov approximations for the scattering problem.
- Scattering by randomly rough surfaces and propagation in inhomogeneous media.
- Ill-posedness of the inverse scattering problem, and the Moore-Penrose generalised inverse.
- Regularisation methods and methods for solving some inverse scattering problems

- Time reversal and focusing in inhomogeneous media.

Prerequisites

This course assumes basic knowledge of PDEs, and of Fourier transforms. Some familiarity with linear algebra and with basic concepts in functional analysis is helpful, though by no means necessary.

The Part III course Inverse Problems is a useful complement to part of this course, although not a prerequisite.

Preliminary Reading

These can provide some useful background:

1. C.W. Groetsch *Inverse Problems in the Mathematical Sciences.* Braunschweig 1993.
2. L.D. Landau and E.M. Lifschitz *Fluid Dynamics,* [Chapter 8]. Pergamon, 1987. Also available at https://users-phys.au.dk/srf/hydro/Landau+Lifschitz.pdf.

Literature

1. D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory.* Springer, 1992.
2. D.G. Crighton et al, *Modern Methods in Analytical Acoustics.* Springer, 1992.
3. H.W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problem.* , Kluwer, 2000
4. BA. Ishimaru, *Wave Propagation and Scattering in Random Media.* Academic Press, 1978.
5. B. Uscinski, *The Elements of Wave Propagation in Random Media.* McGraw-Hill, 1977

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Demonstrations in Fluid Mechanics. (L8)Prof. S.B. Dalziel

While the equations governing most fluid flows are well known, they are often very difficult or impossible to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The main purpose of this demonstration course is to help develop an intuitive 'feeling' for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed. Although the course is entitled 'demonstrations', the course has a history of including some novel experiments that have not, to our knowledge, been performed anywhere previously. Some of these have subsequently led to PhD projects. The course thus offers an introduction into how experiments can provide insight into previously unexplored (or undiscovered) phenomena.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves, internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- avalanches, granular flows, sedimentation and resuspension;
- flows in porous, absorbent and elastic media;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low-Reynolds-number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account, although participation is encouraged.

Prerequisites

Undergraduate Fluid Dynamics.

Literature

1. M. Van Dyke. *An Album of Fluid Motion.* Parabolic Press.
2. G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. *Multimedia Fluid Mechanics.* (Multilingual Version CD-ROM), CUP.
3. M. Samimy, K. Breuer, P. Steen, & L. G. Leal. *A Gallery of Fluid Motion.* CUP.

Fluid dynamics of the solid Earth (L24)

Prof. M. Grae Worster

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth's core is formed by the solidification of a mixture of molten iron and various lighter elements, a process that drives compositional convection in the liquid outer core, thus producing the geodynamo responsible for Earth's magnetic field. On million-year timescales, the solid mantle convects, and as it upwells to the surface it partially melts leading to volcanism. At the surface, convection drives the motion of brittle plates, which are responsible for the Earth's topography as can be felt and imaged through the seismic record. In the Earth's crust, fluids flow through porous rocks, forming groundwater aquifers that feed streams and rivers that erode the solid surface. On the Earth's surface, similar physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of polar oceans to form sea ice to the flow of glacial ice over land and the ocean.

This course will use observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through and deformation of porous materials. The focus will be on the generation and solution of mathematical models motivated by observational data.

The course will cover a selection of topics including:

- Growth of Earth's inner core
- Groundwater flows
- Formation and evolution of sea ice
- Carbon capture and storage
- Stability of marine ice sheets

Prerequisites

Mathematical methods, particularly the solution of ordinary and partial differential equations. An understanding of viscous fluid dynamics.

Literature

1. M.G. Worster. *Solidification of Fluids.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP, 2000.
2. H.E. Huppert. *Geological fluid mechanics.* In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP, 2000.
3. D.L. Turcotte, G. Schubert. Geodynamics, second edition. CUP, 2002.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Hydrodynamic Stability (L16)

Professor R. Kerswell

Developing an understanding of how "small" perturbations grow, saturate and modify fluid flows is central to addressing many challenges of interest in fluid mechanics. Furthermore, many applied mathematical tools of much broader relevance have been developed to solve hydrodynamic stability problems, and hydrodynamic stability theory remains an exceptionally active area of research, with several exciting new developments being reported over the last few years.

In this course, an overview of some of these recent developments will be presented. After a brief introduction to the general concepts of flow instability, presenting a range of examples, the major content of this course will be focussed on the broad class of flow instabilities where velocity "shear" and fluid inertia play key dynamical roles. Such flows, typically characterised by sufficiently "high" Reynolds number Ud/ν, where U and d are characteristic velocity and length scales of the flow, and ν is the kinematic viscosity of the fluid, are central to modelling flows in the environment and industry.

A hierarchy of mathematical approaches will be discussed to address a range of stability problems, from more classical concepts of normal mode growth on laminar parallel shear flows and their subsequent weakly nonlinear behaviour, to transient but significant growth of infinitesimal perturbations and finite amplitude instabilities.

Pre-requisites

Undergraduate fluid mechanics, linear algebra, complex analysis and asymptotic methods.

Literature

1. F. Charru *Hydrodynamic Instabilities* CUP 2011.
2. P. G. Drazin & W. H. Reid *Hydrodynamic Stability* 2nd edition. CUP 2004.
3. P. J. Schmid & D. S. Henningson, *Stability and transition in shear flows.* Springer, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be an hour and a half revision class in the Easter Term.

Fluid Dynamics of the Environment (24)

Dr R. K. Bhagat and Dr Daria Frank

Understanding the environment is a critical challenge. Whether we are concerned about climate change, pollution or sediment transport, the high-Reynolds-number fluid dynamics of buoyant flows plays a vital role. In this course, we will focus on flows that occur over sufficiently small time and length scales to be largely unaffected by the Earth's rotation.

The course will begin by considering internal structure of stratified flows. In particular, we will discuss internal gravity waves, their reflections, energy propagation, as well applications to a stratified atmosphere. We will then proceed by studying shallow water flows, a class of 'thin' flows where near-horizontal motion is driven by density differences. These shallow flows can be steady or time dependent, with the density differences arising from temperature, composition, suspended particles, or combinations of these. The cold air entering a room when you open the door on a cold day and the dust cloud from demolishing a building are just two examples. The flows can propagate along interfaces within the fluid, horizontal or sloping boundaries. In many cases, especially as the slope of the boundary increases, turbulence plays an important role, reducing the density differences driving the flow.

Thin flows also arise where there is a localised source of buoyancy without a boundary along which it can flow. Rather than forming a near-horizontal shallow water flow, a vertical flow in the form of a buoyant turbulent plume or a buoyant thermal develops. Such flows are widespread in both the natural and built environment (e.g. from a volcanic eruption or above a radiator in your college room).

Drawing together these thin flows, we will delve further into the fluid dynamics of indoor environments. With an increasing fraction of the world's population residing in cities and spending 90% of their time indoors, creating sustainable urban spaces is essential. Combating the climate crisis through decreasing energy consumption is one imperative, but another is improving indoor air quality due to its impact on many aspects of health and productivity. As poor ventilation played an important role in disease transmission during the recent Covid-19 pandemic, we introduce approaches to modelling the airborne transmission that is is important for many of the most infectious diseases.

Although this course is not primarily about turbulence, we find that turbulence inevitably plays a role in both indoor and outdoor environmental flows and so we shall touch on key aspects of turbulence and how it is modified by stratification at various points during the course

Pre-requisites

Undergraduate fluid dynamics is desirable.

Literature

- J. S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Fluid Dynamics of Climate (M24)

Prof. PH Haynes, Dr A Ming and Prof JR Taylor

Understanding the Earth's climate and predicting its future evolution is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the climate system, helping to make the planet hospitable for life. The dynamical complexity of this fluid motion and the wide range of space and time scales involved is one of the most difficult aspects of climate prediction.

This course, focusing on the large-scale behaviour of stratified and rotating flows, provides an introduction to the fluid dynamics necessary to build mathematical models of the climate system. The course begins by considering flows which evolve on a timescale which is long compared with a day, where the Earth's rotation plays an important role. The rotation is felt through the Coriolis force (a fictitious force arising from use of a frame of reference rotating with the Earth) which causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics, particularly in combination with stable density stratification. Canonical models are introduced and studied to illustrate phenomena such as adjustment to a state of geostrophic balance, where Coriolis force balances pressure gradient, new wave modes that can communicate dynamical information on both regional and global scales, and new hydrodynamic instabilities that lead to atmospheric weather systems and ocean eddies.

The course then moves on apply these basic ideas to important aspects of the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the structure and hence the effects of ocean eddies and atmospheric weather systems, the dynamics of the meridional (north/south) circulation in the ocean and atmosphere and the associated transport of heat and of chemical and biological tracers. The final part of the course will consider the special dynamics of tropical regions which give rise to phenomena such as El Nino.

Prerequisites

Undergraduate fluid dynamics

Literature

The following provide some introductory reading and will also complement the detailed presentation of material given in lectures.

1. G.K. Vallis, *Atmospheric and Oceanic Fluid Dynamics.* Cambridge University Press, 2017.

2. A.E. Gill, *Atmosphere-Ocean Dynamics.* Academic Press, 1982.

3. J. Marshall and R.A. Plumb, *Atmosphere, Ocean, and Climate Dynamics.* Academic Press, 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Perturbation Methods (M16)

Prof. I.D. Abrahams

This course will deal with the asymptotic solution to problems in applied mathematics when some parameter or coordinate assumes large or small values. Many problems of physical interest are covered by such asymptotic limits; indeed, they play a crucial role in the field of mathematical modelling. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases can provide accurate predictions even when the parameter or coordinate has only moderately large or small values.

Some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, with the remaining lectures devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the ‘divide & conquer’ strategy, Laplace’s method, stationary phase and steepest descents. [6]

- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the 'WKB[JLG]' method can be viewed as a special case.) It is a systematic method, capable of extension in many ways, and includes such ideas as those of 'averaging' and 'time scale distortion' in a natural way. A number of applications will be studied, as time allows, potentially including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium.) [5]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. The method was first developed in fluid mechanics, where it is referred to as boundary-layer theory, but it has since been greatly extended and applied to many fields. [5]

Prerequisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no (or only elementary) knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward ordinary & partial differential equations and evaluate reasonably simple integrals.

Literature

Relevant Textbooks (not required for the course, but which may assist)

1. C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers.* McGraw-Hill, 1978. *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to 'Stokes' lines as 'anti-Stokes' lines, and vice versa. The course will use Stokes' convention.*

2. E.J. Hinch. *Perturbation Methods.* Cambridge University Press, 1991. *The lecture course was originally based on this book; some may view it as somewhat concise.*

3. M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics.* Parabolic Press, Stanford, 1975. *This is the original book on perturbation methods; somewhat dated, but still an excellent read.*

4. J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer, 1981. *A useful and accessible book, covering a range of asymptotic methods, and applications.*

Additional Reading on Topics Beyond the Syllabus

1. M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. A, **427**, 265–280 (1990).

2. J.P. Boyd. *The Devil's invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, 56, 1-98 (1999). Also available at http://hdl.handle.net/2027.42/41670 and http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf.

Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.

Slow Viscous Flow (M24)

Prof. J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media, such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Pre-requisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the Einstein summation convention is particularly useful for the first half of the course.

Preliminary Reading

1. D.J. Acheson. *Elementary Fluid Dynamics.* OUP (1990). Chapter 7
2. G.K. Batchelor. *An Introduction to Fluid Dynamics.* CUP (1970). pp.216–255.
3. L.G. Leal. *Laminar flow and convective transport processes.* Butterworth (1992). Chapters 4 & 5.

Literature

1. J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics.* Kluwer (1965).

2. S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications.* (1993)

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

8 Soft Matter and Biological Physics

Theoretical Physics of Soft Condensed Matter (L16)

Mike Cates

Soft Condensed Matter refers to liquid crystals, emulsions, colloidal suspensions and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, generally with noise so that the equations of motion are stochastic PDEs. The form of these equations is helpfully constrained by a requirement that the Boltzmann distribution is regained in the steady state. Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a 'smectic liquid crystal' (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field (and one step short of the renormalization group, which we leave to the Statistical Field Theory course).

Important dynamical models of soft matter include diffusive ϕ^4 field theory ('Model B'), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates 'Model H', a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth. We will also explore the continuum theory of nematic and polar liquid crystals, which spontaneously break rotational but not translational symmetry, focusing on topological defects and their associated mathematical structure such as homotopy classes.

Finally, the course will briefly analyse soft-matter systems whose microscopic dynamics involves the continuous dissipation of energy, such as self-propelled colloidal swimmers. We will discuss how their absence of time-reversal symmetry leads to qualitative changes in dynamical behaviour.

Prerequisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. It is not necessary to attend any other particular part III courses but there is potential synergy with Statistical Field Theory, Slow Viscous Flow, Stochastic Processes in Biology, and to a lesser extent some other courses too. For field theorists, this is a good opportunity to see what statistical field theory looks like when addressing time evolution rather than the purely static properties set by the Boltzmann distribution. For people interested in fluid mechanics, it is a good opportunity to see what this looks like beyond Navier-Stokes fluids, in systems with thermal noise and/or structural order parameters. In previous years the audience has included a mix of students whose main specialism is either field theory or fluid dynamics. People with these differing backgrounds may find different parts of the course easier or harder, but the intention is to create a roughly level playing field so that all can enjoy learning about this interdisciplinary field.

Literature

1. D. Tong *Lectures on Statistical Physics*

   http://www.damtp.cam.ac.uk/user/tong/statphys.html

   Before embarking on this course you need to understand the equation $F = -k_B T \ln Z$ and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Acquaintance with the Landau theory of phase transitions is helpful. David Tong's lecture notes are an excellent resource for revising and reviewing the key material.

2. M. E. Cates and E. Tjhung *Theories of binary fluid mixtures: from phase-separation kinetics to active emulsions.* J. Fluid Mech. (2018), **836**, pp 1-66.

https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/theories-of-binary-fluid-mixtures-from-phaseseparation-kinetics-to-active-emulsions/5BD133CB20D89F47E724D77C296FEF80/share/106fd30f307db12134745de39fd568fbbaa3f9d2

This JFM perspectives article has significant overlap (perhaps 40%) with the course but takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics as needed. It gives a flavour of the types of problem we will address and some of the methodologies involved. However we will not have time to cover much of the material it contains on active systems.

3. P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics.* Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, and topological defects all feature in the description of condensed matter systems at $\hbar = 0$. This is more for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

4. Unofficial lecture notes were taken several years ago:

   https://dec41.user.srcf.net/h/III$_L/theoretical_physics_of_soft_condensed_matter$

These notes are far from perfect but are mentioned here so that all students are aware of their existence.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a 90-minute revision class in the Easter Term. The lecturer will be available at other times by arrangement.

Note on past papers

Starting in the 2016/7 academic year, this course was given in 16-lecture format for 3 years, then as a 24-lecture course for 2 years, before reverting to 16 lectures from the 2021/2 academic year onwards. This means the summer 2021 tripos paper examined some material that lies beyond the current syllabus. (There was no honours paper in 2020.)

Biological Physics and Fluid Dynamics (M24)

Professor R.E. Goldstein

This course will provide an overview of the physics and fluid dynamics of living systems, with an emphasis on their mathematical description. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology.

The topics to be covered span the range of length scales from molecular to ecological, with emphasis on key paradigms. They include:

- Microscopic Physics –van der Waals forces, screened electrostatics, interactions of extended bodies;
- Fluctuations and Brownian Motion –Equipartition, fluctuating continuous objects, diffusion, polymers and entropic forces;
- Biological Filaments –differential geometry, elasticity, constraints, variational principles, elastohydrodynamics, instabilities;
- Membranes –differential geometry, Helfrich model, fluctuations, shapes and shape transformations;
- Cellular Motion –cell swimming, flagella and low-Re locomotion, hydrodynamic interactions;
- Biological Pattern Formation –chemical kinetics, reaction-diffusion systems, Turing instability, chemotaxis.

Prerequisites

Statistical physics, fluid mechanics, electromagnetism, and dynamical systems constitute the appropriate background for this course.

Literature

1. R.E. Goldstein and E. Lauga. *Biological Physics and Fluid Dynamics: A Graduate Course in 24 Lectures* (2023). Available at https://www.damtp.cam.ac.uk/user/gold/pdfs/teach
2. P. Nelson. *Biological Physics.* W.H. Freeman 2007.
3. J.D. Murray. *Mathematical Biology I. & II.* Springer 2007, 2008.
4. J.N. Israelachvili. *Intermolecular and Surface Forces.* 2nd edition. Academic Press 1992.

5. E.J.W. Verwey and J.Th.G. Overbeek. *Theory of the Stability of Lyophobic Colloids*. Elsevier 1948.

6. M. Doi and S.F. Edwards. *The Theory of Polymer Dynamics*. OUP 1986.

7. E. Lauga. *The Fluid Dynamics of Cell Motility*. CUP 2020.

8. D. Bray. *Cell Movements*. Garland 2000.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Non-Equilibrium Statistical Field Theory (M8)

Non-Examinable (Part III Level)

Dr R. Garcia-Milan

This course introduces Master Equations – an ODE for time dependent probability distributions – as a model for reaction systems on lattices. After briefly exploring its basic properties and models such as random walkers as well as branching and coagulation processes, they are transformed into a second-quantized form using bosonic ladder operators and coherent states. The meaning and properties associated with shifts of the creation and annihilation operators are explored as well as the interpretation of observables. Then, the second quantized equation is formally solved in a path integral. At this point, Doi-Peliti field theory starts and a few example models are presented: diffusion, branching, coagulation as well as coupled reactions. Feynman diagrams, loop corrections, n-point correlation functions are all concepts which are introduced and Part III students taking *SFT* will hopefully enjoy a non-equilibrium point of view on them.

Prerequisites

It will be very beneficial to take the Part III course *Statistical Field Theory* (*SFT*) alongside this course. Although both courses talk about different statistical field theories, they explore many of the same concepts.

Literature

1. N.G. van Kampen, *Stochastic Processes in Physics and Chemistry* Elsevier, 1992

2. Uwe Täuber *Critical Dynamics* 1st edition. Cambridge University Press, 2014.

3. John Cardy, *Lecture Notes on Field Theory and Nonequilibrium Statistical Mechanics*, Available at

   https://www-thphys.physics.ox.ac.uk/people/JohnCardy/

4. Gunnar Pruessner *Lecture Notes on Non-equilibrium Statistical Mechanics* Available at

   http://wwwf.imperial.ac.uk/ pruess/publications/Gunnar$_P ruessner_f ield_t heory_n otes.pdf$

Additional support

Lecture notes and voluntary example sheets will be provided. However, since this is a non-examinable course, there will be no example classes.

Stochastic Processes in Biology (L16)

Dr M. Bruna and Dr T. Plesa

Stochastic processes are applicable to biology across a wide range of scales: from modelling molecules which interact via chemical reactions and diffusion, cells which proliferate, differentiate, die and move using active transport, all the way to modelling large groups of animals which interact and move in space according to complex rules. This course will cover mathematical methods and simulation algorithms for the analysis of such continuous-time stochastic models, including the following topics:

- Chemical reaction networks.
- Slow-fast systems and reduction to effective models.
- Random walks, Brownian motion, and reaction-diffusion processes.
- Metastability and exit problems.
- Interacting diffusion processes and collective behaviour.

For each of the topics, we will introduce the relevant stochastic simulation algorithms and how are they implemented in practice (there will be live coding examples and computational problems in the example sheets). We will also present the corresponding probabilistic descriptions in the form of systems of ODEs (chemical master equations) and PDEs (Fokker–Planck equation, backward Kolmogorov equation). We will highlight the connections between different modelling approaches (deterministic vs stochastic, discrete vs continuous).

Prerequisites

This course assumes basic knowledge of probability, ODEs and PDEs. While the computational component of the course is not assessed, some basic proficiency in programming will be useful.

Literature

1. R. Erban & S. J. Chapman, *Stochastic Modelling of Reaction-Diffusion Processes.* Cambridge University Press, 2019.
2. L. J. S. Allen, *An Introduction to Stochastic Processes with Applications to Biology.* CRC Press, 2010.
3. G. A. Pavliotis, *Stochastic Processes and Applications.* Springer, 2015.
4. C. W. Gardiner, *Handbook of Stochastic Methods.* Springer, 1985.

Additional support

Three examples sheets will be provided, and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term. There will also be one drop-off coding session in the first week.

9 Differential Geometry and Topology

Algebraic Topology (M24)

Professor I. Smith

Algebraic Topology studies topological spaces by associating to them algebraic invariants, primarily abelian groups and commutative rings. It permeates modern pure mathematics and theoretical physics. This course will focus on cohomology, with an emphasis on applications to the topology of smooth manifolds. Topics will include:

- basic theory of (co)chain complexes
- singular and cellular (co)homology
- cup-products
- vector bundles
- Thom isomorphism and the Euler class

- Poincaré duality
- and possibly cobordism.

The course will not strictly assume prior knowledge of algebraic topology, but will go quite fast in order to reach more interesting material, so some previous exposure to chain complexes and simplicial homology would definitely be helpful. Some material from the Differential Geometry course (including basics of smooth manifolds and tangent bundles) will also be used at certain points, and may need to be black-boxed by students not taking that course.

Prerequisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. Some knowledge of the fundamental group would be helpful though not a requirement. The books by Bott and Tu and by Hatcher are especially recommended, but there are many other suitable texts and many online resources.

Literature

1. Bott, R. and Tu, L. *Differential forms in algebraic topology.* Springer, 1982.
2. Hatcher, A. *Algebraic Topology.* Cambridge Univ. Press, 2002.
3. May, P. *A concise course in algebraic topology.* Univ. of Chicago Press, 1999.
4. Sutherland, W. *Introduction to metric and topological spaces.* Oxford Univ. Press, 1999.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Differential Geometry (M24)

A. Kovalev

This course is intended as an introduction to modern differential geometry. It can be taken with a view of further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are e.g. in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds.* Definition and examples of manifolds, matrix Lie groups. Tangent vectors, the tangent and cotangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Exterior algebra of differential forms. Orientability of manifolds. Partition of unity and integration on manifolds, Stokes' Theorem. De Rham cohomology.

- *Vector Bundles.* Structure group, principal bundles. The example of Hopf bundle. Bundle morphisms. Three views on connections: vertical and horizontal subspaces, Christoffel symbols, covariant derivative. The curvature form and second Bianchi identity.

- *Riemannian Geometry.* Connections on manifolds, torsion. Riemannian metrics, the Levi–Civita connection. Geodesics, the exponential map, Gauss' Lemma. Decomposition of the curvature of a Riemannian manifold, Ricci and scalar curvature, low-dimensional examples. The Hodge star and Laplace–Beltrami operator. Statement of the Hodge decomposition theorem (with a sketch-proof, time permitting).

Prerequisites

An essential prerequisite is a working knowledge of linear algebra (including dual vector spaces and bilinear forms) and of multivariate calculus (e.g. differentiation in several variables and the inverse function theorem). Previous exposure to some ideas of classical differential geometry (of curves and surfaces in $\mathbf{R}^3$) would be useful.

Literature

[1] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer, 1983.

[2] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry.* Springer-Verlag, 1990.

[3] R.W.R. Darling, *Differential forms and connections.* CUP, 1994.

[4] V. Guillemin, A. Pollack, *Differential topology.* Prentice-Hall Inc., 1974.

Roughly, half of the course material is found in [1]. The book [4] covers the required topology. On the other hand, [3] which has a chapter on vector bundles and on connections assumes no knowledge of topology. Both [2] and [3] have a lot of worked examples. There are many other good differential geometry texts.

Additional support

The lectures will be supplemented by four example classes, based on four example sheets. There will be a one-hour revision class in the Easter Term. Printed notes will be available from
https://www.dpmms.cam.ac.uk/ agk22/teaching.html

Geometric group theory (L24)

Henry Wilton

The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to the geometric features of the spaces on which the group acts. This course will provide an introduction to the basic ideas of the subject.

We will cover the following topics.

1. Free groups, group presentations and Cayley graphs
2. Group actions and quasi-isometry
3. Amalgams and Bass–Serre theory
4. Fuchsian groups
5. Hyperbolic groups

The books listed below could be used to provide alternative perspectives on some of this material.

Pre-requisites

Part IB Geometry and Part II Algebraic Topology (or equivalents) are required.

Literature

1. M. R. Bridson and A. Haefliger *Metric spaces of non-positive curvature.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999. xxii+643 pp.
2. C. Druţu and M. Kapovich, *Geometric group theory.* With an appendix by Bogdan Nica. American Mathematical Society Colloquium Publications, 63. American Mathematical Society, Providence, RI, 2018. xx+819 pp.

3. P. de la Harpe, *Topics in geometric group theory.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000. vi+310 pp.

4. C. Löh, *Geometric group theory – an introduction.* Universitext. Springer, 2017. xi+389 pp.

5. J. Meier, *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups.* London Mathematical Society Student Texts, 73. Cambridge University Press, Cambridge, 2008. xii+231 pp.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Symplectic Topology (L16)

Dr A. Ward

This course is intended as a first introduction to symplectic topology, the study of smooth manifolds equipped with a closed and non-degenerate differential 2-form. This class of spaces generalizes the phase spaces that appear in the study of classical mechanics; other instances of symplectic manifolds include real surfaces equipped with a volume form, complex algebraic varieties, and co-adjoint orbits of Lie groups. As this collection of examples might suggest, symplectic topology has important connections with many other fields of study such as algebraic geometry and low-dimensional topology. It is also interesting in its own right, and is an active area of mathematical research today.

The topics covered in this course will include:

- Symplectic linear algebra. Symplectic manifolds, symplectic/Lagrangian submanifolds. Almost complex structures and Chern classes.
- Symplectomorphisms, Hamiltonian dynamics, the flux homomorphism.
- Moser's trick, Darboux's theorem, Weinstein's neighborhood theorem.
- Constructions of symplectic manifolds including symplectic blow-ups, Lefschetz pencils, symplectic reductions.
- Introduction to pseudo-holomorphic curves. Gromov's non-squeezing theorem.

Pre-requisites

Some familiarity with complex analysis, vector bundles on smooth manifolds, and de Rham cohomology will be essential. Differential geometry and algebraic topology from Part III (or the equivalent) are strongly recommended.

Literature

1. Cannas da Silva, A. *Lectures on Symplectic Geometry.* Springer, 2001.
2. McDuff, D and Salamon, D. *Introduction to Symplectic Topology (3rd Edition).* Oxford Univ. Press, 2017.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.

10 Foundations

Category Theory (M24)

Peter Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

- **Categories, functors and natural transformations**. Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]

- **Locally small categories**. The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]
- **Adjunctions**. Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]
- **Limits**. Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]
- **Monads**. The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck's Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

- **Filtered colimits**. Finitary functors, finitely-presentable objects. Applications to universal algebra.
- **Regular categories**. Embedding theorems. Categories of relations, introduction to allegories.
- **Abelian categories**. Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.
- **Monoidal categories**. Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.
- **Fibrations**. Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Prerequisites

There are no specific prerequisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. S. Mac Lane *Categories for the Working Mathematician.* Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.

2. S. Awodey *Category Theory.* Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.

3. T. Leinster *Basic Category Theory.* Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.

4. E. Riehl *Category Theory in Context.* Dover Publications 2016. A recent account of the subject by someone who first encountered it as a Part III student fifteen years ago.

5. F. Borceux *Handbook of Categorical Algebra.* Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

A full set of written notes for the course will be provided; these contain exercises for the reader, and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Model Theory and Non-Classical Logic (M24)

Dr J. Siqueira

This course builds upon the foundations laid out by the Part II course 'Logic and Set Theory', introducing some important concepts and tools in modern Logic. It is broadly divided into a chapter dedicated to Model Theory, and another to Intuitionistic Logic.

There are two prevailing analogies that aim to explain Model Theory. The first says that 'model theory = universal algebra + logic', viewing it as a generalisation of algebra that allows for relations instead of just operations. The other sees it instead as geometrically flavoured, with logical formulae playing a role akin to equations in defining your structure of interest: 'model theory = algebraic geometry − fields'. Regardless of the point of view, the techniques we will discuss will give us a finer control over the structures described by a first-order theory.

Intuitionistic logic departs from classical logic by adopting a new interpretation for the logical connectives and quantifiers that makes it constructive. We will discuss its syntax and semantics (which requires new notions of model with accompanying completeness theorems), and see how it compares to its classical counterpart.

We will cover most or all of the following topics:

- Definable sets, elementary substructures, and the Tarski-Vaught test;
- Universal theories;
- Skolemisation and the Löwenheim-Skolem Theorem;
- Quantifier elimination;
- Ultraproducts and Łoś's Theorem;
- Nonstandard models of arithmetic;
- The Omitting Types Theorem;
- The Ehrenfeucht–Mostowski Theorem;
- Natural deduction and intuitionistic logic;
- The simply typed λ-calculus and the Curry-Howard correspondence;
- Heyting and Kripke semantics for intuitionistic logic;
- Negative translation;
- Recursive models and Tennenbaum's Theorem.

The course will primarily be of interest to students in Foundations, Combinatorics, and Algebraic Geometry.

Prerequisites

Familiarity with the contents of the Part II course *Logic and Set Theory* (or equivalent) is essential. The course will assume acquaintance with first-order logic, including its syntax and basic results (e.g., completeness, compactness), Zorn's Lemma, ordinals, and cardinals.

Knowledge of basic computability theory (recursive and recursively enumerable sets, the Halting Problem, the s-m-n theorem) to the level of the Part II course *Automata & Formal Languages* (or equivalent) is desirable, but not essential. The relevant facts and definitions will be revised at a fast pace if needed.

Literature

Students are likely to find the following useful, particularly references 2, 3, and 8:

1. C.C. Chang, H.J. Keisler, *Model Theory*, Studies in Logic and the Foundations of Mathematics, Elsevier, Volume 73, 1990.
2. W. Hodges. *Model theory*, Encyclopedia of Mathematics and its Applications, Cambridge University Press, 1993.
3. D. Marker. *Model Theory: An Introduction*, Graduate Texts in Mathematics, Springer, New York, 2002.
4. J. L. Bell, A. B. Slomson. *Models and Ultraproducts: An Introduction*, Dover Books on Mathematics Series, Dover Publications, 2013.
5. A. S. Troelstra, D. van Dalen. *Constructivism in Mathematics: an introduction*, Vol. 1, Elsevier, 1988.
6. J. Girard, Y. Lafont, P. Taylor. *Proofs and Types*, Cambridge University Press, 1989.
7. R. Kaye. *Models Of Peano Arithmetic*, Oxford Logic Guides **15**, Oxford University Press, 1991.
8. D. van Dalen. *Logic and Structure*. Fourth edition. Springer, 1997.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Forcing and the Continuum Hypothesis (L24)

Dr R. Matthews

An early result in the history of set theory is Cantor's famous theorem that the real numbers cannot be put into one-to-one correspondence with the natural numbers. This naturally leads to the question of whether these are the only two possible sizes for infinite sets of reals, with the Continuum Hypothesis being the assertion that this is indeed the case.

(CH) Every infinite set of reals is either equinumerous with the set of natural numbers or equinumerous with the set of all real numbers. Equivalently, $2^{\aleph_0} = \aleph_1$.

This question was considered so important to the foundations of mathematics that David Hilbert posed it as the first of his twenty-three problems for the 20th Century in 1900. In 1938, Kurt Gödel invented the *Constructible Universe* and used this to show that CH cannot be disproved in ZFC. Then, in 1963, Paul Cohen invented the *method of forcing* to show that CH cannot be proved in ZFC. From these two results it follows that CH is independent from the standard axioms of set theory.

In this course we shall study the concept of independence results in set theory. The course will cover a selection of topics including:

- **Models of Set Theory.** Transitive models. Absoluteness. Reflection Principles.
- **Inner Models.** Definability. The Constructible Universe. Condensation. Gödel's proof of the consistency of CH.
- **Forcing.** Generic extensions. The Forcing Theorems. Adding reals. Cohen's proof of the consistency of $\neg$CH.

Prerequisites

The Part II course *Logic & Set Theory* or an equivalent course is essential.

The Part III course *Model Theory & Non-Classical Logic* (Michaelmas 2023) is not directly related to this course but will contain various mathematical topics that are useful in understanding the material in this course.

This course will run concurrently with the Part III course *Large Cardinals* (Lent 2024). The two courses are self-contained and can be taken separately, but given the close relationship between the underlying topics, it is strongly suggested that students consider taking both courses.

Literature

1. Kenneth Kunen *Set Theory An Introduction to Independence Proofs.* Elsevier, 1980.
2. Thomas Jech *Set Theory.* The Third Millenium Edition, revised and expanded, Springer, 2003.
3. Jon Barwise *Admissible sets and structures.* Cambridge University Press, 2017.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Large Cardinals (L16)

Professor B. Löwe

The following definitions and facts should be familiar from an introductory course on set theory: a cardinal κ is called *regular* if every unbounded subset of κ has cardinality κ; successor cardinals, i.e., cardinals of the form $\aleph_{\alpha+1}$, are always regular; the usual limit cardinals, e.g., $\aleph_\omega$, $\aleph_{\omega+\omega}$, or $\aleph_{\omega_1}$, are not.

Thus, the following is a natural question:

"Are there any uncountable regular limit cardinals?".

If they exist, they must be very large, in particular, much larger than any of the mentioned limit cardinals.

It turns out that this question is intricately connected with the incompleteness phenomenon in set theory: if there is an uncountable regular limit cardinal, then there is a model of ZFC; therefore, ZFC is consistent, and hence (by Gödel's Second Incompleteness Theorem) ZFC cannot prove the existence of these cardinals (unless, of course, it is inconsistent).

Regular limit cardinals (a.k.a. *weakly inaccessible cardinals*) are the smallest examples of set-theoretic notions called *large cardinals*: cardinals with properties that imply that they must be very big and whose existence cannot be proved in ZFC.

In this course, we shall get to know a number of these large cardinals, study their behaviour, observe consequences of their existence for set theory, and develop techniques to determine the logical strength of large cardinals (the so-called *consistency strength hierarchy*). In modern set theory, large cardinals are used as the standard way to calibrate logical strength of extensions of ZFC.

Prerequisites

The Part II course *Logic & Set Theory* or an equivalent course is essential.

This course shares some content with the Part III courses *Model Theory & Nonclassical Logic* (Michaelmas 2023) and *Forcing & the Continuum Hypothesis* (Lent 2024) that will be proved in the other courses and used in this course. Following these two courses is not necessary, but recommended.

Literature

1. Thomas Jech, *Set Theory*, The Third Millenium Edition, revised and expanded, Springer, 2003 [Springer Monographs in Mathematics]

2. Akihiro Kanamori, *The Higher Infinite. Large Cardinals in Set Theory from Their Beginnings.* Springer, 2003 [Springer Monographs in Mathematics]

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

11 Information and Finance

Information Theory (M16)

Ioannis Kontoyiannis

Information theory is the mathematical foundation of the ideas and tools required for the quantitative description and analysis of the notion of information. The origin of the theory in Claude Shannon's landmark 1948 paper is motivated by questions in communications engineering, but since then information theory has forged deep connections with many areas of mathematics, most notably with probability and statistics. This will be an introductory course to the main information-theoretic ideas, results, and techniques. We will discuss entropy as a measure of information, relative entropy as a natural distance between probability distributions, and mutual information as a universal dependence measure between random variables. Their properties will be established (monotonicity, chain rules, data processing inequalities, asymptotic equipartition) and different versions of Shannon's source coding theorems will be proved, including their extensions to universal data compression by Shtarkov and Rissanen. Connections with large deviations theory and with additive combinatorics will also be developed.

Pre-requisites

The only pre-requisite is knowledge of basic probability, although a certain level of maturity and familiarity with the use of probabilistic techniques will be helpful. Knowledge of advanced (including measure-theoretic) probability is not necessary.

Literature

1. Mainstream introduction to information theory:
   T.M. Cover and J. Thomas. *Elements of Information Theory.* 2nd edition.

Wiley-Interscience, 2006.

2. A more applied perspective:
   D. MacKay. *Information Theory, Inference, and Learning Algorithms.* Cambridge University Press, 2003. Available free online at: https://www.inference.org.uk/itprnn/book.pc

3. Theoretical connections with ergodic theory and probability...
   P. Billingsley. *Ergodic Theory and Information.* J. Wiley, 1965.

4. ... and: P. Shields. *The Ergodic Theory of Discrete Sample Paths.* American Mathematical Society, 1996.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Advanced Financial Models (L24)

Dr M.R. Tehranchi

This course is an introduction to financial mathematics, with a focus on the pricing and hedging of contingent claims. It complements the material in Advanced Probability and Stochastic Calculus & Applications.

The course will cover a selection of topics including:

- *Discrete-time models.* Arbitrage, martingale deflators, the fundamental theorem of asset pricing. Numéraires, equivalent martingale measures. Forwards, options, futures, bonds, interest rates. Attainable claims, market completeness. The Breeden–Litzenberger formula. Fourier pricing. American claims.

- *Continuous-time models.* Admissible strategies. Absolute and relative arbitrage. Existence of replicating strategies. Pricing and hedging via partial differential equations. The implied volatility surface. Dupire's formula. Stochastic volatility models. The HJM approach to term structure. Merton's problem.

Prerequisites

Familiarity with measure-theoretic probability will be assumed.

Literature

1. M. Baxter & A. Rennie. *Financial calculus: an introduction to derivative pricing.* Cambridge University Press, 1996.

2. M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling.* Springer, 2006.

3. D. Kennedy. *Stochasic Financial Models.* Chapman & Hall, 2010.

4. D. Lamberton & B. Lapeyre.*Introduction to stochastic calculus applied to finance.* Chapman & Hall, 1996.

5. S. Shreve. *Stochastic Calculus for Finance: Vol. 1 and 2.* Springer-Finance, 2005.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Concentration Inequalities (L16)

Dr V. Jog

The topic of this course is the study of concentration inequalities and their applications. Concentration inequalities quantify the random fluctuations of functions of independent random variables and find applications in many domains including statistics, learning theory, discrete mathematics, random matrix theory, information theory, and high-dimensional geometry.

The course will cover a selection of topics including:

- Markov's inequality and Chebyshev's inequality; inequalities for sub-Gaussian random variables; Hoeffding's inequality and related inequalities

- Efron-Stein inequality and Poincaré inequality; basic information-theoretic inequalities; log-Sobolev inequalities; the entropy method

- Talagrand's inequality; the transport method; Marton's transport cost inequalities

Applications of the above inequalities to dimensionality reduction, random matrices, Boolean analysis, combinatorics, and learning theory will be considered throughout the course.

Prerequisites

The only pre-requisite is knowledge of basic probability, although a certain level of maturity and familiarity with the use of probabilistic techniques will be helpful. Knowledge of advanced (including measure-theoretic) probability is not necessary.

Literature

1. S. Boucheron, G. Lugosi and P. Massart, *Concentration Inequalities: A Non-Asymptotic Theory of Independence*, Oxford University Press, 2013.
2. M. Raginsky and I. Sason, *Concentration of Measure Inequalities in Information Theory, Communications and Coding*, Foundations and Trends in Communications and Information Theory, 2013.
3. M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

12 Number Theory

Modular Forms (M24)

Professor J. Thorne

Modular forms are holomorphic functions on the complex upper half plane which are invariant under an action of the group $SL_2(\mathbf{Z})$ (or a finite index subgroup). They feature in many different parts of mathematics, but are most famous in number theory for their role in the proof of Fermat's Last Theorem, itself based on a proof of the Shimura–Taniyama conjecture concerning the modularity of elliptic curves over $\mathbf{Q}$.

In this course we will introduce the theory of modular forms from a number-theoretic point of view, including the theory of Eisenstein series, Hecke operators, and the connection with L-functions.

Prerequisites

Essential: IB Complex Analysis , IB Groups, Rings & Modules (or equivalents)

Useful: Part II Riemann Surfaces (or equivalent)

Literature

1. J.-P. Serre, *A Course in Arithmetic*, Graduate Texts in Maths. 7, Springer, New York, 1973
2. F. Diamond, J. Shurman, *A First Course in Modular Forms*, Graduate Texts in Maths. 228, Springer, New York, 2005
3. J. Milne, *Modular Functions and Modular Forms*, Lecture notes available at

   https://www.jmilne.org/math/CourseNotes/MF.pdf

4. M. Ram Murty, Applications of symmetric power L-functions. In *Lectures on automorphic L-functions*, pp. 203–283, Fields Inst. Monogr., 20, Amer. Math. Soc., Providence, RI, 2004

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Algebraic Number Theory (L24)

Dr H. Wiersema

This course constitutes a second course in algebraic number theory and will cover various topics in the subject. One of the main topics we will discuss is global class field theory, which aims to classify abelian extensions of number fields. We will not prove the main statements, but we will discuss both the ideal and the idele theoretic version of class field theory. After introducing the necessary concepts in each version, we will see how these two are compatible.

We will study other objects central to current research in number theory, such as ζ-functions and L-functions. Moreover, we will discuss density and density results like the Chebotarev density theorem.

There is plenty of literature in this area. We focus on the books [3], [4] and [7].

Topics likely to be included (not in order):

- Dedekind ζ-functions, analytic class number formula.
- L-functions.
- Ideles and adeles.
- Statements of global class field theory.
- Dirichlet density, Chebotarev density theorem.

Prerequisites

Galois Theory, Number Fields, Local Fields.

Literature

1. J.W.S. Cassels and A. Fröhlich (eds.), *Algebraic Number Theory*, Academic Press, 1967.
2. N. Childress, *Class Field Theory*, Springer New York, 2008.
3. D.A. Cox, *Primes of the Form* $p = x^2 + ny^2$ *: Fermat, Class Field Theory, and Complex Multiplication*, John Wiley & Sons, 1989.
4. G.J. Janusz, *Algebraic Number Fields*, Academic Press, 1973.
5. S. Lang, *Algebraic Number Theory*, Springer GTM 110, 1970.
6. D. Marcus, *Number Fields*, Springer-Verlag, 1977.
7. J. Neukirch, *Algebraic Number Theory*, Springer–Verlag, 1999.
8. J. Neukirch, *Class Field Theory. The Bonn Lectures*. Online edition. Available via https://www.mathi.uni-heidelberg.de/ schmidt/Neukirch-en/
9. H.P.F. Swinnerton-Dyer, *A Brief Guide to Algebraic Number Theory* (London Mathematical Society Student Texts), Cambridge University Press, 2001.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Elliptic Curves (L24)

Professor T. A. Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points.

The following topics will be covered, and possibly others if time is available.

- **Weierstrass equations and the group law.** Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process.
- **Isogenies.** The degree of an isogeny is a quadratic form. The invariant differential and separability. The torsion subgroup over an algebraically closed field.
- **Elliptic curves over finite fields.** Hasse's theorem and zeta functions.
- **Elliptic curves over local fields.** Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p, and the formal group of an elliptic curve. Singular Weierstrass equations.
- **Elliptic curves over number fields.** The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Prerequisites

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. The first few lectures will include a brief review of the necessary geometric background, but this will assume some previous knowledge of algebraic curves, at the level of the Part II course *Algebraic Geometry* or the first two chapters of [3]. Later in the course, some basic knowledge of the Part III course *Local Fields* will be assumed.

Preliminary Reading

1. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Literature

2. J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.
3. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

Additional support

Four example sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Local Fields (M24)

Rong Zhou

The p-adic numbers Q_p were introduced by Hensel at the end of the 19th century and are now a ubiquitous tool in modern number theory as well as many other fields including algebraic topology, representation theory and algebraic geometry. The idea is to consider the completion of Q with respect to the absolute value defined by $|x|_p = p^{-n}$ for non-zero $x \in Q$ where $x = p^n \frac{a}{b}$ with $a, b, n \in Z$ and a, b coprime to p. The resulting field Q_p gives a neat way of packaging the information of congruences modulo p^n for all n and is the basic example of a local field. From this point of view, local fields are objects lying on the interface between algebra and analysis and the techniques used to study them involve an interesting mix of the two subjects.

This course will cover the basic theory of local fields, as well as some more advanced topics such as local class field theory, and is likely to be useful for students interested in studying other Part III courses on number theory. Topics likely to be covered include:

Absolute values on fields; valuations; Hensel's Lemma; structure of local fields;

Extensions of complete fields; Dedekind domains; Decomposition groups;

Ramification theory; higher ramification groups; Weil group;

Statements of local class field theory; formal groups; Lubin–Tate theory.

Prerequisites

Basic algebra up to and including Part II Galois theory as well as knowledge of concepts in point set topology and metric spaces are essential pre-requisites. It will be assumed that students have attended a first course in algebraic number fields.

Literature

1. J.W.S. Cassels, *Local fields*, Cambridge University Press, 1986. .
2. J.W.S. Cassels; A. Fröhlich, *Algebraic Number Theory*, Academic Press, 1967
3. J. Neukirch, *Algebraic Number Theory*, Springer–Verlag, 1999.
4. J. P. Serre, *Local fields*, Graduate Texts in Mathematics, 67. Springer–Verlag, 1979.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

13 Particle Physics and Quantum Fields

Applications of Quantum Field Theory (E16)

Sean Hartnoll

This course will explore quantum field theories that emerge as the long distance, continuum, description of quantum many-body systems. These quantum field theories share important conceptual tools with those encountered in particle physics, such as renormalisation group flow, spontaneous symmetry breaking and diagrammatic perturbation theory. However, due to the immense diversity of many-body systems, which can have 1, 2 or 3 spatial dimensions, are not Lorentz invariant, and admit a variety of different collective modes, the quantum field theories to be considered in this course exhibit a range of rich and novel behaviours. Furthermore, while collider physics occurs in the vacuum, many-body physics takes place in states with a large number of particles and often at nonzero temperature. It becomes crucial to understand the interplay between quantum and thermal fluctuations.

Some subset of the following topics will be covered, as time allows:

- Spontaneous symmetry breaking in quantum many-body physics. Effective field theories of superfluids and magnets. Nonlinear sigma models. Coherent state path integral. Spin chains, topological terms and large S expansion.
- Nonzero temperature. Periodicity in imaginary time. Matsubara frequencies. Spectral functions. Fluctuation-dissipation theorem.

- Quantum Criticality. Quantum and themal fluctuations. Large N expansion.
- Effective field theory of Fermi surfaces. Renormalisation group picture of Fermi liquids. Superconducting instability of Fermi surfaces. Non-Fermi liquids.
- Dimerized and valence-bond solid phases. Spin liquids, spinons and emergent gauge fields. Deconfined quantum criticality and monopoles.

Prerequisites

Understanding of quantum field theory at the level of the Advanced Quantum Field Theory course is required. In particular, familiarity with path integrals and Wilsonian renormalisation will be assumed. Exposure to material in the Statistical Field Theory course (in particular, classical Landau-Ginzburg theory) and The Standard Model course (in particular, spontaneous symmetry breaking in quantum field theory) may be useful but will not be assumed. No prior exposure to solid state physics will be assumed.

Literature

1. S. Sachdev, *Quantum Phase Transitions*, 2nd Edition, CUP 2011.
2. X-G. Wen, *Quantum Field Theory of Many-Body Systems*, OUP 2004.
3. A. Zee, *Quantum Field Theory in a Nutshell*, PUP, 2003.
4. A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd Edition, CUP 2010.
5. S. Sachdev, *Quantum Phases of Matter*, CUP 2023.

Additional support

Two examples sheets will be provided and associated examples classes will be given. There will be a revision session before the exam.

Advanced Quantum Field Theory (L24)

Dr R. A. Reid-Edwards

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess and is the basic theoretical framework for describing elementary particles and their interactions (excluding gravity). QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course introduces techniques of path integrals and functional methods to study QFT including gauge theories. Gauge Theories are a generalization of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie group at every point in space-time and yet still describe the same physics. Quantizing a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry.

A further major component of the course is to study renormalization. Wilson's picture of renormalization is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the renormalization group (RG) flow. The course explores renormalization systematically using dimensional regularization in perturbative loop integrals. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a nonabelian gauge theory can effectively become small at high energies. Known as *asymptotic freedom*, this phenomenon revolutionized our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrize possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Prerequisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be helpful for the section on nonabelian gauge theory. The section on the renormalization group has some overlap with Statistical Field Theory but no knowledge of this course will be assumed.

Literature

1. Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).

2. Srednicki, M., *Quantum Field Theory*, CUP (2007) A pre-publication draft is available at http://web.physics.ucsb.edu/ mark/qft.html.

3. Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Field Theory (M24)

Alejandra Castro

Quantum Field Theory is a marriage of quantum mechanics and special relativity which provides the mathematical framework for describing the interactions of elementary particles. This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalars, (Dirac) spinors, and gauge fields. The relativistic covariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem. The quantisation of free fields is developed using Hamiltonian methods in terms of operators which create and annihilate particles and anti-particles. The Fock space of quantum physical states and the associated realisation of particle statistics are described.

Interacting field theory is developed next using the LSZ formula for the scattering amplitude. Perturbation theory is formulated by applying the Schwinger-Dyson equation and Wick's theorem; Feynman diagrams are then introduced as an efficient way of organising the resulting calculations. Physically measurable quantities such as decay rates and cross-sections are defined. Spinors and the Dirac equation are explored in detail, along with parity and chirality. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions.

We review the relativistic formulation of Maxwell's equations formulating a corresponding action principle. The significance of gauge invariance is discussed. Lorentz gauge quantisation of the electromagnetic field is accomplished by imposing the Gupta-Bleuler condition. Finally, quantum electrodynamics (QED) is developed. Interactions between photons and charged matter are introduced via the principle of minimal coupling. We discuss the calculation of tree-level scattering amplitudes in QED.

Prerequisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics including Dirac's "bra/ket" notation. A basic knowledge of group theory is also very useful for this course.

Literature

1. David Tong, *Lectures on Quantum Field Theory*, http://www.damtp.cam.ac.uk/user/tong/qft.html**link.**
2. M. D. Schwartz, *Quantum Field Theory and the Standard Model,* Cambridge University Press, 2014.
3. M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory,* Harper Collins Publishers, 1995.
4. T. Zee, *Quantum Field Theory in a Nutshell,* Princeton University Press, 2003.
5. M. Srednicki, *Quantum Field Theory,* Cambridge University Press, 2007. http://web.physics.ucsb.edu/ mark/qft.html**link**.
6. P. Ramond, *Field Theory: A Modern Primer,* Benjamin/Cummings Pub. Co.,1981.

Quantum field theory is an extremely vast subject. The course will mainly follow David Tong's lecture notes, and specific chapters in Schwartz's book listed above. Every book listed here has strengths and weaknesses, so it is useful to be familiar with several. For instance, Zee's book is very good for a conceptual and intuitive introduction, but lacks details on how to do computations. Srednicki's book on the other hand is very explicit on how to carry out computations, but there are few explanations and motivations. Schwartz's book gives a modern understanding of effective field theories and balances well formal aspects, explicit derivations and explanations. Ramond's book is short but has wonderful explanations which I personally find useful.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The Standard Model (L24)

David Tong

The Standard Model is, by any measure, the most successful scientific theory of all time. The culmination of 350 years of scientific enquiry, it correctly explains every experiment that we've ever done (at least if you're willing to turn a blind eye to gravity for now). The purpose of this course is to explain the ingredients that go into the Standard Model, and the way in which they fit together.

In many ways, this course can be thought of as “QFT 3”. We will cover many concepts that are widely applicable to any quantum field theory. In particular, there will be a focus on symmetries, including global symmetries, gauge symmetries, Poincaré symmetries, and discrete symmetries such as C,P and T. Associated to many of these symmetries is the idea of “symmetry breaking” and, to this end, we will prove Goldstone’s theorem and explain the Higgs mechanism.

There will also be a focus on mathematical consistency. A phenomenon known as “anomaly cancellation” means that certain features of the Standard Model cannot be any other way: they are rigid and fixed. So you cannot, for example, have an electron-type particle without also having quarks and neutrinos. Other parts of the Standard Model appear to be much less rigid, especially those concerning the various interactions with the Higgs boson, usually referred to as “flavour”, and these too will be described in detail.

Prerequisites

You should be comfortable with the material from the Michaelmas term courses on QFT and Symmetries. It will also be useful to attend Advanced Quantum Field Theory in Lent.

Literature

Lecture notes for this course are available at

https://www.damtp.cam.ac.uk/user/tong/standardmodel.htmlhttps://www.damtp.cam.ac.uk/user/tong/stan

In addition, I previously wrote lecture notes for the CERN summer school which cover the material using only high school maths:

http://www.damtp.cam.ac.uk/user/tong/particle.htmlhttp://www.damtp.cam.ac.uk/user/tong/particle.html.

Some books thay may be worth consulting include

1. David Griffiths, “*Introduction to Elementary Particles*”, Wiley 2008
2. Matt Schwartz, “*Quantum Field Theory and the Standard Model*”. Cambridge University Press, 2014.
3. Cliff Burgess and Guy Moore, “*The Standard Model: A Primer*”, Cambridge University Press, 2007.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistical Field Theory (M16)

Prof. C. E. Thomas

This course introduces the renormalization group, focusing on statistical systems such as spin models with further connections to quantum field theory.

After presenting the Ising Model, Landau's mean field theory is introduced and used to describe phase transitions. The extension to the Landau-Ginzburg theory reveals broader aspects of fluctuations whilst consolidating connections to quantum field theory. At second order phase transitions, also known as 'critical points', renormalisation group methods play a starring role. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems, including those with continuous symmetries.

Prerequisites

Background knowledge of statistical mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Literature

1. N Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Addison-Wesley 1992.
2. M Kardar, *Statistical Physics of Fields*, Cambridge University Press 2007.
3. J Cardy, *Scaling and Renormalisation in Statistical Physics*, Cambridge University Press 1996.
4. J M Yeomans, *Statistical Mechanics of Phase Transitions*, Clarendon Press 1992.
5. M Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press 1991.
6. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press 1992.

7. D Amit and V Martin-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific 2005.

8. L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press 1996.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in Easter Term.

String Theory (L24)

Professor D. Skinner

String theory is the quantum theory of interacting one-dimensional extended objects (strings). What makes it so appealing is that it's a quantum theory that includes gravity, thus providing the first tentative steps towards a full quantum theory of gravity. It has become clear that string theory is also much more than this. It is a framework in which to reinterpret problems in quantum field theory in terms of geometry, and acts as a crucible for new ideas in mathematics.

This course provides an introduction to String Theory. We begin by generalising the worldline of a particle to the two-dimensional surface swept out by a string. The quantum theory of the embedding of these surfaces in space-time is governed by a two-dimensional quantum field theory and we shall study the simplest example – the bosonic string – in detail.

An introduction to relevant ideas in Conformal Field Theory (CFT) will be given. The quantisation of the string will be studied, its spectrum obtained, and the relationship between states on the two dimensional CFT and fields in space-time will be discussed. We will see the necessity of the critical dimension of space-time.

The path integral approach to the theory will be discussed. Fadeev-Popov methods will be introduced to deal with the redundancies that appear in the theory. Vertex operators will be introduced and scattering amplitudes will be computed at tree level. Perturbation theory at higher loops and the role played by moduli space of Riemann surfaces will be sketched.

The course will focus on closed strings, but time permitting, open strings and the role of D-branes may be discussed. There may also be some discussion of more stringy phenomena such as symmetry enhancement and duality.

Prerequisites

Knowledge of the Quantum Field Theory course in Michaelmas term is assumed. Advanced Quantum Field Theory will complement this course but will not be assumed.

Literature

1. J. Polchinski, *String Theory, Vol. 1: An Introduction to the Bosonic String*, CUP 1998.
2. M. Green, J. Schwarz and E. Witten, *Superstring Theory, Vol. 1: Introduction*, CUP 1987.
3. R. Blumenhagen, D. Lüst and S. Theisen, *Basic Concepts of String Theory*, Springer 2013.
4. D. Tong, *Lectures on String Theory*, https://arxiv.org/abs/0908.0333

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Supersymmetry (L16)

Ben Allanach

This course provides an introduction to the use of supersymmetry in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables, thus relating fermions to bosons.

Physical and theoretical motivations for supersymmetry are provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced.

Three examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries, Particles and Fields courses, or be familiar with the material covered in them.

Introductory Reading

1. The first chapters of `http://arxiv.org/abs/hep-ph/0505105`

Reading to complement course material

For more advanced topics later in the course, it will helpful to have a knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model in order to understand the final lecture on the minimal supersymmetric standard model.

Beware: most of the supersymmetry references contain errors in minus signs.

1. Supersymmetric Gauge Field Theory and String Theory, Bailin and Love, IoP Publishing (1994) has nice explanations of the physics.
2. Introduction to supersymmetry, J.D. Lykken, `hep-th/9612114`. This introduction is good for extended supersymmetry and more mathematical aspects.
3. Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this terse and more mathematical book has the opposite sign of metric to the course.
4. A supersymmetry primer, S.P. Martin, `hep-ph/9709256` is detailed on phenomenological aspects, although it has the opposite sign metric to the course.
5. From Spinors to Supersymmetry, Dreiner, Haber and Martin, CUP publishing (2023) is an extensive (over 1000 pages) and definitive supersymmetry text.

Symmetries, Particles, and Fields (M24)

M B Wingate

Lie groups and Lie algebras are important in the construction of quantum field theories which describe interactions between particles. Particle states and quantum fields are described in terms of irreducible representations of the Poincaré group, a Lie group. Gauge theories, which describe interactions in the Standard Model of particle physics, rely on Lie groups.

This course will introduce the mathematics necessary for employing the Lie groups and algebras in high energy physics and beyond. The focus will be on the mathematical concepts and key applications. Mathematical proofs will be given where straightforward or useful for a deeper understanding; however, proof of some facts will remain beyond the scope of this course.

After preliminary remarks, we introduce Lie matrix groups, which depend on continuous parameters. Differentially, these act as a Lie algebra. The exponential map connects the Lie algebra to the Lie group. We then introduce group and algebra representations as linear maps on vector spaces.

The group of rotations in 3-dimensional space, $SO(3)$, is examined, along with $SU(2)$ and the connection to angular momentum states in quantum theory. Representations of these are covered and used later in the course. The relativistic symmetries (Lorentz group and Poincaré group in four dimensions) are studied from the point of view of their group elements and Lie algebras. We shall cover irreducible induced representations of the Poincaré group, which are used to describe one-particle states and fields.

Analysis of simple Lie algebras and their finite representations comes from mapping them to a geometrical picture involving roots and weights via the Cartan matrix. An overview of the results of the Cartan classification of simple Lie algebras is included.

An application in terms of representations of a global $SU(3)_F$ flavour symmetry explains some features of the spectrum of hadronic particles. Further properties of the spectrum lead one to introduce an additional local $SU(3)_c$ colour symmetry leading a particular gauge theory called quantum chromodynamics. We shall close by covering gauge theory for general Lie groups and different representations of matter.

Prerequisites

Linear algebra, including direct sums and tensor products of vector spaces. Special relativity and quantum theory, including angular momentum theory and Pauli spin matrices. Brief reminders of some definitions are given; however familiarity with undergraduate quantum mechanics will be important for fully appreciating some of the physical applications of the mathematics in this course.

Literature

1. For a more basic introduction to the use of groups in physics, *Group Theory in a Nutshell for Physicists*, A. Zee (Princeton University Press, 2016).

2. *Lie Algebras in Particle Physics* by H. Georgi (Westview Press, 1999) gives an introduction aimed at physicists.

3. *Groups, Representations, and Physics*, H. F. Jones (Taylor and Francis, 1998). The first half covers discrete groups; later chapters cover some material in this course.

4. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, B. C. Hall (Springer, 2015). A more mathematical text, useful reference for detail beyond this course.

5. *Symmetries and Group Theory in Particle Physics*, G. Costa and G. Fogli (Springer, 2012) has nice geometrical aspects.

Additional support

Four Examples Sheets will be provided and four associated Examples Classes will be given (the final one in Lent term). There will be a Revision Class in the Easter Term.

14 Probability

Lattice Models (M16)

Professor W. Werner

Lattice models are probabilistic models defined on $\mathbf{Z}^d$ (or other lattices). Examples include the Ising model, percolation, the Potts models, dimer models, the self-avoiding walk, uniform spanning trees and forests, the Gaussian Free Field and others. Most of these models are motivated from statistical physics. They are all simple to define, yet they often have very rich behaviour, and a number of fundamental and easy-to-state questions remain open.

This course will present a small selection of ideas and results. We will discuss aspects of the phase transition for percolation, Wilson's algorithm to construct uniform spanning trees, and properties of the discrete Gaussian Free Field. We will also describe some features specific to models in the plane, such as Smirnov's proof of conformal invariance for percolation on the triangular lattice.

Prerequisites

This course assumes familiarity with probability, measure theory, and analysis at Part II level. Parallel participation in Advanced Probability is encouraged.

Literature

The course will use material from various sources. Some helpful references are:

1. H. Duminil-Copin, *Introduction to percolation theory.*
   Availiable here: https://www.unige.ch/ duminil/publi/2017percolation.pdf

2. G. Grimmett, *Probability on Graphs*, Cambridge University Press
   Available here: http://www.statslab.cam.ac.uk/ grg/books/pgs.html
3. W. Werner and E. Powell, Lecture notes on the Gaussian Free Field, Cours Spécialisés S.M.F.
   Available here: https://arxiv.org/abs/2004.04720

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Stochastic Calculus (L24)

Professor J. Miller

This course will be an introduction to Itô calculus and will aim to cover the following topics.

- *Brownian motion.* Existence and sample path properties.
- *Stochastic calculus for continuous processes.* Martingales, local martingales, semi-martingales, quadratic variation and cross-variation, Itô's isometry, definition of the stochastic integral, Kunita-Watanabe theorem, and Itô's formula.
- *Applications to Brownian motion and martingales.* Lévy characterization of Brownian motion, Dubins-Schwartz theorem, martingale representation, Girsanov theorem, conformal invariance of planar Brownian motion, and Dirichlet problems.
- *Stochastic differential equations.* Strong and weak solutions, notions of existence and uniqueness, Yamada-Watanabe theorem, strong Markov property, and relation to second order partial differential equations.
- *Stroock–Varadhan theory.* Diffusions, martingale problems, equivalence with SDEs, approximations of diffusions by Markov chains.

Prerequisites

We will assume knowledge of measure theoretic probability as taught in Part III Advanced Probability. In particular we assume familiarity with discrete-time martingales and Brownian motion.

Literature

1. R. Durrett *Probability: theory and examples.* Cambridge, 2010.
2. I. Karatzas and S. Shreve *Brownian Motion and Stochastic Calculus.* Springer, 1998.
3. P. Morters and Y. Peres *Brownian Motion.* Cambridge, 2010.
4. D. Revuz and M. Yor, *Continuous martingales and Brownian motion.* Springer, 1999.
5. L.C. Rogers and D. Williams *Diffusions, Markov Processes, and Martingales.* Cambridge, 2000.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Schramm-Loewner Evolutions (L16)

Dr Y. Yuan

Schramm-Loewner evolution (SLE), a family of random fractal curves on planar domains, was introduced in the early 2000s to describe scaling limits of interfaces in critical two-dimensional lattice models (such as the Ising model, percolation, Gaussian free field...). Soon it became apparent that it plays a major role in two-dimensional probability (more precisely, conformally invariant processes). One famous example is that SLE arguments have been successfully applied to prove several conjectures about two-dimensional Brownian motion and simple random walks (intersection exponents, Mandelbrot conjecture). Of course, SLE is highly interesting in its own right, combining ideas from complex analysis and probability, and giving rise to beautiful mathematics.

The main goal of this course is to define and study SLE. The following topics will be covered:

- conformal maps: relations to planar Brownian motion, distortion estimates, and the Loewner differential equation,
- the definition of SLE, basic properties and their geometry,
- relation to the Gaussian free field.

Prerequisites

To follow this course, the student should be familiar with

- measure-theoretic probability (as taught in the course Advanced Probability),
- basic complex analysis, in particular conformal maps and the Riemann mapping theorem,
- stochastic calculus (can be attended in parallel to this course).

Literature

The course will follow the lecture notes by Jason Miller: http://www.statslab.cam.ac.uk/ jpm205/teaching/lent2

For extra material on SLE the student can consult:

1. Wendelin Werner, *Random planar curves and Schramm-Loewner evolutions*, 2004. Also available at https://arxiv.org/abs/math/0303354.
2. Gregory F. Lawler, *Conformally Invariant Processes in the Plane*, volume 114 of Mathe- matical Surveys and Monographs. American Mathematical Society, Providence, RI, 2005.

For extra material on conformal maps, see the following: (Note that the references below are beyond the prerequisites of the course. We will cover what is needed but the interested student will enjoy the books.)

1. Dimitry Beliaev, *Conformal maps and geometry*, Advanced Textbook in Mathematics. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2020.
2. John B. Garnett and Donald Marshall, *Harmonic Measure*, New Mathematical Mono- graphs: 2. CUP, 2005.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

15 Quantum Computation, Information and Foundations

Quantum Entanglement in Many-body Physics (LT16)

F. Verstraete

The theory of quantum entanglement provides a new language for describing correlations in quantum many-body systems, thé central problem in theoretical physics. One of the big challenges is to develop the grammar and semantics of that new language. This course will provide an introduction to this novel but striving field of research which spans a wide array of fields such as quantum computation, condensed matter physics, quantum field theory and quantum chemistry.

The course will cover a selection of topics including:

- Introduction to the theory of entanglement
- The relevant physical corner of Hilbert space
- Lieb-Robinson bounds and the spreading of entanglement
- Area laws of the entanglement entropy
- Quantum tensor networks
- Entanglement-based Variational methods for strongly correlated systems
- Topological quantum order and entanglement patterns
- Notions of entanglement in condensed matter physics, quantum field theory and quantum chemistry.

Prerequisites

It will be assumed that you have taken an advanced quantum mechanics course, similar to Part II Principles of Quantum Mechanics. Familiarity with quantum computation (e.g., from Part II and Part III Quantum Information and Computation) will be beneficial.

Literature

1. B Zeng, X Chen, DL Zhou, XG Wen, *Quantum information meets quantum matter*, Springer.

2. R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Quantum entanglement*, Reviews of modern physics, 17;81(2):865 (2009).

3. M.B. Hastings, *Locality in Quantum Systems*, arXiv:1008.5137.

4. J.I. Cirac, D. Perez-Garcia, N. Schuch, F. Verstraete, *Matrix product states and projected entangled pair states: Concepts, symmetries, theorems*, Reviews of Modern Physics 93 (4), 045003 (2021).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Information, Foundations and Gravity (L16)

Prof. Adrian Kent

The course gives an introduction to foundational questions in quantum theory and quantum information, with a focus on the relationship between quantum theory and gravity. We will examine arguments that the gravitational field must necessarily be quantized, together with counter-arguments, review old ideas of semi-classical theories of gravity and their problems. Then we will discuss recently proposed mass interferometry experiments that aim to use the properties of quantum information to test whether gravity is indeed mediated by quantum particle exchange.

Topics covered in the course will include:

- Basic concepts in quantum theory and quantum information: pure and mixed states, density matrix, the Schmidt decomposition, entanglement, the no-cloning theorem.
- Relationship between quantum theory and relativity: the no-signalling principle; quantum state summoning and the no-summoning theorem; possible and impossible summoning tasks.
- Bell's theorem and Bell nonlocality.
- Entanglement measures and entanglement witnesses.

- Brief review of Everettian quantum theory and collapse-based versions of quantum theory.
- Basic notions of semi-classical gravity; the Page-Geilker experiment.
- Quantum theory and gravity: the Colella, Overhauser and Werner experiment; the Eppley-Hannah argument.
- Non-linear extensions of quantum theory; refutation of the Eppley-Hannah argument
- The proposed experiments of Bose et al. and Marletto-Vedral (BMV) testing the generation of entanglement between separated mesoscopic quantum systems in superposition.
- Analysis of the implications of possible outcomes of BMV experiments.

If time permits we may also discuss other recent experimental proposals.

Prerequisites

Familiarity with undergraduate level quantum mechanics is essential. Familiarity with a first course in quantum information theory, such as the Cambridge Part II Quantum Information and Computation course, would be highly advantageous.

Some relevant papers

1. Don N Page and CD Geilker. Indirect evidence for quantum gravity. Physical Review Letters, 47(14):979, 1981.
2. Adrian Kent, Simple Refutation of the Eppley-Hannah argument. Classical and Quantum Gravity 35 (24) 245008 (2018)
3. Sougato Bose, Anupam Mazumdar, Gavin W. Morley, Hendrik Ulbricht, Marko Toros, Mauro Paternostro, Andrew A Geraci, Peter F Barker, MS Kim, and Gerard Milburn. Spin entanglement witness for quantum gravity. Physical Review Letters, 119(24):240401, 2017.
4. Chiara Marletto and Vlatko Vedral. Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. Physical Review Letters, 119(24):240402, 2017.
5. Daniel Carney, Philip CE Stamp, and Jacob M Taylor. Tabletop experiments for quantum gravity: a user's manual. Classical and Quantum Gravity, 36(3):034001, 2019.

Additional support

Three examples sheets will be provided and three associated one-hour examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topological Quantum Matter (L16)

Prof. B. Béri

Topologically ordered systems display quantum mechanical features that are insensitive to the local details of the system but are robustly set by its topology. Such features include "anyonic" particles beyond the boson/fermion dichotomy, a "holographic" correspondence between the system's bulk and its boundary, or a topology-dependent ground-state degeneracy. These remarkable properties, fundamentally linked to certain gauge theories, have intriguing potential applications, including the use of topological ground-state degeneracy in quantum error correction or employing so-called non-Abelian anyons in robust schemes for quantum computation.

This course is an introduction to topological order and its quantum computing applications. We begin with some basics of conventional (i.e., symmetry-breaking) order, illustrated using the quantum Ising chain. This will serve as a reference with which to compare topological order.

We then embark on our study of topological order. Concepts we shall aim to discuss include: ground-state degeneracy, anyons, the robustness of topological order, bulk-boundary correspondence, Majorana zero modes, non-Abelian anyons. We shall use the toric code and the Kitaev chain – a boson-fermion dual of the quantum Ising chain – as illustrative examples.

Of the quantum computing applications of topological order, we shall aim to discuss quantum error correction with the toric code, and topological quantum computation using non-Abelian anyons, in particular those based on Majorana zero modes.

Some of our discussions will use second quantisation, the basics of path integrals, and of quantum error correction. These shall be reviewed or introduced at the appropriate points of the course.

Prerequisites

It will be assumed that you have taken an advanced quantum mechanics course, similar to Part II Principles of Quantum Mechanics. Familiarity with the basics of band theory (e.g., from Part II Applications of Quantum Mechanics) and of quantum computation (e.g., from Part II Quantum Information and Computation) will also be useful.

Literature

1. S. H. Simon, *Topological Quantum: Lecture Notes and Proto-Book*, available at Steve Simon's http://www-thphys.physics.ox.ac.uk/people/SteveSimon/website. Chapter 3 and Parts II, V are especially useful for this course.

2. S. Sachdev, *Quantum Phase Transitions*, CUP (2011), parts of chapters 1, 5, 10.

3. B. A. Bernevig with T. L. Hughes, *Topological Insulators and Topological Superconductors*, Princeton University Press (2013), chapter 16.

4. S. Das Sarma, M. Freedman, C. Nayak, *Majorana zero modes and topological quantum computation* https://www.nature.com/articles/npjqi20151npj Quantum Inf. **1**, 15001, (2015).

5. A. Altland and B. Simons, *Condensed Matter Field Theory*, CUP (2010), chapters 2, 3 for background on second quantisation and path integrals.

Additional support

Three example sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

16 Relativity and Cosmology

Cosmology (M24)

Blake Sherwin

This course discusses what we know (and don't know) about the evolution of our universe, from inflationary quantum fluctuations in the first fraction of a second to the formation of galaxies and structures today. It also seeks to illustrate how cosmology can serve as a uniquely powerful laboratory for understanding fundamental physics.

In detail, the course will cover the following topics:

- Geometry and dynamics of our Universe
- Inflation
- Thermal history
- Cosmological perturbation theory

- Structure formation
- Cosmic microwave background basics
- Initial conditions from inflation

Pre-requisites

Although the course is intended to be as self contained as possible, knowledge of relativity, quantum mechanics and statistical mechanics will be very useful. General relativity and quantum field theory courses may allow a deeper understanding of some of the material covered.

Literature

1. D. Baumann,*Cosmology*
2. S. Dodelson, *Modern Cosmology*
3. E. Kolb and M. Turner, *The Early Universe*
4. S. Weinberg, *Cosmology*

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Black Holes (L24)

Professor H. Reall

A black hole is a region of space-time that is causally disconnected from the rest of the Universe. Black holes are of huge importance both in astrophysics and in attempts to develop a quantum theory of gravity.

This course will build on the General Relativity course to develop the mathematics required to study classical properties of black holes. It will then build on the Quantum Field Theory course to discuss quantum aspects of black holes. Topics to be discussed will include:

- Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
- Causal structure, null geodesic congruences, Penrose singularity theorem.

- Penrose diagrams, asymptotic flatness, cosmic censorship.
- Reissner-Nordstrom and Kerr black holes.
- Energy, angular momentum and charge in curved spacetime.
- The laws of black hole mechanics.
- Quantum field theory in curved spacetime. The Hawking effect and its implications.

Prerequisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

1. H. S. Reall, *Part 3 Black Holes*: lecture notes available at www.damtp.cam.ac.uk/user/hsr1000
2. R.M. Wald, *General relativity*, University of Chicago Press, 1984.
3. S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
4. V.P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.
5. N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. R.M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Canonical Gravity (L16)

Non-Examinable (Part III Level)

Professor M.J. Perry

General relativity is often presented in a way that unifies space and time into a complete whole, spacetime. This approach is encapsulated in the Einstein equation

$$R_{ab} - \frac{1}{2} R\, g_{ab} = 8\pi G T_{ab} \tag{1}$$

where there is no separation into space and time. Nevertheless, physics is often formulated in terms of the initial value problem where the initial configuration of spatial geometry, fields and particles together with their time derivatives are specified. One can then compute the time evolution of the system. We apply these ideas to general relativity. Firstly, we need to understand the general picture of constrained Hamiltonian systems and why the idea of constraints is related to gauge degrees of freedom. We will prqctice these ideas for particle mechanics, scalar field theory and then electromagnetism. Finally we apply these methods to general relativity to provide a canonical formulation along the lines first found by Dirac and by Arnowitt, Deser and Misner. More generally, we will then examine the evolution of a spacetime in terms of its trajectories in superspace, the space of all spatial geometries. We then will apply these ideas to a simple cosmological model and subsequently look how these ideas impact the problem of singularities is spacetime and the chaotic behaviour associated with them.

Quantum mechanics is usually presented starting from a Hamiltonian and then developed as a theory in which classical phase space is deformed using the Heisenberg algebra in which classical quantities are replaced by operators. We apply this technique to general relativity and so develop the Wheeler-DeWitt equation. Then we can use this to see how a classical spacetime can emerge from the quantum picture by use of the WKB approximation. Finally, if time permits, we will look at the relationship between the Wheeler-DeWitt equation and the path integral, what happens when the Wheeler-DeWitt equation is used to examine spacetime singularities, and lastly the role of observers to extract measurements in quantum gravity.

Prerequisites

Basic general relativity. Quantum Mechanics.

Additional support

There will be no examples sheets. An office hour will be arranged but perhaps some kind of discussion session can be organised.

Field theory in cosmology (L24)

Professor E. Pajer

This course discusses applications of classical, statistical and quantum field theory to cosmology. The course comprises three interconnected topics:

- Cosmological inflation and primordial quantum perturbations (QFT in curved spacetime)
- The matter and galaxy distribution in the Large Scale Structure of the Universe (statistical field theory)
- The physics of the Cosmic Microwave Background (classical and statistical field theory)

More specifically, the following topics will be discussed:

- Inflation: A brief review of background cosmology and inflation
- Inflation: Free fields on de Sitter spacetime
- Inflation: Interacting fields, the in-in formalism and correlators in $P(X, \phi)$ theories
- Inflation: Quantizing pure gravity, gravity plus matter, symmetries and phenomenology
- Large Scale Structures: Newtonian dynamics
- Large Scale Structures: Standard perturbation theory
- Large Scale Structures: the effective field theory approach
- Cosmic Microwave background: the Boltzmann equation
- Cosmic Microwave background: anisotropies and the angular power spectrum

Prerequisites

Some familiarity with introductory quantum field theory and general relativity, as for example provided by the respective michaelmas courses, is highly recommended. Basic knowledge of introductory cosmology is very helpful. Students who did not attend the Michaelmas course on cosmology may still follow this course, and may want to read Section 1.1 and 1.2 of the notes before the course starts.

Literature

The lecture notes for this course can be found on E. Pajer's website at http://www.damtp.cam.ac.uk/user/ep551

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

General Relativity (M24)

Claude Warnick

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

This is a second course on General Relativity, albeit one that could just about be followed without prior exposure to the subject. The first half of the course will give an introduction to differential geometry, the mathematics that underlies curved spacetime. The second half of the course will discuss the physics of gravity.

Prerequisites

Familiarity with Newtonian gravity, special relativity, finite-dimensional vector spaces and the Euler-Lagrange equations is essential. Knowledge of the relativistic formulation of MaxwellÕs equations is highly desirable.

Most students attending this course have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before then you should read an introductory book (e.g. Hartle or Rindler) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Literature

1. *Gravity: An introduction to Einstein's General Relativity,* J.B. Hartle, Addison-Wesley, 2003.

2. *Relativity: Special, General, and Cosmological*, 2nd ed., W. Rindler, OUP, 2006.

3. *General Relativity,* R.M. Wald, Chicago UP, 1984.
4. *Gravitation,* C.W. Misner, K.S. Thorne and J.A. Wheeler, W.H. Freeman, 1973.
5. *Spacetime and geometry: an introduction to General Relativity,* S.M. Carroll, Addison- Wesley, 2004.
6. *Advanced General Relativity,* J.M. Stewart, CUP, 1993.
7. *Gravitation and Cosmology,* S. Weinberg, Wiley, 1972.

The course will follow a similar approach to Wald and Misner, Thorne and Wheeler. Carroll's book is a very readable introduction at about the same level. Stewart's book is based on a previous version of this course. Weinberg's book gives a good discussion of the Equivalence Principle.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Gravitational Waves and Numerical Relativity (E16)

Ulrich Sperhake

The direct detection of a gravitational-wave signal from the black-hole binary merger GW150914, detected by LIGO in September 2015, has promoted gravitational-wave physics to a rich field combining theory with observation. With about 100 further events observed over the ensuing years, a vast range of physical and astrophysical phenomena are now accessible to a qualitatively new channel of observations. New gravitational-wave observatories on Earth and in Space will greatly enhance the sensitivity and frequency range in future decades. Gravitational-wave observations heavily rely on accurate theoretical predictions that require source modelling in the framework of Einstein's theory of relativity. Due to the complexity of Einstein's field equations and the highly dynamical character of the physical systems under consideration, the calculation of such templates is a formidable challenge that requires the combination of analytic and numerical techniques. In this lecture course, we will discuss the theoretical foundations of gravitational waves (at linear and non-linear level), formulations of the Einstein field equations amenable for a numerical treatment, initial data, gauge conditions and diagnostic tools.

More specifically, the course will cover the following topics.

- Characteristic surfaces and the classification of partial differential equations.
- The structure of the Einstein field equations.
- Gravitational waves in the characteristic (aka *Bondi-Sachs*) formalism of general relativity.
- The space-time split of the Einstein equations and well-posed formulations of the field equations.
- The construction of constraint satisfying initial data representing realistic snapshots of systems of multiple black holes.
- Singularity avoiding coordinate choices employed in contemporary numerical relativity simulations.
- Diagnostic tools to extract the gravitational-wave signal from numerically generated solutions.

Prerequisites

This course assumes knowledge of Part III General Relativity. Some knowledge of Part III Black Holes will be helpful but not essential for following this course.

Literature

1. T. W. Baumgarte and S. L. Shapiro *Numerical Relativity.* Cambridge University Press, 2010.
2. T. W. Baumgarte and S. L. Shapiro *Numerical Relativity: Starting from Scratch.* Cambridge University Press, 2021.
3. M. Alcubierre, *Introduction to 3+1 Numerical Relativity*, Oxford University Press, 2008.
4. E. Gourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, Springer, New York, 2012. See also https://arxiv.org/abs/gr-qc/0703035 .
5. M. Maggiore, Gravitational Waves. Vols. 1 (Theory and Experiments) and 2 (Astrophysics and Cosmology), Oxford University Press, 2018.
6. R. d'Inverno, *Introducing Einstein's Relativity*, Oxford Clarendon Press, 1992.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Solitons, instantons and geometry (L16)

Professor D. Stuart

In classical field theory, solutions which are stable, spatially localized and particle-like are called *solitons*; they exist in many nonlinear field theories. In many quantum field theories, including gauge theories, it is necessary to consider solutions of the field equations that are topologically distinct from the vacuum: solitons of this type are known as topological solitons. We will discuss existence of solitons in various examples, and describe their properties and dynamics.

In a slightly different direction, classical solutions which are also localised in *Euclidean* space-time are called *instantons*, and are studied in connection with tunneling phenomena and the vacuum in field theory.

In certain cases, the (usually) second order field equations can be reduced to first order Bogomolny equations, which makes it easier to analyse the soliton/instanton solutions. The resulting equations are often integrable in some sense, and explicit solutions can sometimes be found. Examples include scalar kinks, magnetic monopoles in Yang-Mills-Higgs theory and anti-self-dual instantons in pure Yang-Mills theory.

An introduction to the mathematical framework behind gauge theory (vector bundles, connections and curvature) will be given, together with some basic ideas from topology and calculus of variations as needed.

Prerequisites

Part II Quantum Mechanics, Electrodynamics and General Relativity (or some introductory Differential Geometry course such as the one in Part II) are essential. Part III General Relativity and Quantum Field Theory will be helpful.

Literature

1. T.D. Lee, *Particle physics and introduction to field theory,* Harwood, 1981. Chapter 7. (A good introduction to the phenomena).

2. R. Rajaraman, *Solitons and instantons, An introduction to solitons and instantons in quantum field theory.* North-Holland Publishing Co. 1982. (As well as general introduction and examples similar to Lee this contains material on semiclassical quantization of solitons.)

3. A. Jaffe and C. Taubes, *Vortices and monopoles,* Birkhauser, 1982. Chapters 1,2 and 3. (A mathematical treatment for those interested in analysis).

4. N.S. Manton and P.M. Sutcliffe, *Topological Solitons,* CUP, 2004, Chapters 3,4,5,6,8 and 10. (Detailed discussion of the physics of solitons, beyond this course.)

5. M. Dunajski, *Solitons, Instantons, and Twistors,* Oxford Graduate Texts in Mathematics, Oxford University Press, 2009. (Includes introduction to integrable aspects from the twistor point of view.)

6. T. Eguchi, P. Gilkey, and A.J. Hanson. *Physics Reports,* 66 (1980) 213-393. (Mathematical background.)

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

17 Statistics

Causal Inference (M16)

Dr. Q. Zhao

From its onset, modern statistics engages in the problem of inferring causality from data. A common mindset is that causal inference is only possible using randomised experiments, but developments in statistics and related fields have shown that this view is too simplistic. We now have a much better understanding of the assumptions and methodologies that enable causal inference from observational, non-experimental data. This course aims to cover some of the most fundamental ideas in causal inference, a vibrant research area where statistical theory meets scientific practice.

1. **Motivations:**

    - Principles of causal inference: motivations; historical perspectives; basic concepts.
    - Randomized experiments: randomization tests, regression adjustment and its asymptotic inference.
    - Path analysis and linear structural equation models (SEMs).

2. **Languages for causality:**

    - Probabilistic directed acyclic graphical (DAG) models: Markov properties, d-separation, structure discovery.

- Counterfactual causal models: nonparametric SEMs; single-world intervention graphs; g-computation formula.
- Causal identification: back-door criterion, front-door criterion; potential outcomes calculus; other examples.

3. **Design and statistical methods:**

   - Observed confounders: matching, randomisation inference, Rosenbaum's sensitivity analysis; influence functions and semiparametric inference.
   - Instrumental variables (IV): core IV assumptions; generalised method of moments; principal stratification.
   - Other selected topics: regression discontinuity design; difference in differences and negative control methods; mediation analysis; longitudinal data and time-varying treatments; meta-analysis and evidence synthesis.

Prerequisites

This course assumes familiarity with undergraduate-level probability and statistics.

Literature

1. Hernán M. A. and Robins, J. M. *Causal Inference: What If.* Chapman & Hall, 2020.
2. Imbens, G. W. and Rubin, D. B. *Causal Inference in Statistics, Social, and Biomedical Sciences.* Cambridge University Press, 2015.
3. Lauritzen, S. L. *Graphical Models.* Clarendon Press, 1996.
4. Angrist, J. D. and Pischke, J. S. *Mostly Harmless Econometrics: An Empiricist's Companion.* Princeton University Press, 2008.

Additional support

Lecture notes will be provided. Three examples sheets will be provided and three associated examples classes will be given. A fourth example class will ask the student to read and present an applied research article. There will also be a revision class in the Easter Term.

Modern Statistical Methods (M24)

Sergio Bacallado

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data where we have huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to study such datasets. We aim to cover some of the following topics.

- Kernel machines: Ridge regression, the kernel trick, reproducing kernel Hilbert spaces, kernel ridge regression, the support vector machine, the hashing trick.
- Penalised regression: Model selection, the Lasso, variants of the Lasso.
- Conditional independence and high-dimensional inference: Conditional independence graphs, the graphical Lasso, conditional independence testing, the debiased Lasso.
- Multiple testing: the closed testing procedure and the Benjamini–Hochberg procedure.

Prerequisites

Basic knowledge of statistics, probability, linear algebra and real analysis. Some background in optimisation would be helpful but is not essential.

Literature

1. T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning.* 2nd edition. Springer, 2001.
2. P. Bühlmann, S. van de Geer, *Statistics for High-Dimensional Data.* Springer, 2011.

3. T. Hastie, R. Tibshirani and M. Wainwright, *Statistical learning with sparsity: the lasso and generalizations.* CRC Press, 2015.

4. C. Giraud, *Introduction to High-Dimensional Statistics.* CRC Press, 2014.

5. M. Wainwright, *High-Dimensional Statistics: A Non-Asymptotic Viewpoint.* Cambridge University Press, 2019.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistical Learning in Practice (L24)

Dr R. Altmeyer

Statistical learning aims to build statistical models for learning relationships and structures from data, mainly with the goal of predicting future outcomes. It blends classical statistics with techniques from modern machine learning and tries to provide concrete algorithms, for example for regression or classification tasks. This course consists of 12 lectures and 12 practical classes. We will get to know some of the most successful and widely used statistical methodologies in modern applications, with a focus on practical aspects and concrete problems. The practical classes will cover an introduction to the programming language R, exploratory data analysis and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models for regression and classification
- Model selection and regularisation
- Bayesian regression
- Mixed effects models
- Linear discriminant analysis and Support Vector Machines
- Introduction to Deep learning and Random forests
- Principal component analysis
- Introduction to time series

Prerequisites

Elementary probability theory, maximum likelihood estimation, hypothesis tests and confidence intervals, linear models. Previous experience with R is not essential and can be obtained during the course (only a good working knowledge of R will be necessary, as provided, for instance, in the lab sessions of the first reference below).

Literature

1. G. James, D. Witten, T. Hastie and R. Tibshirani. *An Introduction to Statistical Learning (with Applications in R).* Springer, 2013. Available at https://www.statlearning.com/.

2. T. Hastie, R. Tibshirani and J. Friedman. *The Elements of Statistical Learning.* Springer, Second Edition.

3. A. Agresti. *Foundations of linear and generalized linear models.* John Wiley & Sons, 2015.

4. C. Robert and G. Casella. *Introducing Monte Carlo Methods with R.* Springer, 2009.

5. P.J. Brockwell and R.A. Davis. *Introduction to time series and forecasting.* Springer, 1996.

6. A.J. Dobson and A. Barnett. *An Introduction to Generalized Linear Models.* Third edition. Chapman & Hall/CRC, 2008.

7. L. Wasserman. *All of Statistics.* Springer, 2003.

Additional support

This course includes practical classes, in which statistical methods are introduced in a practical context and students carry out analysis of datasets using R. In the practical classes, the students have the opportunity to discuss statistical questions with the lecturer. Four example sheets will be provided and there will be four associated example classes (the last one probably to be held in Easter term). There will be a revision class in the Easter Term.

Statistics in Medicine (3 units)

Statistics in Medical Practice (M12)

Lecturers from the MRC Biostatistics Unit

This part of the course includes three modules covering a range of statistical methods and their application in three areas of biostatistics. It is firmly focused on how methods are used in application, rather than rigorous derivations of mathematical results.

Note that this part is examined together with Analysis of Survival Data (Lent), as a single 24-lecture course.

A. Causal Inference in Practice [4 Lectures] **(S. Burgess)**

It is well known that "correlation is not causation". But how then do you assess causal claims? Is it possible to show that X is a cause of Y? What does it even mean to say that X is a cause of Y? In this module, we introduce definitions of causal concepts, starting with the work of Rubin, Pearl, and Robins, and discuss practical approaches for assessing causal claims from observational data.

B. Multi-State Models in Health [4 Lectures] **(C. Jackson, D. De Angelis, P. Birrell)**

(i) Continuous-time multi-state models, applied to modelling the incidence and progression of non-communicable diseases. Defining quantities of applied interest from Markov models. Constructing likelihoods for parametric models given various forms of individual-level data. Interpreting and comparing estimates from fitted models.

(ii) Multi-state modelling to estimate incidence of infectious diseases from population-level data. Backcalculation methods for the estimation of incidence of disease with long incubation periods. Dynamic modelling of infectious disease transmission.

C. Design and Analysis of Clinical Trials [4 Lectures] **(M. Law, D. Couturier, S. Villar, D. Robertson)**

Introductory concepts in clinical trial design and analysis, including sample size estimation. Early phase dose-finding clinical trials using the continual reassessment method and Bayesian parametric models. Types of randomisation procedures. Optimal response-adaptive procedures. Adaptive and multi-stage designs (including group-sequential designs). Treatment effect estimation following a group-sequential trial.

Statistics in Medicine (3 units)

Analysis of Survival Data (L12)

P. Treasure

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

Note that this part is examined together with Statistics in Medical Practice (Michaelmas) as a single 24-lecture course.

D. Time-to-Event Analysis [4 Lectures]

'Survival analysis' is generalised to *time-to-event* analysis. The implications of event times which are unknown or in the future (*censored* data) are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (*Kaplan-Meier*) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (*log-rank* test).

E. Modelling Hazard [4 Lectures]

The *hazard* function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size...) on the time-to-event distribution (*proportional hazards* modelling). Model checking procedures are introduced with an emphasis on excess event (*Martingale*) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (*competing risk* analysis) and (ii) the cause of death is unknown (*net survival*). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar year (e.g. 2019) are addressed: *period* survival analysis.

Statistics in Medicine (3 units)
Additional Information

Prerequisites

Undergraduate-level statistics and probability: including analysis and interpretation of data, maximum likelihood estimation, hypothesis testing, basic stochastic processes.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. Some literature to complement the course material is listed below.

1. Armitage P, Berry G, Matthews JNS, *Statistical Methods in Medical Research.* Wiley-Blackwell, 2001. [A good introductory companion to the whole course]

2. Burgess S, Thompson SG, *Mendelian Randomization: Methods for Using Genetic Variants in Causal Estimation* Chapman and Hall, 2015 [Module A]

3. van den Hout, A, *Multi-State Survival Models for Interval-Censored Data.* Chapman and Hall, 2016 [Module B]

4. Keeling, MJ, & Rohani, P *Modeling Infectious Diseases in Humans and Animals.* Princeton University Press, 2008 [Module B]

5. Senn, S. *Statistical Issues in Drug Development.* Wiley, 2021. [Module C]

6. Jennison C, Turnbull B, *Group Sequential Methods with Applications to Clinical Trials.* Chapman and Hall, 2000. [Module C]

7. Rosenberger, William F., and John M. Lachin. *Randomization in clinical trials: theory and practice.* John Wiley & Sons, 2015. [Module C]

8. Cox DR, Oakes D, *Analysis of Survival Data.* Chapman and Hall, 1984 [Modules D, E, F: the classic text]

9. Collett D, *Modelling Survival Data in Medical Research.* CRC Press, 2023 [Modules D, E, F: modern, applied, supports and extends lectures.]

10. Aalen OO, Borgen Ø, Gjessing HK, *Survival and Event History Analysis: A Process Point of View.* Springer, 2008 [Modules D, E, F: excellent modern approach]

Additional support

[Modules A, B, C] One-hour example classes for each of the three modules, supported by question sheets and solutions, will be given in Michaelmas Term. Additional support can be arranged with individual lecturers.

[Modules D, E, F] A 90-minute example class, supported by question sheets and solutions, will be given in each of the Lent and Easter Terms. A two-hour revision class will be held just before the examination. There will be regular office hours by Zoom.

Topics in Statistical Theory (M16)

Richard J. Samworth

This course will provide an introduction to the theory behind a selection of statistical problems that play a key role in modern statistics. Most undergraduate statistics courses are restricted to the study of parametric models; here we will no longer assume that our distributions belong to finite-dimensional classes and will instead study fundamental nonparametric problems such the estimation of a distribution function, a density function or a regression function. We will also study minimax lower bounds, which characterise the intrinsic difficulty of a statistical problem, and provide benchmarks against which statistical procedures can be compared.

An outline of the course is as follows:

- An introduction to nonparametric statistics: the basics of empirical process theory, Glivenko–Cantelli theorem, Dvoretzky–Kiefer–Wolfowitz inequality. Concentation inequalities, including Hoeffding, Bennett and Bernstein inequalities.

- Kernel density estimation: Definition, bounds on bias and variance, uniform nonasymptotic bounds on MSE and MISE. Bandwidth selection via least squares cross validation and Lepski's method, choice of kernel, multivariate density estimation.

- Nonparametric regression: Local poynomial estimation, bounds on weights, bias, variance and MSE. Cubic splines, natural cubic smoothing splines, choice of smoothing parameter.

- Minimax lower bounds: Reduction to testing, f-divergences, Le Cam's two point lemma, Assoaud's lemma, the data processing inequality, Fano's lemma; examples.

Pre-requisites

A good background in undergraduate probability theory, elements of linear algebra and real analysis. Measure theory is not necessary but may be helpful; similarly for a preliminary course in mathematical statistics. Though the material in the Modern Statistical Methods course will not be needed here, the two courses complement each other well.

Literature

The lecturer is currently writing a book based on the course, and this should be available (if not published) in time for the course. Some of the material is covered in:

1. S. Boucheron, Lugosi, G. and Massart, P. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.
2. A. Tsybakov, *Introduction to Nonparametric Estimation*. Springer, 2009.
3. M. J. Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge University Press, 2019.

Additional support

Three examples sheets will be provided and associated examples classes will be given. There will be weekly office hours during Michaelmas and a revision class in the Easter Term.

Functional Data Analysis (L16)

Professor J. Aston

Functional Data Analysis (FDA) is the study of the statistics of curves and surfaces. In most statistical analyses, the object of study is either a number (one dimensional univariate statistics) or a vector (finite dimensional multivariate statistics). In FDA the object of study is usually considered to be infinite dimensional, often but not always, with some inherent smoothness characteristics. In this course, we will examine both the theory behind such data analysis and the myriad of different places that these data arise.

The course will likely cover the following topics:

- Definition of Functional Data

- Functional Principal Component Analysis
- Registration
- Covariance Operators
- Functional Linear Models

Prerequisites

Basic knowledge of statistics, probability, linear algebra and real analysis.

Literature

1. Ramsay, J.O. and Silverman B.W. *Functional Data Analysis* 2nd edition, Springer, 2005.
2. Horvath, L. and Kokoszka, P. *Inference for Functional Data with Applications*, Springer, 2012.
3. Kokoszka, P. and Reimherr, M. *Introduction to Functional Data Analysis*, Chapman and Hall, 2017.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Robust Statistics (L16)

Professor P. Loh

This is a topics course on the use of robustness in estimation and inference. We will begin with an overview of robustness theory from classical statistics, including notions such as the influence function and breakdown point, and the asymptotic theory of M-estimators. We will then survey a variety of more recent directions in machine learning and theoretical computer science where ideas in classical robustness theory have been leveraged and expanded.

Below is a tentative list of topics:

- Asymptotic theory of M-estimators and minimax results
- Influence functions, optimal robust estimators

- Robust linear regression
- Robust hypothesis testing
- Estimation under adversarial contamination
- Heavy-tailed estimation

Prerequisites

This course is appropriate for students with a general background in statistics. We will also assume proficiency in linear algebra and basic optimisation. Some familiarity with machine learning will be helpful.

Literature

1. Huber and Ronchetti, *Robust Statistics*, 2011.
2. Hampel, Ronchetti, and Rousseeuw, *Robust Statistics: The Approach Based on Influence Functions*, 2011.
3. Maronna, Martin, and Yohai, *Robust Statistics: Theory and Methods*, 2006.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter Term.