

Mathematical Tripos Part III Lecture Courses in 2024-2025

Department of Pure Mathematics and Mathematical Statistics

Department of Applied Mathematics and Theoretical Physics

1 Algebra

Commutative Algebra (M24)

Dr O. Becker

This course will provide an introduction to the theory of commutative rings and modules over these rings. It should be viewed as a foundational course for Algebraic Geometry and Algebraic Number Theory.

The course will cover a selection of topics including:

- Unital commutative rings, ideals, Hilbert's basis theorem
- Tensor products, flatness, localization
- Nakayama's lemma, integral and finite extensions, going up and going down theorems
- Noether's normalization theorem and Hilbert's nullstellensatz, primary decomposition
- Direct and inverse limits, a brief introduction to completions, the Artin-Rees lemma
- Dimension theory and Hilbert functions
- Discrete valuation rings and Dedekind domains

Prerequisites

You will have attended a first course in ring theory, such as the IB course Groups, Rings and Modules. Experience of more advanced material such as

Part II courses Galois Theory, Representation Theory, Algebraic Geometry or Number Fields is desirable (contributing to your level of algebraic maturity) but not essential.

Literature

1. M.F. Atiyah and I.G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley, 1969.
2. J.S. Milne, *A Primer of Commutative Algebra*, <https://www.jmilne.org/math/xnotes/ca.html>
3. D. Eisenbud, *Commutative Algebra with a view toward Algebraic Geometry*, Springer-Verlag, 1995.
4. H. Matsumura, *Commutative Algebra, second edition*, The Benjamin/Cummings publishing co., 1980.
5. H. Matsumura, *Commutative Ring Theory*, Cambridge Studies 8, CUP, 1989.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Coxeter Groups (L16)

Dr A. Jones

This course will examine the theory of Coxeter groups from an algebraic perspective, focusing on their applications within the representation theory of algebraic groups.

Coxeter groups are abstractly defined as groups generated by *involutions*, i.e. elements of order 2. As such, they arise very naturally throughout geometry, as reflectional symmetry groups. In algebra, they also play a major role in Lie theory, in the form of the Weyl group of a semisimple Lie algebra or a reductive Lie group, structures of fundamental importance within modern number theory and physics.

We will cover the following topics, building up the theory of Coxeter groups from their most basic examples, exploring their key properties and some applications.

- Finite and linear reflection groups.

- Definition and basic properties of Coxeter systems, Coxeter diagrams, length functions and irreducibility.
- Geometric representation, and classification of finite Coxeter groups.
- Affine, Spherical, Euclidean and Hyperbolic groups, examples from Lie theory.
- Groups with (B, N) -pairs and buildings.
- Introduction to Hecke algebras and Kazhdan-Lusztig theory.

Prerequisites

Part IA Geometry, Part IA Groups and Part IB Linear Algebra are definite prerequisites. An understanding of representation theory is desirable, so Part IB Groups, Rings and Modules and Part II Representation Theory are recommended, as is very strongly the additional Part III course Lie algebras and their Representations.

Literature

Our main resource will be Humphreys' book [4], but the other references below also give a good overview of the theory, and some further reading, including applications in Lie theory [2], Kazhdan-Lusztig theory [3], and the theory of buildings [1,6] that we will not have time to explore in detail. See [5] for a more geometric approach to the subject.

1. K. S. Brown, *Buildings*, Springer-Verlag, New York (1989).
2. R. W. Carter, *Simple Groups of Lie type*, Wiley Classics Library, John Wiley & Sons, Inc., New York, (1989 reprint).
3. B. Elias; S. Makisumi; U. Thiel; G. Williamson, *Introduction to Soergel Bimodules*, RSME Springer Series Volume 5 (2020).
4. J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge University Press, Cambridge (1990).
5. A. Thomas, *Geometric and Topological aspects of Coxeter groups and Buildings*, European Mathematical Society (2018).
6. M. Ronan, *Lectures on Buildings*, Perspectives in Mathematics **7** (1989).

Additional support

There will be 16 one-hour lectures given in Lent term 2025, as well as four examples classes. Four examples sheets will be provided, to be discussed during the examples classes. There will also be a revision class in Easter Term 2025 before the exam.

Hochschild cohomology and deformation theory (L16)

Non-Examinable (Part III Level)

Dr C.J.B. Brookes

Algebraic deformation theory is primarily concerned with the interplay between homological algebra and the perturbations of algebraic structures. For example one might want to deform a commutative algebra to give a non-commutative one via ‘quantisation’.

In a series of papers Gerstenhaber developed deformation theory for associative algebras. The cohomology theory required in this context is Hochschild cohomology and one finds that 2-cocycles arise from deformations. A more advanced approach is to consider additional algebraic structures, including that of a differential graded Lie algebra defined on the Hochschild cocomplex of the associative algebra. Deformations correspond to Maurer-Cartan elements.

The course will start with a brief introduction to Hochschild cohomology which is the cohomology theory for associative algebras. Possible examples to consider are the Hochschild cohomology rings of polynomial algebras and of finite group algebras. I will then discuss Gerstenhaber’s work. There have been applications in a variety of contexts and I’ll aim to choose one or two according to the interests of the audience. Alternatively I may finish by discussing Hopf algebras and their deformations, and consider some examples of quantum groups.

Prerequisites

I shall assume basic algebraic knowledge with experience of groups, rings, modules and commutative algebra and it will be helpful to have met Lie algebras. I will point out connections with the courses on algebraic D-modules and on modular representations of finite groups.

Preliminary Reading

The introductory article by Fox is a good place to start. It first describes the classical theory of deformations of associative algebras and then moves on to more general algebraic structures. The papers of Gerstenhaber are also very readable.

Literature

1. T. Fox *An introduction to algebraic deformation theory*. J Pure and Applied Algebra, **84**, 17-41 (1999).
2. M. Gerstenhaber *On the deformation theory of rings and algebras*. Ann. Math. **78**, 267-288 (1963).
3. M. Gerstenhaber *On the deformation theory of rings and algebras II*. Ann. Math. **84**, 1-19 (1966).
4. M. Gerstenhaber *On the deformation theory of rings and algebras III*. Ann. Math. **88**, 1-34 (1968).
5. M. Gerstenhaber *On the deformation theory of rings and algebras IV*. Ann. Math. **99**, 257-256 (1974).
6. S.F. Siegel and S.J. Witherspoon *The Hochschild cohomology ring of a group algebra*. Proc. London Math. Soc. **79**, 131–157 (1999).
7. S.J. Witherspoon, *Hochschild cohomology for algebras*. Graduate studies in mathematics **204**, A.M.S. 2019.

Lie algebras and their Representations (M24)

Jef Laga

This course is an introduction to the properties and representations of semisimple complex Lie algebras. The structure and representation theory of semisimple Lie algebras is one of the most beautiful and wide-reaching subjects in mathematics. It has applications to number theory, topology, algebraic geometry, and theoretical physics, to name a few examples. Lie algebras arise as tangent spaces to certain manifolds called Lie groups, yet they can be defined purely algebraically, which simplifies their study. The representation theory of a complex simple Lie algebra can be understood in terms of combinatorial data (e.g. roots, weights, and Weyl groups). Understanding this data will be at the heart of this course.

The following is an outline of the topics that will be covered in the course:

1. The basics of (general) Lie algebras and their representations.
2. Representation theory of $\mathfrak{sl}_2$.
3. Root systems, Weyl groups, and the classification of simple complex Lie algebras.
4. The classification of finite-dimensional representations of simple complex Lie algebras, Verma modules, and the Weyl character formula.

Prerequisites

The only real prerequisite is linear algebra. It will be beneficial, but not strictly necessary, to have some familiarity with group actions, modules and basic representation theory, as well as tensor products, symmetric powers, and exterior powers of vector spaces.

Literature

1. Humphreys, James E. *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, 9, Springer-Verlag, New York-Berlin, 1978. (The canonical reference, which is closest to the route we will take.)
2. Fulton, William and Harris, Joe. *Representation theory*, Graduate Texts in Mathematics, 129, Springer-Verlag, New York, 1991. (Contains a wealth of examples. Lecture 1 is a good introduction to representation theory of finite groups that motivates some of the results that we will discuss.)
3. Carter, Roger W. *Simple groups of Lie type*, *Wiley Classics Library*, John Wiley & Sons, Inc., New York, 1989.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Modular Representation of Finite Groups (L16)

Non-Examinable (Part III Level)

Stuart Martin

Modular representation theory is the representation theory of finite groups over a field of prime characteristic p . It is very different from the characteristic

zero theory which most undergraduates are exposed to in a first course, though there is a rich interplay between the two. If G is a finite group of Lie type such as $\mathrm{GL}_n(\mathbf{F}_q)$ then the normaliser of a Sylow subgroup is a Borel subgroup, and more generally parabolic subgroups are normalisers of certain p -subgroups. Since the parabolic subgroups play such an important role in other areas of representation theory (including the theory of automorphic forms) it is not surprising that the normalisers of p -groups should play an important role in the representation theory of arbitrary finite groups. What is surprising is the Brauer correspondence, a deep connection between the representation theory of G and normalisers of p -subgroups, an altogether new phenomenon in the modular theory. More recently, beginning with work of Lascoux, Leclerc and Thibon, deep connections have been found with the representation theory of quantum groups and modular representation theory of (for example) of the symmetric group.

Taking a step back, the first really major developments came with Richard Brauer, who from 1935 onwards exploited this rich and virtually untapped area of mathematics. Brauer's methods were mainly character-theoretic. What is now called the theory of Brauer characters determines the composition factors of the representation but, unlike the situation over $\mathbf{C}$, not the equivalence type. One of Brauer's main goals was finding numerical constraints on the orders and internal structure of finite simple groups, and his methods were later used by Glauberman and others in the Classification program for finite simple groups.

The next revolution in the theory came with Sandy Green, who considered the modules themselves. His techniques were completely different to those of Brauer, and his main goals lay in understanding the modules ring-theoretically, rather than Brauer who worked mostly with characters and their blocks (a set of equivalence classes of characters). The aim of this course is to give a flavour of both classical techniques of Brauer and the more modern techniques of Green. We'll discuss Brauer characters, defect groups, blocks, decomposition numbers as well as projective and injective modules (as time allows).

I'll aim to cover some of the following:

- Review of assumed and basic material: group algebras, representations and modules, reducibility and decomposability, Maschke's theorem, tensors and homs, Frobenius reciprocity, Mackey's theorem, maximal and primitive ideals, Jacobson radical, complete reducibility, semisimple rings, Artin-Wedderburn structure theorem.
- Modular character theory: p -singular and p -regular elements, Brauer characters, Grothendieck groups.
- Irreducible, projective and injective modules: DVRs, (splitting) p -modular systems, forms, decomposition numbers and the decomposition matrix, counting modular irreducible modules.

- Projective indecomposable modules and their characters: the Higman criterion, primitive orthogonal idempotents, idempotent refinement.
- Block theory: the socle and the head, Cartan matrix, lifting projectives, central idempotents.
- Central characters: the central character of an ordinary irreducible mod p determines the block.
- (If time permits) Further block theory: defect groups, relative projectivity and blocks of defect 0. (Statement of) the Alperin conjecture, some comments on local–global conjectures.

Prerequisites

Basic group theory; ordinary representations/character theory from the Part II (or equivalent) course; Sylow theory for finite groups; commutative algebra from the relevant Part III course (rings, ideals, completions, local rings, primality, maximality, the Artin-Wedderburn theorem); some categorical nonsense. The Part III course on the representations of complex semisimple Lie algebras is complementary to this course.

Preliminary Reading

1. C.W. Curtis, *Pioneers of representation theory: Frobenius, Burnside, Schur and Brauer* (AMS 1999)
2. P. Etingof *et al*, *Introduction to representation theory* (AMS 2011)
3. I.M. Isaacs, *Character theory of finite groups* (Dover reprint 1994)
4. G.D. James & M.W. Liebeck, *Representations and characters of groups* (CUP, 2nd edn 2001)
5. W. Ledermann, *Introduction to group characters* (CUP, 2nd edn, 1987)

Literature

1. J.L. Alperin, *Local representation theory* (CUP 1986)
2. D.J. Benson, *Representations and cohomology, Vol 1* (CUP 1991)
3. D.A. Craven, *Representation theory of finite groups: a guidebook* (Springer 2019) - it has 574 references!
4. P. Schneider, *Modular representation theory of finite groups*, (Springer 2013)
5. P. J. Webb, *A course in finite group representation theory* (CUP 2016)

Amenability

Non-Examinable (Part III Level)

Dr. F. Fournier-Facio

Amenability was originally introduced by von Neumann in an attempt to explain the Hausdorff–Banach–Tarski paradox. Since then, it has become the cornerstone of analytic group theory, and one of the most important topics in the theory of infinite discrete groups in general. The goal of the course is to introduce you to this topic, and to the techniques of analytic group theory along the way.

One of the most fascinating things about amenability is that it has a shocking number of equivalent characterisations. These involve: invariant measures, paradoxical decompositions, fixed point theorems, almost invariant sets, almost invariant vectors, harmonic functions, operator algebras, bounded cohomology, symbolic dynamics... The core of the course will consist in proving some of these equivalent characterisations.

Additional topics will be covered depending on the interests of the audience. These will be chosen among the following.

- Counterexamples to the von Neumann–Day problem.
- Quasi-isometries, orbit equivalence, and the Ornstein–Weiss Theorem.
- Relation to dynamics, groups acting on the circle and the Margulis alternative.
- Amenability over non-Archimedean fields.
- Gromov’s polynomial growth Theorem.
- Antipodes of amenability: property (T).

Prerequisites

A first course in group theory and a first course in topology (beyond metric spaces) will be assumed throughout.

Some basics of functional analysis are fundamental to the study of amenability. Specifically: normed vector spaces, the weak-* topology, the Banach–Alaoglu Theorem and the Hahn–Banach Theorem, possibly nets and subnets. Since not everyone has done a functional analysis course in their undergrad, I will quickly review these notions and results when they are needed, but you are encouraged to catch up on them on your own before the course starts. These are all basic enough that wikipedia will do, otherwise any functional analysis textbook (Rudin / Conway / Zimmer) will have all you need.

Literature

I will write lecture notes throughout the semester and make them available week by week. Here are some general references on amenability, where you will be able to find everything that the course will cover and more.

1. T. Ceccherini-Silberstein and M. Coornaert, *Cellular automata and groups*. Springer, 2010.
2. K. Juschenko, *Amenability of discrete groups by examples*. AMS, 2022.
3. T. Cohen, T. Gelander. *An invitation to analytic group theory*. arXiv, 2024.
4. V. Runde. *Lectures on amenability*. Springer, 2002.

Additional support

Some steps in the course will be left as exercises. These will not have accompanying solutions or be graded, but you are encouraged to give them a try and there will be dedicated office hours where we can talk about them.

Introduction to Geometric Representation Theory (L24)

Ian Grojnowski

This is a course about algebra and representation theory. Given a group, for example SL_2 , or perhaps a Lie algebra, such as sl_2 , one would like to understand how it acts in the world — to describe each of its representations. We want to know how many there are, what their dimension is as a vector space, how they break up into representations of natural subgroups or algebras — for a representation of a finite group that means determining its character — and so on.

Sometimes one can do this just with linear algebra. You will have seen this in the Lie algebra course, where the finite dimensional representations of a complex semisimple Lie algebra are completely described.

Most of the time this is not possible. For example, you might want to understand how a Verma module, which is an ‘induced’ representation of a simple Lie algebra, decomposes into irreducibles. Strenuous computations in small examples such as sl_4 produce answers that appear quite mysterious.

About 45 years ago it was discovered (the ‘Kazhdan-Lusztig conjecture’) that these mysterious answers are in fact topological invariants — they are determined by the cohomology of natural algebraic varieties associated to the representation theory. This was proved by Beilinson and Bernstein, using D -modules, perverse sheaves and the Weil conjectures.

Since then it has been understood that for most interesting algebras or groups – even finite groups like $SL_n(F_q)$ – that certain categories of sheaves on algebraic varieties control their representation theory.

The two widely used names for this area of study are ‘geometric representation theory’, and ‘the Geometric Langlands programme’.

This is intended to be a *gentle* Part III introductory course on geometric representation theory. I’ll assume you know some very basic algebraic geometry, and not much else. The Lie algebra course will be helpful though not entirely necessary, as most of the theorems and constructions are already interesting for sl_2 . Hopefully the course will motivate you to pick up any technical background you need but do not have.

We will begin with the Lie algebra representations of sl_2 and the Beilinson-Bernstein theorem, by hand (this will be really good revision for the part III algebraic geometry course!) then continue with a few other topics.

These may include 1) the Borel-Weil-Bott theorem and introduction to the geometry of the flag variety, 2) the Beilinson-Bernstein theorem describing representations of semisimple complex Lie algebras, 3) a brief introduction to complex representations of Chevalley groups, such as $SL_2(F_q)$ and Deligne-Lusztig theory, 4) the mod p representations of these same groups, and 5) Jantzen’s theorems on reduction mod p of the complex representations.

In each case we will do SL_2 by hand, with very little technical background required, and then only state and prove some indicative theorems about the general case (as this is an introductory course).

If time remains we will continue with more recent and harder subjects, which will not be examinable. (When lectures are over the course moodle will list the examinable topics).

Prerequisites

For the first two topics I’ll assume you know what a Lie algebra is and have some basic algebraic geometry, but not a lot else. For the next few topics, you need to know what a character of a representation is (the Part II representation theory class is more than sufficient).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

2 Algebraic Geometry

Algebraic Geometry (M24)

Professor Mark Gross

The course will be an introduction to modern algebraic geometry. The first part of the course will develop the basic theory of sheaves, and use this to give a treatment of schemes, morphisms between them, and fibre products. We will then explore morphisms the basic properties of morphisms, including separatedness and properness. After this, we turn to the study of quasi-coherent sheaves, with a particular focus on line bundles, the Picard group, and other proximate notions. The final few lectures in the course will give a rapid introduction to sheaf cohomology “for the working mathematician”.

The course will develop the basic toolkit used by practicing algebraic geometers, and will also be appropriate for anyone with an interest in acquiring a working knowledge of algebraic geometry, including those in nearby fields such as differential geometry, topology, and number theory.

Prerequisites

The basic theory of rings and modules, and elementary topology will all be assumed. It is recommended that students have had a previous course in Commutative Algebra, take one in parallel, or familiarize themselves with the concepts from Atiyah–MacDonald’s text on the subject. In the latter case, the crucial content is in Chapters 1–3 and 5–7. The key ideas from algebra that we will need are ideals, modules, tensor products, and localization. Basic intuition is provided from topology and the theory of manifolds, and this may be helpful but is not essential.

A prior course in the algebraic geometry of affine and projective varieties is not logically necessary for the course, i.e. the contents of the Part II Algebraic Geometry course. However, students should be aware that certain key examples in scheme theory do come from classical algebraic geometry. Students with no prior exposure to algebraic geometry may occasionally need to look up such constructions. However, this will be the exception rather than the rule, and ample references will be provided.

Literature

The following books complement the course material. The first and second are roughly interchangeable. The final text provides valuable intuition for the many unfamiliar concepts in scheme theory.

1. R. Hartshorne, *Algebraic Geometry*, Springer (1977).
2. R. Vakil, *The rising sea. Foundations of algebraic geometry*. Available online.
3. D. Eisenbud and J. Harris, *Geometry of schemes*, Springer (2000).

Additional support

Four examples sheets will be provided and four associated examples classes will be given; these will be essential. Additional problems and examples may be provided to clarify more difficult concepts. There will be a revision class in the Easter Term.

Abelian Varieties (L24)

Professor A. J. Scholl

An abelian variety is an irreducible projective variety over a field which is also a group. The simplest example is an elliptic curve. Other examples include the Jacobian variety of a (smooth projective) curve, whose points parametrise divisor classes of degree zero. There are beautiful analytic and algebraic theories of such objects. They also crop up in other areas of geometry and in number theory.

Although elliptic curves can be studied using fairly elementary algebraic geometry and explicit equations, for abelian varieties of higher dimension scheme theory is pretty much essential, and gives a much better understanding (as well as a good illustration of how powerful scheme theory is). This course will therefore be in 2 parts.

The first 3–4 weeks will be a continuation of the Algebraic Geometry course from Michaelmas. We will cover various topics in abstract algebraic geometry. One important topic will be the study of how cohomology varies in a family (cohomology and basechange).

In the remainder of the lectures, we will introduce group schemes, and develop some of the algebraic theory of abelian varieties, approximately covering Chapter 2 of Mumford's book. If time permits, we may also give the construction of the Jacobian of a curve.

Prerequisites

Algebraic Geometry (MT) and Commutative Algebra (MT) or equivalent are essential.

Literature

1. David Mumford: *Abelian varieties*. Second edition. Tata Institute of Fundamental Research, 1974.
2. Edixhoven, Moonen and Van Der Geer: draft book, available at <http://van-der-geer.nl/gerard/AV.pdf> or <https://www.math.ru.nl/bmoonen/research.html#bookabvar>
3. Robin Harsthorne: *Algebraic geometry*. Springer, corrected edition. (For the first half of the course)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Intersection Theory (L24)

Dr. Alessio Cela

Intersection theory plays a similar role in algebraic geometry as cohomology does in the theory of manifolds. It is the starting point for many areas of modern research, notably enumerative geometry. The first part of the course will develop the theory of Chow groups, including cycles, rational equivalence, flat pullbacks, Chern classes, Segre classes, and intersection products. Along the way, we will introduce important concepts in algebraic geometry, such as flatness, projective bundles, and Grassmannians, as needed by the theory. We will conclude the course by proving the Riemann-Hurwitz formula, computing the Chow ring of Grassmannians, and, time permitting, counting 27 lines on a cubic surface.

Prerequisites

It is required that the students have a previous knowledge of the language of schemes and sheaves, i.e. the contents of the Part III Algebraic Geometry course. Algebraic topology may be helpful for context but is not essential.

Literature

1. W. Fulton *Intersection Theory*. 2nd edition. Springer.
2. D. Eisenbud and J. Harris, *3264 & All That. Intersection Theory in Algebraic Geometry*. Cambridge University Press.

Additional support

Four examples sheets will be provided.

3 Analysis and PDEs

Analysis of Partial Differential Equations (M24)

Professor C. Mouhot

This course serves as an introduction to the mathematical study of Partial Differential Equations (PDEs). The theory of PDEs is presently a huge area of active research, and it goes back to the very birth of mathematical analysis in the 18th and 19th centuries. The subject lies at the crossroads of physics and many areas of pure and applied mathematics.

The course will mostly focus on developing the theory and methods of the modern approach to PDE theory. Emphasis will be given to functional analytic techniques, relying on a priori estimates rather than explicit solutions. The course will primarily focus on approaches to linear, elliptic and evolutionary problems through energy estimates, with the prototypical examples being Laplace's equation, the heat equation, the transport equation, the wave equation and Schrödinger's equation.

The following concepts will be studied:

- well-posedness in the sense of Hadamard
- the Cauchy problem for general (non-linear) PDE
- characteristics
- Sobolev spaces
- elliptic boundary value problems (solvability and regularity)
- evolutionary problems (hyperbolic, parabolic and dispersive PDE)

Prerequisites

There are no specific prerequisites beyond a standard undergraduate analysis background, in particular a familiarity with measure theory and integration. The course will be mostly selfcontained and can be used as a first introductory course in PDEs for students wishing to continue with some specialised PDE Part III courses in the Lent and Easter terms.

Literature

As a preliminary reading and revision, students are encouraged to read through the online notes of one of the incarnations of the Part II Analysis of Functions course. Previous lecture notes for this Part III course are available at <https://cmouhot.wordpress.com/teaching/> and <https://www.dpmms.cam.ac.uk/~cmw50/Teaching/>.

1. Klainerman, S., Partial Differential Equations, *Princeton Companion to Mathematics* (editor T. Gowers), Princeton University Press, 2008.
2. Evans, L. C., *Partial Differential Equations*, Springer, 2010.
3. Brezis, H., *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer, 2010.
4. John, F., *Partial Differential Equations*, Springer, 1991.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Fourier restriction theory and applications (L24)

Professor D. Maldague

Decoupling and restriction theory is a recent branch of Fourier analysis that has applications in number theory, incidence geometry, and partial differential equations. This course seeks to establish fundamental intuitions and give some sense of the current state of research.

The course will cover a selection of topics including:

- The Kakeya conjecture: (Borel) subsets of R^n containing a line in every direction must have Hausdorff dimension n . Discrete variants. Equivalent variants. Multilinear variants.

- Basics of Fourier analysis: common properties of the Fourier transform, convolution, some stationary/non-stationary phase estimates.
- Tools from modern Fourier restriction theory: locally constant property (i.e. *uncertainty principle*), local L^2 orthogonality, wave packet decomposition, multi-scale analysis (induction on scales).
- Hierarchy of conjectures related to the Fourier restriction conjecture, Keakeya conjecture, square function conjecture, Bochner-Riesz conjecture, etc.
- Decoupling theory with some applications, like to the Vinogradov Mean Value Theorem and Strichartz estimates for the periodic Schrödinger equation over irrational tori.

Prerequisites

This course assumes you are familiar with the basics of analysis, like L^p spaces, Hölder's inequality, Plancherel's theorem, Schwartz functions, interpolation, etc.

Literature

1. Demeter C. Fourier Restriction, Decoupling, and Applications. Cambridge University Press; 2020.
2. Larry Guth, course notes available at <https://math.mit.edu/~lguth/Math118.html>.
3. Terry Tao, course notes available at <https://www.math.ucla.edu/~tao/254b.1.99s/>.
4. Tom Wolff, course notes available at https://personal.math.ubc.ca/~ilaba/wolff/notes_march2002.pdf.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Elliptic Partial Differential Equations (L24)

Greg Tautjanskas

This course will provide an introduction to the theory of linear second order elliptic partial differential equations. Second order elliptic equations play a fundamental role in many areas of mathematics including Calculus of Variations, Riemannian Geometry and Mathematical Physics. Linear elliptic theory provides the foundation for studying a number of non-linear problems arising in

these fields, such as minimal submanifolds, harmonic maps and evolution problems in Geometry and Mathematical Physics.

The course will provide a rigorous treatment, based on a priori estimates, of both classical and weak solutions to linear elliptic equations, focusing on the question of existence and uniqueness of solutions to the Dirichlet problem and the question of regularity of solutions. Specific topics include:

- harmonic functions
- maximum principles for general second order equations
- Schauder estimates (via L. Simon's scaling argument)
- the continuity method for existence of solutions
- solvability of the Dirichlet problem in balls
- Perron's method, divergence form operators
- De Giorgi–Nash–Moser estimates and the Harnack theory
- (as time permits) a brief discussion of the quasilinear theory centred around the prototypical example of the Minimal Surface Equation

Prerequisites

Part III *Analysis of PDEs*.

Literature

1. D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*.
2. L. Simon, *Schauder estimates by scaling*. Calc. Var. & PDE, **5**, (1997), 391–407.
3. P. Minter, Lecture notes for the courses *Elliptic PDEs* and *The Minimal Surface Equation and Related Topics*, <https://minterscompactness.wordpress.com/lecture-notes/>

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Functional Analysis (M24)

Dr A Zsák

This course covers many of the major theorems of abstract Functional Analysis. It is intended to provide a foundation for several areas of pure and applied mathematics. We will cover the following topics:

- Hahn–Banach Theorems on the extension of linear functionals. Locally convex spaces.
- Duals of the spaces $L_p(\mu)$ and $C(K)$. The Radon–Nikodym Theorem and the Riesz Representation Theorem.
- Weak and weak-* topologies. Theorems of Mazur, Goldstine, Banach–Alaoglu. Reflexivity and local reflexivity.
- Hahn–Banach Theorems on separation of convex sets. Extreme points and the Krein–Milman theorem. Partial converse and the Banach–Stone Theorem.
- Banach algebras, elementary spectral theory. Commutative Banach algebras and the Gelfand representation theorem. Holomorphic functional calculus.
- Hilbert space operators, C^* -algebras. The Gelfand–Naimark theorem. Spectral theorem for commutative C^* -algebras. Spectral theorem and Borel functional calculus for normal operators.
- Some additional topics time permitting. For example, uniform convexity and smoothness, ultraproducts, the Fréchet–Kolmogorov Theorem, weakly compact subsets of $L_1(\mu)$, the Eberlein–Šmulian and the Krein–Šmulian theorems, the Gelfand–Naimark–Segal construction.

Prerequisites

Thorough grounding in basic topology and analysis. Some knowledge of basic functional analysis and basic measure theory (much of which will be recalled either in lectures or via handouts). In Spectral Theory we will make use of basic complex analysis. For example, Cauchy’s Theorem, Cauchy’s Integral Formula and the Maximum Modulus Principle.

Literature

1. Allan, Graham R. *Introduction to Banach spaces and algebras (prepared for publication by H. Garth Dales)*. Oxford University Press, 2011.

2. Bollobás, Béla *Linear analysis: an introductory course*. Cambridge University Press, 1990.
3. Murphy, Gerard J. *C^* -Algebras and Operator Theory*. Academic Press, Inc., 1990.
4. Rudin, Walter *Real & Complex Analysis*. McGraw-Hill, 1987.
5. Rudin, Walter *Functional Analysis*. McGraw-Hill, 1990.
6. Taylor, S. J. *Introduction to measure and integration*. Cambridge University Press 1973.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term. There will be some material as well as examples sheets and announcements available at www.dpmms.cam.ac.uk/~az10000/

4 Applied and Computational Analysis

Distribution Theory and Applications (L16)

Dr. A. Ashton

This course will give an introduction to the theory of distributions and its application to the study of linear PDEs. We aim to make mathematical sense of objects like the Dirac delta function and find out how to meaningfully take the Fourier transform of a polynomial. The course will focus on the *use* of distributions, rather than the functional-analytic foundations of the theory.

First we will cover the basic definitions for distributions and related spaces of test functions. Then we look at operations such as differentiation, translation, convolution and the Fourier transform. We will introduce the Sobolev spaces $H^s(\mathbf{R}^n)$ and $H_{\text{loc}}^s(X)$ and describe them in terms of Fourier transforms for tempered distributions. The material that follows will address questions such as

- What does a generic distribution look like?
- Why are solutions to Laplace's equation always infinitely differentiable?
- Which functions are the Fourier transform of a distribution with compact support?

i.e. structure theorems, elliptic regularity, Paley-Wiener-Schwartz. The final section of the course will be concerned with Hörmander's theory of oscillatory integrals.

Prerequisites

Elementary concepts from undergraduate real analysis. Some knowledge of complex analysis would be advantageous (e.g. the level of IB complex methods or analysis). Knowledge of measure theory and functional analysis will *not* be needed, but might be complementary.

Preliminary Reading

1. F.G. Friedlander & M.S. Joshi, *Introduction to the Theory of Distributions*, C.U.P, 1998.
2. M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions*, C.U.P, 1958.
3. G.B. Folland, *Introduction to Partial Differential Equations*, Princeton Univ Pr, 1995.

Literature

1. L. Hörmander, *The Analysis of Linear Partial Differential Operators: Vols I-II*, Springer, 1985.
2. M. Reed & B. Simon, *Methods of Modern Mathematical Physics: Vols I-II*, Academic Press, 1979.
3. F. Trèves, *Linear Partial Differential Equations with Constant Coefficients*, Routledge, 1966.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. Model solutions will be made available. There will be a revision class in the Easter Term.

Geometric Numerical Analysis and Deep Learning (L16)

Non-Examinable (Part III Level)

Davide Murari

Differential equations are fundamental to mathematically describe physical phenomena and play a crucial role in many disciplines. Finding an exact solution to these equations is generally impossible; thus, reliable numerical approximations are essential. Among the methods providing such approximations, some aim not only to obtain a quantitatively accurate result but also to ensure a qualitatively accurate representation of the underlying dynamics. These methods are generally called *geometric* or *structure-preserving*.

This non-examinable course focuses on a selection of geometric numerical methods for Ordinary Differential Equations (ODEs), including

- Symplectic numerical methods for Hamiltonian systems, and
- Methods reproducing the non-expansive nature of some ordinary differential equations, such as negative gradient flows of convex potentials.

In addition to traditional applications, the numerical approximation of ODE solutions has also demonstrated importance in the analysis and design of neural networks. A neural network $\mathcal{N}_\theta = F_{\theta_L} \circ \dots \circ F_{\theta_1}$ can be seen as the composition of parametric maps, called layers of the network, where the parameters in $\theta = (\theta_1, \dots, \theta_L)$ are chosen so that the resulting function solves sufficiently accurately a task of interest. When the number of network layers is larger than two, $\mathcal{N}_\theta$ is called a *deep neural network*, and *deep learning* is the field studying these deep networks.

Using geometric numerical methods to design neural networks has proven to be a valuable approach to structure them as desired, such as making them less sensitive to input perturbations. This course will include some applications of the presented geometric numerical methods in the context of deep learning, highlighting their direct relevance and importance in the field.

Prerequisites

This course assumes some familiarity with numerical methods for ordinary differential equations, particularly Runge–Kutta methods. The numerical experiments presented in the course will be based on the Python programming language. Some experience with Python is beneficial but not necessary. No knowledge about neural networks is required.

Literature

1. Iserles, A., *A first course in the numerical analysis of differential equations*. Cambridge University Press, 2009

2. Hairer, E., Lubich, C., and Wanner, G., *Geometric Numerical Integration. Structure-Preserving Algorithms for Ordinary Differential Equations*, 2006
3. Weinan, E., *A proposal on machine learning via dynamical systems*. Communications in Mathematics and Statistics, 1(5), 1-11, 2017.
4. Celledoni, E., Ehrhardt, M. J., Etmann, C., McLachlan, R. I., Owren, B., Schonlieb, C. B., and Sherry, F., *Structure-preserving deep learning*. European journal of applied mathematics, 32(5), 888-936, 2021.

Inverse Problems (M24)

Dr F. J. Sherry

Whenever we have a model for how to take measurements of the state of some underlying object, there is a natural question of whether this state can be recovered from the corresponding measurements. The field of inverse problems concerns itself with exactly this question. Inverse problems arise naturally in many scientific and engineering settings, such as medical imaging (reconstruction problems in magnetic resonance imaging, computed tomography etc.), seismic imaging (e.g. full waveform inversion) and computer vision (e.g. optical flow estimation).

Interesting inverse problems are usually *ill-posed*, meaning that an inverse doesn't exist or, even if a naïve inversion can be defined, its output is unstable, making it useless in applications: measurements are generally corrupted by noise, which will be blown up by the inversion.

In this course, we will study methods that can be used to overcome this ill-posedness, with a focus on applications to imaging. The course will consist of two main parts, the first of which studies deterministic methods, through the classical *regularisation theory* of (linear) inverse problems. In this part, we will pay special attention to so-called *variational regularisation* methods, which can be efficiently solved using modern (convex) optimisation methods. In the second part, we will take a *Bayesian* statistical perspective on inverse problems, which is a natural framework for studying stochastic methods for inverse problems.

Prerequisites

This course assumes basic knowledge of linear algebra, analysis (as treated in Part II Linear Analysis or Analysis of Functions) and probability theory (as treated in Part II Probability and Measure). Additional knowledge in convex analysis and Bayesian statistics is beneficial, but not strictly necessary.

Literature

1. H. W. Engl, M. Hanke and A. Neubauer. *Regularization of Inverse Problems*. Vol. 375, Springer Science & Business Media, 1996, ISBN: 9780792341574.
2. O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier and F. Lenzen. *Variational Methods in Imaging*. Applied Mathematical Sciences, Springer New York, 2008, ISBN: 9780387309316.
3. M. Dashti and A. M. Stuart. *The Bayesian Approach to Inverse Problems*, in: *Handbook of Uncertainty Quantification*. Springer Cham, 2017, ISBN: 9783319123844.
4. A. M. Stuart, *Inverse problems: a Bayesian perspective*. Acta Numerica, **19**, pp. 451–559, 2010. <https://doi.org/10.1017/S0962492910000061>.
5. M. Benning, M. Burger, *Modern Regularization Methods for Inverse Problems*. Acta Numerica, **27**, pp. 1—111, 2018. <https://doi.org/10.1017/S0962492918000016>.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Spectral Computations in Infinite Dimensions (L16)

Matthew Colbrook

The spectral theory of operators is a fundamental area of mathematics with profound implications across diverse fields. It plays a pivotal role in quantum mechanics, where the spectrum of an operator corresponds to potential measurement outcomes. Furthermore, in areas such as signal processing, control theory, data-driven discovery in dynamical systems, and partial differential equations (PDEs), spectral analysis methods are essential for solving complex problems.

This course aims to offer a comprehensive understanding of operator theory on separable Hilbert spaces, with a particular emphasis on the foundations of computing spectra of operators. The course is ideally suited for graduate students in mathematics or physics who have a keen interest in advanced functional analysis concepts. A significant aspect of the course involves learning how to manage the error and ensure convergence when reducing an infinite-dimensional problem to a series of finite-dimensional ones.

The course will cover a selection of topics including:

- Frameworks for establishing the foundations of computation and classifying problem difficulty. For example, we will explore the concept of the Solvability Complexity Index and discuss developing canonical embedding problems for problem classification.
- Computing spectra of operators via computing the resolvent operator and addressing the fundamental issue of spectral pollution (spurious eigenvalues).
- Spectral measures and continuous spectra of self-adjoint operators, convolution methods for their computation, and contour projection methods.
- Contemporary applications in data-driven study of so-called Koopman operators and transfer operators for nonlinear dynamical systems.
- Time permitting, we will discuss the broader implications of these techniques beyond pure spectral theory, including areas such as computer-assisted proofs and PDEs.

Prerequisites

This course presumes basic knowledge in linear algebra, analysis, and linear analysis. Other useful Part III courses include *Functional Analysis*. However, no formal background in numerical analysis courses is required.

Literature

1. W. Rudin, *Functional analysis* 2nd edition. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991.
2. M. Reed, B. Simon, *Methods of modern mathematical physics. I*, Academic Press, Inc., New York, 1980.
3. J.B. Conway, *A course in functional analysis*, Vol. 96. Springer, 2019.
4. E.B. Davies, *Linear operators and their spectra*, Vol. 106. Cambridge University Press, 2007.
5. M. Colbrook, A. Horning, A. Townsend, *Computing spectral measures of self-adjoint operators*, SIAM review 63.3 (2021): 489-524.
6. M. Colbrook, *On the computation of geometric features of spectra of linear operators on Hilbert spaces*, Foundations of Computational Mathematics (2022): 1-82.

Additional support

Three examples sheets will be provided, accompanied by three associated examples classes. A one-hour revision class will also be offered in the Easter Term.

Approximation Theory (M16)

Professor A. Shadrin

The course will give an overview of basic concepts of Approximation Theory, i.e., best and good approximation of a large family of functions by a smaller set (usually finitely generated, linear or nonlinear) in certain normed spaces (such as C and L_p), construction of good approximants (in various settings), finding approximation order for smooth functions (say, order n^{-k} for approximation by polynomials of degree n).

The course consists of three parts.

- We start with the classical approximation by polynomials, which includes the Weierstrass theorem, positive linear operators, and direct and inverse theorems for trigonometric approximation.
- We move then the emphasis to univariate splines which are piecewise polynomial functions. Here we study representation through the B-spline basis, spline interpolation theory and norm-minimization property of splines via orthogonal spline projector.
- Finally, we make a tour into wavelets which will cover the multiresolution analysis and Daubechies orthogonal wavelets with a compact support.

Prerequisites

The course is mostly self-contained, but it assumes a standard mathematical analysis background (say, normed linear spaces, inner products, Fourier series) and some linear algebra (solution of linear systems of equations). Some functional analysis tools (e.g., Hahn-Banach theorem) appear in comments and advanced problems, so they are helpful but not required.

Literature

To a large extent, the course follows the Lecture Notes by C. de Boor where one can find much more details on each topic. In the first part, it is also based on Cheney's book.

1. C. de Boor, Notes on Approximation Theory
pages.cs.wisc.edu/~deboor/ma887.html;
2. E. W. Cheney, Approximation theory, McGraw-Hill, New-York, 1966.
3. R. A. DeVore, G. G. Lorents, Constructive Approximation, Springer-Verlag, Berlin, 1993.
4. I. Daubechies, Ten lectures on wavelets, CBMS-NSF Regional Conference Series in Applied Mathematics, 61, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1992.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in the Easter term.

Mathematical Analysis of the Incompressible Navier-Stokes Equations (M24) – Examinable

Edriss S. Titi

This course will introduce basic mathematical analytical tools for studying non-linear Parabolic Partial Differential Equations. It will focus the Navier-Stokes equations of viscous incompressible fluids as a prototype of such system of evolution equations. The Navier-Stokes equations appear in a wide range of physical and biological applications, varying from oceanic and atmospheric dynamics, to combustion theory and body-fluid transport. On the one hand, from the mathematical analysis point of view these equations have been recognized to be among the most challenging problems in Applied Analysis. On the other hand, from the computational point of view they are prohibitively expensive to simulate for very large Reynolds numbers, and out of reach even for most powerful state-of-the-art computers.

The course will cover a selection of topics including:

- Introducing the Navier-Stokes and Euler equations of incompressible fluids.
- Sobolev spaces, Sobolev inequalities, interpolation inequalities; and the relevant Functional Spaces.
- Helmholtz Decomposition.
- Steady state solutions to the incompressible Navier-Stokes equations and their regularity.

- Time dependent Leray-Hopf weak solutions to the incompressible Navier-Stokes equations.
- Global regularity of strong solutions for the two-dimensional incompressible Navier-Stokes equations.
- Short time existence of strong solutions in the three-dimensional incompressible Navier-Stokes equations.
- Weak-strong uniqueness and the role of the energy inequality in the Leray-Hopf weak solutions.
- Time analyticity and Gevrey regularity (spatial analyticity) of strong solutions to incompressible Navier-Stokes equations.

Prerequisites

Basic knowledge of Real and Functional Analysis. The rest of the course will be, to a large extent, self-contained.

Literature

1. P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, 1988.
2. J. C. Robinson, *Infinite-dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics.
3. R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North-Holland. New print published by the AMS 2001.
4. R. Temam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF Regional Conference Series in Applied Math 66, SIAM, 2nd Ed, 1995.

Additional References

1. A. Chorin and J. Marsden, *A Mathematical Introduction to Fluid Mechanics*, Springer-Verlag.
2. C. Doering and J. Gibbon, *Applied Analysis of the Navier-Stokes Equations*, Cambridge University Press.
3. C. Foias, O. Manley, R. Rosa and R. Temam, *Navier-Stokes Equations and Turbulence*, Cambridge University Press, Cambridge, 2001.

4. O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Mathematics and Its Applications, 2 (Revised Second ed.), (1969) [1963], New York; London; Paris; Montreux; Tokyo; Melbourne: Gordon and Breach.
5. A. Majda and A. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge University Press, 2002.
6. H. Sohr, *The Navier–Stokes Equations, An Elementary Functional Analytic Approach*, Birkhäuser Verlag, Basel, 2001.
7. R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, 2nd Ed, Applied Math Sci. 68, Springer-Verlag, 1997.

Additional support

There will be some examples sheets associated with this course, but some exercise will be assigned during the lectures. Office hours will be by appointment.

5 Astrophysics

Astrophysical Fluid Dynamics (M24)

Professor G. I. Ogilvie

Fluid dynamics is involved in a very wide range of astrophysical phenomena, such as the formation and internal dynamics of stars and giant planets, the workings of accretion flows, winds and jets around stars and black holes, and the dynamics of the expanding Universe. Effects that can be important in astrophysical fluids include compressibility, self-gravitation and the dynamical influence of the magnetic field that is ‘frozen in’ to a highly conducting plasma.

The basic models introduced and applied in this course are Newtonian gas dynamics and magnetohydrodynamics (MHD) for an ideal compressible fluid. The mathematical structure of the governing equations and the associated conservation laws will be explored in some detail because of their importance for both analytical and numerical methods of solution, as well as for physical interpretation. Linear and nonlinear waves, including shocks and other discontinuities, will be analysed. Steady solutions with spherical or axial symmetry reveal the physics of winds and jets from stars and discs. The linearized equations determine the oscillation modes of astrophysical bodies, as well as their stability and their response to tidal forcing.

The course will aim to cover a selection of topics including:

- Overview of astrophysical fluid dynamics and its applications.
- Equations of ideal gas dynamics and MHD, including compressibility, thermodynamic relations and self-gravitation.
- Physical interpretation of ideal MHD. Magnetostatic equilibria and the Grad–Shafranov equation.
- Conservation laws, stress tensor and virial theorem.
- Linear waves in homogeneous media. Nonlinear waves, astrophysical shocks (supernova explosions) and other discontinuities.
- Spherically symmetric steady flows: stellar winds and accretion.
- Axisymmetric rotating magnetized flows: winds and jets from rotating stars and discs.
- Stellar oscillations. Introduction to asteroseismology and astrophysical tides.

Prerequisites

This course is suitable for both astrophysicists and fluid dynamicists. Knowledge of key elements of vector calculus, fluid dynamics, thermodynamics and electromagnetism will be assumed.

Literature

1. Choudhuri, A. R. (1998). *The Physics of Fluids and Plasmas*. Cambridge University Press.
2. Landau, L. D., & Lifshitz, E. M. (1987). *Fluid Mechanics*, 2nd ed. Butterworth–Heinemann.
3. Pringle, J. E., & King, A. R. (2007). *Astrophysical Flows*. Cambridge University Press.
4. Shu, F. H. (1992). *The Physics of Astrophysics*, vol. 2: *Gas Dynamics*. University Science Books.
5. Thompson, M. J. (2006). *An Introduction to Astrophysical Fluid Dynamics*. Imperial College Press.
6. Ogilvie, G. I. (2016). *Lecture Notes: Astrophysical Fluid Dynamics*. J. Plasma Phys. **82**, 205820301.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Dynamics of Astrophysical Discs (L16)

Henrik Latter

Discs are ubiquitous in astrophysics and participate in some of its most important processes. Most, but not all, feed a central mass: by facilitating the transfer of angular momentum, they permit the accretion of material that would otherwise remain in orbit. As a consequence, discs are essential to star, planet, and satellite formation. They also regulate the growth of supermassive black holes and thus indirectly influence galactic structure and the intracluster medium. Although astrophysical discs can vary by ten orders of magnitude in size and differ hugely in composition, all share the same basic dynamics and many physical phenomena.

The theoretical study of astrophysical discs combines aspects of orbital dynamics and continuum mechanics (fluid dynamics or magnetohydrodynamics). In fact, an astrophysical disc is a rotating shear flow that can sustain various instabilities, waves, and turbulence, which allow the necessary accretion. On the other hand, the resonant gravitational interaction of a planet within the disc generates waves that leads to orbital evolution of the planet and is one of the processes shaping the observed distribution of extrasolar planets.

Provisional synopsis:

- Occurrence of discs in various astronomical systems, basic physical and observational properties.
- Orbital dynamics, characteristic frequencies, precession, elementary mechanics of accretion.
- Viscous evolution of an accretion disc.
- Vertical disc structure: thin-disc approximations, thermal instability in cataclysmic variables.
- The shearing sheet, symmetries, shearing waves.
- Incompressible dynamics: hydrodynamic stability, vortices and dust in protoplanetary discs.
- Compressible dynamics: density waves, gravitational instability and ‘gravitoturbulence’ in planetary rings and protoplanetary discs.

- Satellite-disc interaction, impulse approximation, gap opening by embedded planets.
- Magnetorotational instability, ‘dead zones’ and outbursts in protoplanetary discs, magnetised outflows and jets

Pre-requisites

Newtonian mechanics and basic fluid dynamics. Some knowledge of magnetohydrodynamics is helpful for the topic of magnetorotational instability.

Literature

Astrophysical background can be found in reference [1] below, while the classical theory is summarised in the two review articles [2,3].

1. Frank, J., King, A. & Raine, D. (2002), *Accretion Power in Astrophysics*, 3rd edn, CUP.
2. Pringle, J. E. (1981), *Annu. Rev. Astron. Astrophys.* 19, 137.
3. Papaloizou, J. C. B. & Lin, D. N. C. (1995), *Annu. Rev. Astron. Astrophys.* 33, 505.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Extrasolar Planets: Atmospheres and Interiors (L24)

Nikku Madhusudhan

The field of extrasolar planets (or ‘exoplanets’) is one of the most dynamic frontiers of modern astronomy. Exoplanets are planets orbiting stars beyond the solar system. Thousands of exoplanets are now known with a wide range of masses, sizes, temperatures, and orbital parameters, spanning gas giants, ice giants, rocky planets and more. The field is now in a new era of detailed exoplanet characterization, which involves understanding the atmospheres, interiors, and formation mechanisms of exoplanets, and ultimately finding potential biosignatures in the atmospheres of habitable exoplanets. These efforts are aided by

both high-precision spectroscopic observations as well as detailed theoretical models of exoplanets.

The present course will cover the theory and observations of exoplanetary atmospheres and interiors. Topics in theory will include (1) physicochemical processes in exoplanetary atmospheres (e.g. radiative transfer, energy transport, temperature profiles and thermal inversions, equilibrium/non-equilibrium chemistry, atmospheric dynamics and clouds/hazes) (2) models of exoplanetary atmospheres and observable spectra (1-D and 3-D self-consistent models, parametric models and retrieval techniques) (3) exoplanetary interiors (equations of state, mass-radius relations, and internal structures of giant planets, super-Earths/mini-Neptunes, and rocky exoplanets), and (4) habitability and biosignatures. Topics in observations will cover observing techniques and state-of-the-art instruments used to observe exoplanetary atmospheres. The latest observational constraints on all the above-mentioned theoretical aspects will be discussed.

Prerequisites

The course material should be accessible to students in physics or mathematics at the masters and doctoral level, and to astronomers and applied mathematicians in general. Knowledge of basic radiative transfer and chemistry is preferable but not necessary. The course is self-contained and basic concepts will be introduced as required.

Literature

1. Seager, S., *Exoplanet Atmospheres: Physical Processes*, Princeton Series in Astrophysics (2010).
2. *Exoplanets*, University of Arizona Press (2011), ed. S. Seager.
3. de Pater, I. and Lissauer J., *Planetary Sciences*, Cambridge University Press (2010).
4. Chapters on exoplanetary atmospheres and interiors in the book *Protostars and Planets VI*, University of Arizona Press (2014), eds. H. Beuther, R. Klessen, C. Dullemond, Th. Henning. Available publicly on astro-ph arXiv (e.g., arXiv:1402.1169, arXiv:1401.4738).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Formation of Galaxy (M24)

N W Evans

This course describes our current state of knowledge of galaxy formation and evolution in a cold dark matter cosmology. We will start with structure formation in the non-linear regime, the formation and evolution of dark matter haloes and Press-Schechter theory. We will cover physical processes (shock heating, radiative cooling and star formation) as well as dynamical transformations (dynamical friction, tidal shocking, accretion and mergers) that are responsible for the shapes and properties of the galaxies we see today. We will end with a study of the formation and current day attributes of disk galaxies (Sersic profiles, thin and thick disks, stellar haloes) and elliptical galaxies (fast/slow rotators, major/minor mergers, Faber-Jackson relation). Recent discoveries on the structure of the Local Group and the Milky Way galaxy will be used as illustrative examples of formation processes throughout the course.

Prerequisites

This Part III course assumes that you have taken undergraduate courses in cosmology, relativity and dynamics.

Literature

1. J. Binney and S. Tremaine *Galactic Dynamics* 2nd edition, Princeton University Press, 2008
2. J. Bland-Hawthorn, K. Freeman *The Origin of the Galaxy and the Local Group*, Springer, 2014
3. A. Loeb *How Did the First Stars and Galaxies Form*, Princeton, 2010 (Background reading)
4. M. Longair, *Galaxy Formation* 2nd edition, Springer, 2008
5. H. Mo, F. van den Bosch and S. White, *Galaxy Formation and Evolution*, Cambridge University Press, 2010
6. S. Phillips, *The Structure and Evolution of Galaxies*, Wiley, 2005
7. L. Sparke, J. Gallagher, *Galaxies in the Universe*, 2nd edition, Cambridge University Press, 2007 (Background reading)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Planetary System Dynamics (M24)

Mark Wyatt

This course will cover the principles of celestial mechanics and their application to the Solar System and to extrasolar planetary systems. These principles have been developed over the centuries since the time of Newton, but this field continues to be invigorated by ongoing observational discoveries in the Solar System, such as the reservoir of comets in the Kuiper belt, and by the rapidly growing inventory of 1000s of extrasolar planets and debris discs that are providing new applications of these principles and the emergence of a new set of dynamical phenomena. The course will consider gravitational interactions between components of all sizes in planetary systems (i.e., planets, asteroids, comets and dust) as well as the effects of collisions and other perturbing forces. The resulting theory has numerous applications that will be elaborated in the course, including the growth of planets in the protoplanetary disc, the dynamical interaction between planets and how their orbits evolve, the sculpting of debris discs by interactions with planets and the destruction of those discs in collisions, and the evolution of circumplanetary ring and satellite systems.

Specific topics to be covered include:

1. Planetary system architecture: overview of Solar System and extrasolar systems, detectability, planet formation
2. Two-body problem: equation of motion, orbital elements, barycentric motion, Kepler's equation, perturbed orbits
3. Small body forces: stellar radiation, optical properties, radiation pressure, Poynting-Robertson drag, planetocentric orbits, stellar wind drag, Yarkovsky forces, gas drag, motion in protoplanetary disc, minimum mass solar nebula, settling, radial drift
4. Three-body problem: restricted equations of motion, Jacobi integral, Lagrange equilibrium points, stability, tadpole and horseshoe orbits
5. Close approaches: hyperbolic orbits, gravity assist, patched conics, escape velocity, gravitational focussing, dynamical friction, Tisserand parameter, cometary dynamics, Galactic tide
6. Collisions: accretion, coagulation equation, runaway and oligarchic growth, isolation mass, viscous stirring, collisional damping, fragmentation and collisional cascade, size distributions, collision rates, steady state, long term evolution, effect of radiation forces
7. Disturbing function: elliptic expansions, expansion using Legendre polynomials and Laplace coefficients, Lagrange's planetary equations, classification of arguments

8. Secular perturbations: Laplace coefficients, Laplace-Lagrange theory, test particles, secular resonances, Kozai cycles, hierarchical systems
9. Resonant perturbations: geometry of resonance, physics of resonance, pendulum model, libration width, resonant encounters and trapping, evolution in resonance, asymmetric libration, resonance overlap

Pre-requisites

This course is self-contained.

Literature

1. Murray C. D. and Dermott S. F., *Solar System Dynamics*. Cambridge University Press, 1999.
2. Tremaine S., *Dynamics of Planetary Systems*. Princeton University Press, 2023.
3. Armitage P. J., *Astrophysics of Planet Formation*. Cambridge University Press, 2010.
4. de Pater I. and Lissauer J. J., *Planetary Sciences*. Cambridge University Press, 2010.
5. Valtonen M. and Karttunen H., *The Three-Body Problem*. Cambridge University Press, 2006.
6. Seager S., *Exoplanets*. University of Arizona Press, 2011.
7. Perryman M., *The Exoplanet Handbook*. Cambridge University Press, 2011.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Radiative processes in astrophysical plasma

Non-Examinable (Part III Level)

Dr. G. Del Zanna

The main aim of this non-examinable 16-lecture Part-III course is to provide a broad overview of the physics underlying the main radiative processes occurring

in astrophysical plasma. We will focus first on emission line diagnostics, which mostly covers the lower-energies, then discuss the continuum emission. We provide general background theory but devote a good fraction of the lectures to applications, to provide an insight on current research in astrophysics based on high-resolution spectroscopy

The course will cover the following topics:

- Introduction to Radiation quantities and examples of spectra of astrophysical plasma, from the X-rays to the near infrared. An introduction to the background theory and the plasma diagnostics using line and continuum emission.
- Radiation and matter in thermodynamical equilibrium: black-body radiation, thermal plasma, Maxwell, Boltzmann, Saha distributions.
- Basics of radiative transfer. The interaction between atoms and radiation: spontaneous emission and absorption, stimulated emission, Einstein's coefficients and their relation with the absorption and emission coefficients. Spectral line profiles and line broadening effects. Optical depth and opacity of a spectral line.
- Basics of atomic structure for complex atoms. Non-relativistic Breit-Pauli vs. relativistic (Dirac) approach. Central field approximation, coupling schemes, good quantum numbers. Transition probabilities, dipole approximation, selection rules, forbidden lines. The formation of the periodic table.
- Atomic data (wavelengths, A-values, identifications) and their uncertainties. Examples.
- Fundamental processes affecting the level populations within an ion. Collisional excitation due to impact by electrons and protons and reverse processes. Basics of scattering calculations. Cross-sections and rates.
- Statistical equilibrium equations. LTE vs. NLTE. Level population within an atom. Two-level atom. Metastable levels and the principle of measuring electron densities from line ratios.
- Photoexcitation, resonant scattering, Doppler dimming in moving plasma.
- Fundamental processes affecting the ion charge state distributions in plasmas. Collisional ionization by electrons and protons, photoionization. Dielectronic and radiative recombination. Cross-sections and reverse processes. Rates. Ion charge state distributions.
- Density-dependent effects on the ion charge state distributions. Non-equilibrium effects: ionising and recombining plasma, non-Maxwellian electron distributions.

- Measuring electron densities and temperatures in low-density plasma using emission lines e.g. in planetary nebulae, galaxy cluster hot plasma, stellar coronae.
- Techniques for measuring temperature distributions and chemical abundances using emission lines. Examples of solar/stellar spectra.
- Theory of the X-ray satellite lines for He-like ions, formed by inner-shell excitation and dielectronic recombination. Diagnostic applications: how to directly measure electron temperatures, the ionization state of the plasma and non-thermal electrons. The Perseus cluster observed by Hitomi.
- Continuum radiation. Free-free, free-bound. Measuring electron temperatures from continuum emission.
- Cyclotron and synchrotron emission and absorption. Examples of synchrotron emission in astrophysical plasma. Thompson scattering, Compton scattering, inverse Compton scattering.
- Active Galactic Nuclei: accretion disks, broad and narrow emission line spectra, Compton reflection, X-ray emission and the 6 keV iron feature.

Prerequisites

his course assumes you are familiar with Principles of Quantum Mechanics. It would be useful to have knowledge of Atomic structure (at least the Hydrogen atom), and of Elements of quantum field theory.

Literature

1. Del Zanna, G. and Mason, H.E., *Solar UV and X-Ray Spectral Diagnostics*, Living Reviews in Solar Physics, 2018, 15,5, DOI: 10.1007/s41116-018-0015-3
<https://link.springer.com/article/10.1007/s41116-018-0015-3>
2. Landi Degl' Innocenti, E., *Atomic Spectroscopy and Radiative Processes*, 2014, Springer.
3. Rybicki, G. B. and Lightman, A. P., *Radiative processes in astrophysics.*, 1979, New York, Wiley-Interscience.
4. Ghisellini, G., *Radiative processes in high energy astrophysics*, 2013.
<http://adsabs.harvard.edu/abs/2013LNP...873.....G>

Structure and Evolution of Stars (M24)

Professor C. A. Tout

The structure of a star can be mathematically described by certain differential equations which can be derived from the principles of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics. Some familiarity with these theories will be assumed.

The basic equations of a spherical star will be derived in detail and the mode of energy transport, the equation of state, and the physics of the opacity sources and the nuclear reactions will be discussed.

Approximate solutions of the equations for stellar structure will be given. Attention will be given to the virial theorem, polytropic gas spheres and homology principles. The procedure for numerical solution of the equations will be mentioned briefly.

The evolution of a star will be discussed with reference to its main-sequence evolution, the exhaustion of various nuclear fuels and the end points of evolution such as white dwarfs, neutron stars and black holes.

Close binary stars are two stars in a gravitationally bound system in which the ratio of orbital separation to stellar radius is sufficiently small for the stars to interact through tides and mass transfer. The concept of Roche lobe overflow and its stability will be discussed, as well as the effects on the stellar evolution. Algols and Cataclysmic Variables will be considered as observed examples.

Throughout the course, reference will be made to the observational properties of stars and these will be discussed at appropriate times with particular reference to the Hertzsprung–Russell diagram, the mass-luminosity law and spectroscopic information.

There will be four examples sheets each of which will be discussed during an examples.

Prerequisites

A basic understanding of the principles of hydrodynamics, electromagnetic theory, thermodynamics, quantum mechanics, atomic and nuclear physics.

Preliminary Reading

1. Shu, F. *The Physical Universe*, W. H. Freeman University Science Books, 1991.
2. Phillips, A. *The Physics of Stars*, Wiley, 1999.

Literature

1. J. J. Eldridge and C. A. Tout, *The Structure and Evolution of Stars* World Scientific, 2019.
2. Prialnik, D. *An Introduction to the Theory of Stellar Structure and Stellar Evolution*, CUP, 2000.
3. J. E. Pringle and R. A. Wade *Interacting Binary Stars*, CUP, 1985.

Additional support

There will be four examples sheets each of which will be discussed during an examples class. There will be a revision class in the Easter term.

Further Stellar Evolution (L16)

Dr A. N. Żytkow

We will continue with the investigation of stellar structure and evolution following on from the course “Structure and Evolution of Stars” in the Michaelmas Term.

The only way we know how to form heavy elements is through nuclear processes in stars in both their steady evolution and in their possible final catastrophic moments. Thus stars are central in building the world around us. Insights into the structure and evolution of stars rely on a mathematical description of the physical processes which determine their nature. Stellar structure and evolution theory can then be tested through the comparison of theoretical results to observations.

The main subjects to be covered will be an introduction to asteroseismology, stellar rotation, further details regarding mass loss from stars as well as some examples of stellar models and also a little about nuclear processes. All these are essential if we endeavour to understand, interpret and study current observational data.

Prerequisites

Some understanding of hydrodynamics, electromagnetic theory, thermodynamics, nuclear physics, quantum mechanics as well as theory of structure and evolution of stars, although a detailed knowledge of all of these is not necessary.

Literature

1. Kippenhahn, R. and Weigert, A. *Stellar Structure and Evolution*, Second Edition, Springer- Verlag, 2012.
2. Cox, J. P. and Giuli, R. T. *Principles of Stellar Structure*, Gordon and Breach, 1968.
3. Padmanabhan, T. *Theoretical Astrophysics*, Volume II: Stars and Stellar Systems, Cambridge University Press, 2001

Astrostatistics (L24)

Professor K. Mandel

This interdisciplinary course will cover applied statistical methods necessary to properly interpret today's increasingly complex datasets in astronomy. Particular emphasis will be placed on principled statistical modelling of astrophysical data and statistical computation of inferences of scientific interest. Statistical techniques, such as latent variable and measurement error models, Bayesian inference, Monte Carlo sampling methods, hierarchical models, Gaussian processes, and model selection, will be examined in the context of applications to modern astronomical data analysis. Topics and examples will be motivated by case studies across astrophysics and cosmology.

Prerequisites

Students of astrophysics, physics, statistics or mathematics are welcome. Astronomical context will be provided when necessary. Students who may need to brush up on their statistics background should familiarise themselves with the material in Feigelson & Babu, Chapters 1-4, and Ivezić et al., Chapters 1, 3-5, (listed below) by the beginning of the course. (Note that the two textbooks cover many of the same topics). These texts are freely available online to Cambridge students via the library website. Alternatively, they may find it useful to audit the relevant Michaelmas-term courses in the MPhil for Data-Intensive Science.

Literature

1. E. Feigelson and G. Babu. *Modern statistical methods for astronomy: with R applications*. Cambridge University Press, 2012.
2. Z. Ivezić, A. Connolly, J. VanderPlas & A. Gray. *Statistics, Data Mining, and Machine Learning in Astronomy*. Princeton University Press, 2014.

3. Gelman et al. *Bayesian Data Analysis*. 3rd edition, Chapman & Hall, CRC Press, 2013. Also available at: <http://www.stat.columbia.edu/gelman/book/>
4. C. Bishop. *Pattern Recognition & Machine Learning*. Springer-Verlag, 2006. Also available at: <https://www.microsoft.com/en-us/research/people/cmbishop/prml-book>
5. D. MacKay. *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press, 2003. Also available at: <http://www.inference.org.uk/mackay/itila/book.htm>
6. C. Schafer. *A Framework for Statistical Inference in Astrophysics*. 2015, Annual Review of Statistics and Its Application, 2: 141-162.
7. E. Feigelson, et al. *21st Century Statistical and Computational Challenges in Astrophysics*. 2020, Annual Review of Statistics and Its Application, 8. <https://arxiv.org/abs/2005.13025>

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

6 Combinatorics

Combinatorics (M16)

Prof I.B.Leader

The flavour of the course is similar to that of the Part II Graph Theory course, although we shall not rely on many of the results from that course.

We shall study collections of subsets of a finite set, with special emphasis on size, intersection and containment. There are many very natural and fundamental questions to ask about families of subsets; although many of these remain unsolved, several have been answered using a great variety of elegant techniques.

We shall cover a number of ‘classical’ extremal theorems, such as those of Erdős-Ko-Rado and Kruskal-Katona, together with more recent results concerning isoperimetric inequalities and intersecting families. There will be several indications of open problems.

We hope to cover the following material.

Set Systems

Definitions. Antichains; Sperner's lemma and related results. Shadows. Compression operators and the Kruskal-Katona theorem. Intersecting families; the Erdős-Ko-Rado theorem.

Isoperimetric Inequalities

Harper's theorem and the edge-isoperimetric inequality in the cube. Inequalities in the grid. The classical isoperimetric inequality on the sphere. The 'concentration of measure' phenomenon. Applications.

Intersecting Families

Katona's t -intersecting theorem. The Ahlswede-Khachatrian theorem. Restricted intersections. The Kahn-Kalai counterexample to Borsuk's conjecture.

Desirable Previous Knowledge

The only prerequisites are the very basic concepts of graph theory.

Introductory Reading

1. Bollobás, B., *Combinatorics*, C.U.P. 1986.

Additional support

Three examples sheets will be provided and associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Entropy methods in combinatorics (L16)

W. T. Gowers

The entropy $H[X]$ of a discrete random variables X taking values in a set A is defined to be $\sum_{a \in A} p_a \log_2(1/p_a)$, where p_a denotes the probability that $X = a$. For example, if X is uniformly distributed over all n -bit 01-strings, then $H[X] = n$. More generally, one can think of $H[X]$ as being something like "the number of bits one typically needs to specify X ".

This definition, introduced by Claude Shannon in 1948, is fundamental to information theory, coding theory, and important to several other disciplines such as machine learning and biology. Rather more surprisingly, it is a useful tool in extremal combinatorics, and has been at the centre of several recent breakthroughs. It is not that easy to give a unified explanation of just why this is, so the main aim of the course will simply be to demonstrate the use of entropy in all its variety, identifying common themes where possible.

The following topics will approximate the content of the course.

- The axiomatic approach to entropy.
- A special case of Sidorenko’s conjecture.
- Brégman’s theorem.
- Shearer’s lemma, and applications.
- Bounds for the union-closed conjecture.
- Bounds for the sunflower lemma.
- Entropy and sumsets.
- Marton’s conjecture for $\mathbf{F}_2^n$

Prerequisites

This course will start more or less from first principles, but a basic knowledge of extremal graph theory and additive combinatorics will be helpful for setting it in context.

Literature

There is no single text that covers all the topics above. If you search online for “entropy methods in combinatorics” you will find lecture notes that cover the first few sections. For the rest I recommend searching for material on each topic individually.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Introduction to Additive Combinatorics (M16)

Professor J. Wolf

This course provides an introduction to additive combinatorics, a fast-moving area of mathematics motivated by long-standing questions about additive structure in sets of integers. It will attempt to reflect both the elementary appeal of the key problems in the area as well as the breadth of techniques (analytic, combinatorial, algebraic, probabilistic) used to tackle them. By the end of the

course, students should be able to appreciate various recent breakthroughs in the area.

The course will be organised along the following lines.

- Part I: Combinatorial methods
Plünnecke's inequality for sum sets; the Freiman-Ruzsa theorem
- Part II: Fourier-analytic techniques
Bogolyubov's lemma; Roth's theorem on 3-term arithmetic progressions
- Part III: Probabilistic methods
the Balog-Szemerédi-Gowers theorem; Croot-Sisask almost periodicity and applications
- Part IV: Further topics (if time permits)
algebraic constructions and obstructions; beyond 3-term progressions

Prerequisites

This course assumes basic familiarity with finite abelian groups, linear algebra, discrete probability and graph theory, at a level found in most undergraduate mathematics degrees. As such, it should be accessible to a broad range of students.

Literature

The following books give a flavour of the course, but should not be regarded as compulsory or definitive reading.

1. T. Tao and V. Vu. *Additive Combinatorics* (Cambridge Studies in Advanced Mathematics). Cambridge University Press, 2006.
2. Y. Zhao. *Graph Theory and Additive Combinatorics: Exploring Structure and Randomness*. Cambridge University Press, 2023.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Ramsey Theory (M16)

Dr M.R. Ivan

Ramsey theory is concerned with the general question of whether, in a large amount of disorder, one can find regions of order. Alternatively, given a certain structure, if one ‘breaks’ up ‘the universe’ into many many pieces, can we find one piece that contains that structure? A typical example is van der Waerden’s theorem, which states that whenever we partition the natural numbers into finitely many classes there is a class that contains arbitrarily long arithmetic progressions. So the universe is the natural numbers and the structure is the arithmetic progression.

This is a purely combinatorial course. Ramsey theory is remarkably attractive: we study questions that are very natural and easy to appreciate, but whose answers rely on a great variety of beautiful methods. Not only that, but almost no two questions can be solved in the same way – at least one brand new idea is needed.

We shall cover a number of ‘classical’ Ramsey theorems, such as Gallai’s theorem and the Hales-Jewett theorem, as well as some more recent developments. There will also be several indications of open problems. We hope to cover the following material.

- Monochromatic Systems.
Ramsey’s theorem (finite and infinite). Canonical Ramsey theorems. Colourings of the natural numbers. Focusing and van der Waerden’s theorem. Combinatorial lines and the Hales-Jewett theorem. Applications, including Gallai’s theorem.
- Partition Regular Equations.
Definitions and examples. The columns property and Rado’s theorem. Applications. (m, p, c) -sets and Deuber’s theorem. Ultrafilters. the Stone-Ćech compactification. Idempotent ultrafilters and Hindman’s theorem.
- Euclidean Ramsey Theory.
Basic definitions. Ramsey sets and spherical sets. Triangles are Ramsey. Regular polygons are Ramsey. Transitive sets. The block sets conjecture.

Prerequisites

There are almost no prerequisites – the course will start with a review of Ramsey’s theorem, so even prior knowledge of this is not essential. At various places we shall make use of some very basic concepts from topology, such as metric spaces and compactness – however, nothing more than basic knowledge any mathematics undergraduate, regardless of personal interest, should possess.

Literature

1. R. Graham, B. Rothschild and J. Spencer, *Ramsey Theory*. John Wiley, 1990.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

7 Continuum Mechanics

Direct and Inverse Scattering of Waves (L16)

Dr O. Rath Spivack

The study of wave scattering is concerned with how the propagation of waves is affected by objects, and has a variety of applications in many fields, from environmental science to seismology, medicine, telecommunications, materials science, and many others. If we know the nature of the objects and we want to find how an incident wave is scattered, we call this a ‘direct scattering problem’ and practical applications will include for example underwater sound propagation, light transmission through the atmosphere, or the effect of noise in built-up areas. If we measure and know the scattered field produced by an incident wave, but we do not know the nature of the objects that have scattered it, we call this an ‘inverse scattering problem’ and applications will include for example non-destructive testing of materials, remote sensing with radar or lidar, or medical imaging.

This course will provide the basic theory of wave propagation and scattering and an overview of the main mathematical methods and approximations, with particular emphasis on inhomogeneous and random media, and on the regularisation of inverse scattering problems. It is intended as a kind of bridge between the Applied and Computational Analysis courses and the Continuum Mechanics courses and should appeal to students interested in waves as well as in inverse problems.

The course will cover a selection of topics including:

- Boundary value problems and the integral form of the wave equation.
- The parabolic equation and Born and Rytov approximations for the scattering problem.

- Scattering by randomly rough surfaces and propagation in inhomogeneous media.
- Ill-posedness of the inverse scattering problem, and the Moore-Penrose generalised inverse.
- Regularisation methods and methods for solving some inverse scattering problems
- Time reversal and focusing in inhomogeneous media.

Prerequisites

This course assumes basic knowledge of PDEs, and of Fourier transforms. Some familiarity with linear algebra and with basic concepts in functional analysis is helpful, though by no means necessary.

The Part III course Inverse Problems is a useful complement to part of this course, although not a prerequisite.

Preliminary Reading

These can provide some useful background:

1. C.W. Groetsch *Inverse Problems in the Mathematical Sciences*. Braunschweig 1993.
2. L.D. Landau and E.M. Lifschitz *Fluid Dynamics*, [Chapter 8]. Pergamon, 1987. Also available at <https://users-physics.au.dk/srf/hydro/Landau+Lifschitz.pdf>.

Literature

1. D. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*. Springer, 1992.
2. D.G. Crighton et al, *Modern Methods in Analytical Acoustics*. Springer, 1992.
3. H.W. Engl, M. Hanke and A. Neubauer, *Regularization of Inverse Problem.*, Kluwer, 2000
4. B.A. Ishimaru, *Wave Propagation and Scattering in Random Media*. Academic Press, 1978.
5. B. Uscinski, *The Elements of Wave Propagation in Random Media*. McGraw-Hill, 1977

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a two-hour revision class in the Easter Term.

Demonstrations in Fluid Mechanics. (L8)

Non-Examinable (Part III Level)

Prof. S.B. Dalziel

While the equations governing most fluid flows are well known, they are often very difficult or impossible to solve. To make progress it is therefore necessary to introduce various simplifications and assumptions about the nature of the flow and thus derive a simpler set of equations. For this process to be meaningful, it is essential that the relevant physics of the flow is maintained in the simplified equations. Deriving such equations requires a combination of mathematical analysis and physical insight. Laboratory experiments play a role in providing physical insight into the flow and in providing both qualitative and quantitative data against which theoretical and numerical models may be tested.

The main purpose of this demonstration course is to help develop an intuitive ‘feeling’ for fluid flows, how they relate to simplified mathematical models, and how they may best be used to increase our understanding of a flow. Limitations of experimental data will also be encountered and discussed. Although the course is entitled ‘demonstrations’, the course has a history of including some novel experiments that have not, to our knowledge, been performed anywhere previously. Some of these have subsequently led to PhD projects. The course thus offers an introduction into how experiments can provide insight into previously unexplored (or undiscovered) phenomena.

The demonstrations will include a range of flows currently being studied in a range of research projects in addition to classical experiments illustrating some of the flows studied in lectures. The demonstrations are likely to include

- instability of jets, shear layers and boundary layers;
- gravity waves, capillary waves, internal waves and inertial waves;
- thermal convection, double-diffusive convection, thermals and plumes;
- gravity currents, intrusions and hydraulic flows;
- vortices, vortex rings and turbulence;
- bubbles, droplets and multiphase flows;
- avalanches, granular flows, sedimentation and resuspension;

- flows in porous, absorbent and elastic media;
- ventilation and industrial flows;
- rotationally dominated flows;
- non-Newtonian and low-Reynolds-number flows;
- image processing techniques and methods of flow visualisation.

It should be noted that students attending this course are not required to undertake laboratory work on their own account, although participation is encouraged.

Pre-requisites

Undergraduate Fluid Dynamics.

Literature

1. M. Van Dyke. An Album of Fluid Motion. Parabolic Press.
2. G. M. Homsy, H. Aref, K. S. Breuer, S. Hochgreb, J. R. Koseff, B. R. Munson, K. G. Powell, C. R. Robertson, S. T. Thoroddsen. Multimedia Fluid Mechanics (Multilingual Version CD-ROM). CUP.
3. M. Samimy, K. Breuer, P. Steen, & L. G. Leal. A Gallery of Fluid Motion. CUP.

Fluid Dynamics of Living Systems (M16)

Non-Examinable (Part III Level)

Eric Lauga

Biology is dominated by transport problems involving fluid flows, from the transport of nutrients and locomotion to flows around plants and the circulatory system of animals. These problems occur over a broad spectrum of time and length scales, making them relevant to fluid mechanics research over the entire range of Reynolds numbers, from turbulent wakes behind flapping birds, to unsteady flows in the heart and Stokes flows inside cells. In this course, we will take a journey through the natural world and study several examples where flows are directly relevant to biology at the levels of cells and organisms. The emphasis will be using mathematical modelling, aiming to link experimental phenomenology of biological flows to solvable mathematical models.

The course topics will include, time permitting:

Extracellular flows

1. Blood flow and the microcirculation
2. Waving transport of fluids
3. Active suspensions
4. Locomotion of swimming microorganisms

Intracellular flows

5. Cytoplasmic streaming
6. Flows and transport in membranes
7. Polymeric fluids and cellular rheology
8. Capillary forces inside cells.

Prerequisites

Undergraduate fluid dynamics, vector calculus and mathematical methods.

Literature

Whenever possible the link will be made with research in each of the topics discussed in the course, so the lectures will be accompanied by a reading list of recent papers. Some relevant articles and books include:

1. Secomb (2017) *Annu. Rev. Fluid Mech.* **49**, 443.
2. Jaffrin & Shapiro (1971) *Annu. Rev. Fluid Mech.* **3**, 13.
3. Saintillan & Shelley (2013) *C. R. Physique* **14**, 497.
4. Lauga (2020) *The Fluid Dynamics of Cell Motility*, Cambridge University Press, UK.
5. Goldstein & Van De Meent (2015) A physical perspective on cytoplasmic streaming *Interface Focus* **5**, 20150030.
6. Saffman & Delbrück (1975) *Proc. Natl. Acad. Sci. USA* **72**, 3111.
7. Spagnolie (Editor) (2015) *Complex Fluids in Biological Systems*, Springer, UK.
8. Gouveia et al. (2022) *Nature* **609**, 255.

Fluid dynamics of the Solid Earth (L24)

Prof JA Neufeld & Prof MG Worster

The dynamic evolution of the solid Earth is governed by a rich variety of physical processes occurring on a wide range of length and time scales. The Earth's core is formed by the solidification of a mixture of molten iron and various lighter elements, a process that drives compositional convection in the liquid outer core, thus producing the geodynamo responsible for Earth's magnetic field. On million-year timescales, the solid mantle convects, and as it upwells to the surface it partially melts leading to volcanism. At the surface, convection drives the motion of brittle plates, which are responsible for the Earth's topography as can be felt and imaged through the seismic record. In the Earth's crust, fluids flow through porous rocks, forming groundwater aquifers that feed streams and rivers that erode the solid surface. On the Earth's surface, similar physical processes of viscous and elastic deformation coupled to phase changes govern the evolution of the Earth's cryosphere, from the solidification of polar oceans to form sea ice to the flow of glacial ice over land and the ocean.

This course will use observations of the solid Earth to motivate mathematical models of physical processes that play key roles in many other environmental and industrial processes. Mathematical topics will include the onset and scaling of convection, the coupling of fluid motions with changes of phase at a boundary, the thermodynamic and mechanical evolution of multicomponent or multiphase systems, the coupling of fluid flow and elastic flexure or deformation, and the flow of fluids through and deformation of porous materials. The focus will be on the generation and solution of mathematical models motivated by observational data.

The course will cover a selection of topics including:

- Growth of Earth's inner core
- Groundwater flows
- Formation and evolution of sea ice
- Carbon capture and storage
- Stability of marine ice sheets

Prerequisites

Mathematical methods, particularly the solution of ordinary and partial differential equations. An understanding of viscous fluid dynamics.

Literature

1. M.G. Worster. *Solidification of Fluids*. In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
2. H.E. Huppert. *Geological fluid mechanics*. In Perspectives in Fluid Dynamics: a Collective Introduction to Current Research. Edited by G.K. Batchelor, H.K. Moffatt and M.G. Worster. pp. 393–446. CUP (2000)
3. D.L. Turcotte, G. Schubert. *Geodynamics*, second edition. CUP (2002)

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Fluid Dynamics of the Environment (L24)

Dr R. K. Bhagat and Dr Daria Frank

Understanding the environment is a critical challenge. Whether we are concerned about climate change, pollution or sediment transport, the high-Reynolds-number fluid dynamics of buoyant flows plays a vital role. In this course, we will focus on flows that occur over sufficiently small time and length scales to be largely unaffected by the Earth's rotation.

The course will begin by considering internal structure of stratified flows. In particular, we will discuss internal gravity waves, their reflections, energy propagation, as well applications to a stratified atmosphere. We will then proceed by studying shallow water flows, a class of ‘thin’ flows where near-horizontal motion is driven by density differences. These shallow flows can be steady or time dependent, with the density differences arising from temperature, composition, suspended particles, or combinations of these. The cold air entering a room when you open the door on a cold day and the dust cloud from demolishing a building are just two examples. The flows can propagate along interfaces within the fluid, horizontal or sloping boundaries. In many cases, especially as the slope of the boundary increases, turbulence plays an important role, reducing the density differences driving the flow.

Thin flows also arise where there is a localised source of buoyancy without a boundary along which it can flow. Rather than forming a near-horizontal shallow water flow, a vertical flow in the form of a buoyant turbulent plume or a buoyant thermal develops. Such flows are widespread in both the natural and built environment (e.g. from a volcanic eruption or above a radiator in your college room).

Drawing together these thin flows, we will delve further into the fluid dynamics of indoor environments. With an increasing fraction of the world's population residing in cities and spending 90% of their time indoors, creating sustainable urban spaces is essential. Combating the climate crisis through decreasing energy consumption is one imperative, but another is improving indoor air quality due to its impact on many aspects of health and productivity. As poor ventilation played an important role in disease transmission during the recent Covid-19 pandemic, we introduce approaches to modelling the airborne transmission that is important for many of the most infectious diseases.

Although this course is not primarily about turbulence, we find that turbulence inevitably plays a role in both indoor and outdoor environmental flows and so we shall touch on key aspects of turbulence and how it is modified by stratification at various points during the course

Pre-requisites

Undergraduate fluid dynamics is desirable.

Literature

- J. S. Turner, Buoyancy Effects in Fluids, Cambridge University Press (1979).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Fluid Dynamics of Climate (L24)

Prof. PH Haynes, Dr A Ming and Prof JR Taylor

Understanding the Earth's climate and predicting its future evolution is one of the great scientific challenges of our times. Fluid motion in the ocean and atmosphere plays a vital role in regulating the climate system, helping to make the planet hospitable for life. The dynamical complexity of this fluid motion and the wide range of space and time scales involved is one of the most difficult aspects of climate prediction.

This course, focusing on the large-scale behaviour of stratified and rotating flows, provides an introduction to the fluid dynamics necessary to build mathematical models of the climate system. The course begins by considering flows which

evolve on a timescale which is long compared with a day, where the Earth's rotation plays an important role. The rotation is felt through the Coriolis force (a fictitious force arising from use of a frame of reference rotating with the Earth) which causes a moving parcel of fluid to experience a force directed to its right in the Northern hemisphere (or its left in the Southern hemisphere), introducing a rich wealth of new dynamics, particularly in combination with stable density stratification. Canonical models are introduced and studied to illustrate phenomena such as adjustment to a state of geostrophic balance, where Coriolis force balances pressure gradient, new wave modes that can communicate dynamical information on both regional and global scales, and new hydrodynamic instabilities that lead to atmospheric weather systems and ocean eddies.

The course then moves on apply these basic ideas to important aspects of the large-scale dynamics of the atmosphere and the oceans that directly impact the global climate system. Specifically, we will examine the structure and hence the effects of ocean eddies and atmospheric weather systems, the dynamics of the meridional (north/south) circulation in the ocean and atmosphere and the associated transport of heat and of chemical and biological tracers. The final part of the course will consider the special dynamics of tropical regions which give rise to phenomena such as El Nino.

Prerequisites

Undergraduate fluid dynamics

Literature

The following provide some introductory reading and will also complement the detailed presentation of material given in lectures.

1. G.K. Vallis, *Atmospheric and Oceanic Fluid Dynamics*. Cambridge University Press, 2017.
2. A.E. Gill, *Atmosphere-Ocean Dynamics*. Academic Press, 1982.
3. J. Marshall and R.A. Plumb, *Atmosphere, Ocean, and Climate Dynamics*. Academic Press, 2008.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

Non-Newtonian fluid mechanics (M16)

Dr D. R. Hewitt

The relationship between the stress applied to a material and its kinematic response is known as its rheology. It is common in fluid mechanics to assume a so-called Newtonian rheology, in which stress and strain rate are linearly related. This assumption is inaccurate for many fluids; attempting to model, predict and understand the wealth of more complex constitutive behaviour that is exhibited by these materials makes up the field of *non-Newtonian fluid mechanics*. Many fluids we experience on a daily basis are non-Newtonian: ketchup, mayonnaise, shampoo and shaving foam all exhibit complex constitutive behaviour encompassing elasticity, plasticity, ageing and jamming. Geophysical and biological materials are also commonly non-Newtonian in character: lava, mud, sand, blood, and mucus, for example, all behave in ways that are not well described by a simple Newtonian rheology.

This course will introduce the mathematical framework for continuum modelling of non-Newtonian fluids in different contexts. After a general introduction to the fascinating and varied phenomenology exhibited by these materials, the course will cover a selection of topics including:

- Non-linear ‘generalised-Newtonian’ fluids;
- Viscoplastic fluids;
- Viscoelastic fluids: linear and non-linear models; convective derivatives;
- Flow problems and applications involving non-Newtonian fluids;
- More complex rheology: memory effects; granular media; mechanics of suspensions.

By the end of this course, students will be equipped with the skills necessary to carry out independent research in academic or industrial fields involving complex fluids and rheology.

Prerequisites

Undergraduate viscous fluid mechanics, vector calculus and mathematical methods.

Literature

1. Barnes, H.A., Hutton, J.F. and Walters, K. *An Introduction to Rheology*, Elsevier 1989.

2. Tanner, R.I. *Engineering Rheology*, 2nd edition. OUP 1999.
3. Bird, R.B., Armstrong, R.C. and Hassager, O. *Dynamics of polymeric liquids, vol.1*, 2nd edition. Wiley 1987.

Additional support

Three examples sheets will be provided, with associated examples classes. There will be a revision class in the Easter Term.

Perturbation Methods (M16)

Prof. I.D. Abrahams

This course will deal with the asymptotic solution to problems in applied mathematics when some parameter or coordinate assumes large or small values. Many problems of physical interest are covered by such asymptotic limits; indeed, they play a crucial role in the field of mathematical modelling. The methods developed have significance, not only in revealing the underlying structure of the solution, but in many cases can provide accurate predictions even when the parameter or coordinate has only moderately large or small values.

Some of the most useful mathematical tools for finding approximate solutions to equations will be covered, and a range of physical applications will be provided. Specifically, the course will start with a brief review of classical asymptotic methods for the evaluation of integrals, with the remaining lectures devoted to singular perturbation problems (including the methods of multiple scales and matched asymptotic expansions), for which straightforward asymptotic methods fail in one of a number of characteristic ways.

More details of the material are as follows, with approximate numbers of lectures in brackets:

- *Methods for Approximating Integrals.* This section will start with a brief review of asymptotic series. This will be followed by various methods for approximating integrals including the ‘divide & conquer’ strategy, Laplace’s method, stationary phase and steepest descents. [6]
- *Multiple Scales.* This method is generally used to study problems in which small effects accumulate over large times or distances to produce significant changes (the ‘WKB[JLG]’ method can be viewed as a special case.) It is a systematic method, capable of extension in many ways, and includes such ideas as those of ‘averaging’ and ‘time scale distortion’ in a natural way. A number of applications will be studied, as time allows, potentially including ray tracing and turning points (e.g. sound or light propagation in an inhomogeneous medium.) [5]

- *Matched Asymptotic Expansions.* This method is applicable, broadly speaking, to problems in which regions of rapid variation occur, and where there is a drastic change in the structure of the problem when the limiting operation is performed. The method was first developed in fluid mechanics, where it is referred to as boundary-layer theory, but it has since been greatly extended and applied to many fields. [5]

Prerequisites

Although many of the techniques and ideas originate from fluid mechanics and classical wave theory, no (or only elementary) knowledge of these fields will be assumed. The only pre-requisites are familiarity with techniques from the theory of complex variables, such as residue calculus and Fourier transforms, and an ability to solve straightforward ordinary & partial differential equations and evaluate reasonably simple integrals.

Literature

Relevant Textbooks (not required for the course, but which may assist)

1. C.M. Bender and S. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, 1978. *This is probably the most comprehensive textbook, but that means that some selective reading is advisable. Note that Bender & Orszag refer to ‘Stokes’ lines as ‘anti-Stokes’ lines, and vice versa. The course will use Stokes’ convention.*
2. E.J. Hinch. *Perturbation Methods*. Cambridge University Press, 1991. *The lecture course was originally based on this book; some may view it as somewhat concise.*
3. M.D. Van Dyke. *Perturbation Methods in Fluid Mechanics*. Parabolic Press, Stanford, 1975. *This is the original book on perturbation methods; somewhat dated, but still an excellent read.*
4. J. Kevorkian and J.D. Cole. *Perturbation Methods in Applied Mathematics*, Springer, 1981. *A useful and accessible book, covering a range of asymptotic methods, and applications.*

Additional Reading on Topics Beyond the Syllabus

1. M.V. Berry. *Waves near Stokes lines*, Proc. R. Soc. A, **427**, 265–280 (1990).
2. J.P. Boyd. *The Devil’s invention: asymptotic, superasymptotic and hyperasymptotic series*, Acta Applicandae, 56, 1-98 (1999). Also available at <http://hdl.handle.net/2027.42/41670> and <http://link.springer.com/content/pdf/10.1023/A:1006145903624.pdf>.

Additional support

In addition to the lectures, three examples sheets will be provided and three associated 2-hour examples classes will run in parallel to the course. There will be a 2-hour revision class in the Easter Term.

Slow Viscous Flow (M24)

Prof. J.R. Lister

In many flows of natural interest or technological importance, the inertia of the fluid is negligible. This may be due to the small scale of the motion, as in the

swimming of micro-organisms and the settling of fine sediments, or due to the high viscosity of the fluid, as in the processing of glass and the convection of the Earth's mantle.

The course will begin by presenting the fundamental principles governing flows of negligible inertia. A number of elegant results and representations of general solutions will be derived for such flows. The motion of rigid particles in a viscous fluid will then be discussed. Many important phenomena arise from the deformation of free boundaries between immiscible liquids under applied or surface-tension forcing. The flows generated by variations in surface tension due to a temperature gradient or contamination by surfactants will be analysed in the context of the translation and deformation of drops and bubbles and in the context of thin films. The small cross-stream lengthscale of thin films renders their inertia negligible and allows them to be analysed by lubrication or extensional-flow approximations. Problems such as the fall of a thread of honey from a spoon and the subsequent spread of the pool of honey will be analysed in this way. Inertia is also negligible in flows through porous media, such as the extraction of oil from sandstone reservoirs, movement of groundwater through soil or the migration of melt through a partially molten mush. Some basic flows in porous media may be discussed.

The course aims to examine a broad range of slow viscous flows and the mathematical methods used to analyse them. The course is thus generally suitable for students of fluid mechanics, and provides background for applied research in geological, biological or rheological fluid mechanics.

Pre-requisites

As described above in the introduction to courses in Continuum Mechanics. Familiarity with basic vector calculus including Cartesian tensors and the Einstein summation convention is particularly useful for the first half of the course.

Preliminary Reading

1. D.J. Acheson. *Elementary Fluid Dynamics*. OUP (1990). Chapter 7
2. G.K. Batchelor. *An Introduction to Fluid Dynamics*. CUP (1970). pp.216–255.
3. L.G. Leal. *Laminar flow and convective transport processes*. Butterworth (1992). Chapters 4 & 5.

Literature

1. J. Happel & H. Brenner. *Low Reynolds Number Hydrodynamics*. Kluwer (1965).

2. S. Kim & J. Karrila. *Microhydrodynamics: Principles and Selected Applications*. (1993)

Additional support

Four two-hour examples classes will be given by the lecturer to cover the four examples sheets. There will be a further revision class in the Easter Term.

Prerequisites

Undergraduate fluid dynamics (Part II Fluid Dynamics or equivalent), vector calculus and Fourier transforms.

Literature

1. Kundu, P. K., Cohen, I. M. & Dowling, D. R. *Fluid Mechanics*. (Chapter 12: Turbulence), 6th edition, Elsevier, 2016.
2. Tennekes, H. & Lumley, J. L. *A First Course in Turbulence*. MIT Press, 1972.
3. Davidson, P. A. *Turbulence. An Introduction for Scientists and Engineers*. (Parts I and II). Oxford University Press, 2004.
4. Frish, U. *Turbulence. The Legacy of A. N. Kolmogorov*. Cambridge University Press, 1995.

8 Soft Matter and Biological Physics

Theoretical Physics of Soft Condensed Matter (L16)

Mike Cates

Soft Condensed Matter refers to liquid crystals, emulsions, colloidal suspensions and other microstructured fluids or semi-solid materials. Alongside many high-tech examples, domestic and biological instances include mayonnaise, toothpaste, engine oil, shaving cream, and the lubricant that stops our joints scraping together. Their behaviour is classical ($\hbar = 0$) but rarely is it deterministic: thermal noise is generally important.

The basic modelling approach therefore involves continuous classical field theories, generally with noise so that the equations of motion are stochastic PDEs.

The form of these equations is helpfully constrained by a requirement that the Boltzmann distribution is regained in the steady state. Both the dynamical and steady-state behaviours have a natural expression in terms of path integrals, defined as weighted sums of trajectories (for dynamics) or configurations (for steady state). These concepts will be introduced in a relatively informal way, focusing on how they can be used for actual calculations.

In many cases mean-field treatments are sufficient, simplifying matters considerably. But we will also meet examples such as the phase transition from an isotropic fluid to a ‘smectic liquid crystal’ (a layered state which is periodic, with solid-like order, in one direction but can flow freely in the other two). Here mean-field theory gets the wrong answer for the order of the transition, but the right one is found in a self-consistent treatment that lies one step beyond mean-field (and one step short of the renormalization group, which we leave to the Statistical Field Theory course).

Important dynamical models of soft matter include diffusive ϕ^4 field theory (‘Model B’), and the noisy Navier-Stokes equation which describes fluid mechanics at colloidal scales, where the noise term is responsible for Brownian motion of suspended particles in a fluid. Coupling these together creates ‘Model H’, a theory that describes the physics of fluid-fluid mixtures (that is, emulsions). We will explore Model B, and then Model H, in some depth. We will also explore the continuum theory of nematic and polar liquid crystals, which spontaneously break rotational but not translational symmetry, focusing on topological defects and their associated mathematical structure such as homotopy classes.

Finally, the course will briefly analyse soft-matter systems whose microscopic dynamics involves the continuous dissipation of energy, such as self-propelled colloidal swimmers. We will discuss how their absence of time-reversal symmetry leads to qualitative changes in dynamical behaviour.

Prerequisites

Knowledge of Statistical Mechanics at an undergraduate level is essential. It is not necessary to attend any other particular part III courses but there is potential synergy with Statistical Field Theory, Slow Viscous Flow, Stochastic Processes in Biology, and to a lesser extent some other courses too. For field theorists, this is a good opportunity to see what statistical field theory looks like when addressing time evolution rather than the purely static properties set by the Boltzmann distribution. For people interested in fluid mechanics, it is a good opportunity to see what this looks like beyond Navier-Stokes fluids, in systems with thermal noise and/or structural order parameters. In previous years the audience has included a mix of students whose main specialism is either field theory or fluid dynamics. People with these differing backgrounds may find different parts of the course easier or harder, but the intention is to create a roughly level playing field so that all can enjoy learning about this interdisciplinary field.

Literature

1. D. Tong *Lectures on Statistical Physics*

<http://www.damtp.cam.ac.uk/user/tong/statphys.html>

Before embarking on this course you need to understand the equation $F = -k_B T \ln Z$ and its implications. This includes knowing what the Boltzmann distribution is, what it describes, and when it is true. You should also have met the concept of chemical potential and the grand canonical ensemble. Acquaintance with the Landau theory of phase transitions is helpful. David Tong's lecture notes are an excellent resource for revising and reviewing the key material.

2. M. E. Cates and E. Tjhung *Theories of binary fluid mixtures: from phase-separation kinetics to active emulsions*. J. Fluid Mech. (2018), **836**, pp 1-66.

<https://www.cambridge.org/core/journals/journal-of-fluid-mechanics/article/theories-of-binary-fluid-mixtures-from-phaseseparation-kinetics-to-active-emulsions/5BD133CB20D89F47E724D77C296FEF80/share/106fd30f307db12134745de39fd568fbbaa3f9d>

This JFM perspectives article has significant overlap (perhaps 40%) with the course but takes fluid mechanics as its starting point whereas we will start from statistical physics and bring in fluid mechanics as needed. It gives a flavour of the types of problem we will address and some of the methodologies involved. However we will not have time to cover much of the material it contains on active systems.

3. P. Chaikin and T. C. Lubensky *Principles of Condensed Matter Physics*. Cambridge University Press, 1995. An authoritative and broad ranging but advanced book, that is worth dipping into to see how hydrodynamics, broken symmetries, and topological defects all feature in the description of condensed matter systems at $\hbar = 0$. This is more for inspiration than information though; this course may help you in understanding the book, but probably not vice versa.

4. Unofficial lecture notes were taken several years ago:

https://dec41.user.srcf.net/h/III_L/theoretical_physics_of_condensed_matter

These notes are far from perfect but are mentioned here so that all students are aware of their existence.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a 90-minute revision class in the Easter Term. The lecturer will be available at other times by arrangement.

Note on past papers

Starting in the 2016/7 academic year, this course was given in 16-lecture format for 3 years, then as a 24-lecture course for 2 years, before reverting to 16 lectures from the 2021/2 academic year onwards. This means the summer 2021 tripos paper examined some material that lies beyond the current syllabus. (There was no honours paper in 2020.)

Biological Physics (M24)

Professor R.E. Goldstein

This course will provide an overview of the physics of living systems, with an emphasis on their mathematical description. The range of subjects and approaches, from phenomenology to detailed calculations, will be of interest to students from applied mathematics, physics, and computational biology.

The topics to be covered span the range of length scales from molecular to ecological, with emphasis on key paradigms. They include: *Microscopic Physics* – van der Waals forces, screened electrostatics, interactions of extended bodies; *Fluctuations and Brownian Motion* – Equipartition, fluctuating continuous objects, diffusion, polymers and entropic forces; *Biological Filaments* – differential geometry, elasticity, constraints, variational principles, elastohydrodynamics, instabilities; *Membranes* – differential geometry, Helfrich model, fluctuations, shapes and shape transformations; *Cellular Motion* – cell swimming, flagella and low-Re locomotion, hydrodynamic interactions; *Biological Pattern Formation* – chemical kinetics, reaction-diffusion systems, Turing instability, chemotaxis.

Prerequisites

Statistical physics, fluid mechanics, electromagnetism, and dynamical systems constitute the appropriate background for this course.

Literature

1. R.E. Goldstein and E. Lauga. *Biological Physics and Fluid Dynamics: A Graduate Course in 24 Lectures* (2023). Available at <https://www.damtp.cam.ac.uk/user/gold/pdfs/teach>
2. P. Nelson. *Biological Physics*. W.H. Freeman (2007).
3. J.D. Murray. *Mathematical Biology I. & II*. Springer (2007, 2008).
4. J.N. Israelachvili. *Intermolecular and Surface Forces*. 2nd edition. Academic Press (1992).

5. E.J.W. Verwey and J.Th.G. Overbeek. *Theory of the Stability of Lyophobic Colloids*. Elsevier (1948).
6. M. Doi and S.F. Edwards. *The Theory of Polymer Dynamics*. OUP (1986).
7. E. Lauga. *The Fluid Dynamics of Cell Motility*. CUP (2020).
8. D. Bray. *Cell Movements*. Garland (2000).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a revision class in the Easter Term.

9 Differential Geometry and Topology

Algebraic Topology (M24)

Dr. J. Steinebrunner

Algebraic Topology studies topological spaces by associating to them algebraic invariants, primarily abelian groups and commutative rings. It permeates modern pure mathematics and theoretical physics. This course will focus on (co)homology, with an emphasis on applications to the topology of manifolds. We will cover:

- singular homology and cohomology and its main calculational devices (eg. Mayer–Vietoris),
- cup-products in cohomology,
- the Künneth theorem,
- vector bundles and the Thom isomorphism theorem,
- the cohomology of manifolds and Poincaré duality.

The course will not assume prior knowledge of algebraic topology, but will start off rather fast in order to reach the more interesting material, so some previous exposure to chain complexes and simplicial homology would definitely be helpful.

Prerequisites

Basic topology: topological spaces, compactness and connectedness, at the level of Sutherland's book. Some knowledge of the fundamental group would be helpful though not a requirement. The book by Hatcher is especially recommended, but there are many other suitable texts and many online resources. The Part III Differential Geometry course will also contain useful, relevant material.

Literature

1. Hatcher, A. *Algebraic Topology*. Cambridge Univ. Press, 2002.
2. May, P. *A concise course in algebraic topology*. Univ. of Chicago Press, 1999.
3. Sutherland, W. *Introduction to metric and topological spaces*. Oxford Univ. Press, 1999.

Additional support

The course will emphasise examples and computations; it will be accompanied by four question sheets with associated Examples Classes, which will again involve applying the general theory to do explicit calculations and solve geometric problems. There will be a one-hour revision class in the Easter term.

Differential Geometry (M24)

A. Kovalev

This course is intended as an introduction to modern differential geometry. It can be taken with a view of further studies in Geometry and Topology and should also be suitable as a supplementary course if your main interests are e.g. in Analysis or Mathematical Physics. A tentative syllabus is as follows.

- *Local Analysis and Differential Manifolds*. Definition and examples of manifolds, matrix Lie groups. Tangent vectors, the tangent and cotangent bundle. Geometric consequences of the implicit function theorem, submanifolds. Exterior algebra of differential forms. Orientability of manifolds. Partition of unity and integration on manifolds, Stokes' Theorem. De Rham cohomology.
- *Vector Bundles*. Structure group, principal bundles. The example of Hopf bundle. Bundle morphisms. Three views on connections: vertical and horizontal subspaces, Christoffel symbols, covariant derivative. The curvature form and second Bianchi identity.

- *Riemannian Geometry*. Connections on manifolds, torsion. Riemannian metrics, the Levi-Civita connection. Geodesics, the exponential map, Gauss' Lemma. Decomposition of the curvature of a Riemannian manifold, Ricci and scalar curvature, low-dimensional examples. The Hodge star and Laplace–Beltrami operator. Statement of the Hodge decomposition theorem (with a sketch-proof, time permitting).

Prerequisites

An essential prerequisite is a working knowledge of linear algebra (including dual vector spaces and bilinear forms) and of multivariate calculus (e.g. differentiation in several variables and the inverse function theorem). Previous exposure to some ideas of classical differential geometry (of curves and surfaces in $\mathbf{R}^3$) would be useful.

Literature

- [1] F.W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer, 1983.
- [2] S. Gallot, D. Hulin, J. Lafontaine, *Riemannian geometry*. Springer-Verlag, 1990.
- [3] R.W.R. Darling, *Differential forms and connections*. CUP, 1994.
- [4] V. Guillemin, A. Pollack, *Differential topology*. Prentice-Hall Inc., 1974.

Roughly, half of the course material is found in [1]. The book [4] covers the required topology. On the other hand, [3] which has a chapter on vector bundles and on connections assumes no knowledge of topology. Both [2] and [3] have a lot of worked examples. There are many other good differential geometry texts.

Additional support

The lectures will be supplemented by four example classes, based on four example sheets. There will be a one-hour revision class in the Easter Term. Printed notes will be available from
<https://www.dpmms.cam.ac.uk/~agk22/teaching.html>

Geometric group theory (L24)

Henry Wilton

The subject of geometric group theory is founded on the observation that the algebraic and algorithmic properties of a discrete group are closely related to

the geometric features of the spaces on which the group acts. This course will provide an introduction to the basic ideas of the subject.

We will cover the following topics.

1. Free groups, group presentations and Cayley graphs
2. Group actions and quasi-isometry
3. Amalgams and Bass–Serre theory
4. Fuchsian groups
5. Hyperbolic groups

The books listed below could be used to provide alternative perspectives on some of this material.

Pre-requisites

Part IB Geometry and Part II Algebraic Topology (or equivalents) are required.

Literature

1. M. R. Bridson and A. Haefliger *Metric spaces of non-positive curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999. xxii+643 pp.
2. C. Druţu and M. Kapovich, *Geometric group theory*. With an appendix by Bogdan Nica. American Mathematical Society Colloquium Publications, 63. American Mathematical Society, Providence, RI, 2018. xx+819 pp.
3. P. de la Harpe, *Topics in geometric group theory*. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 2000. vi+310 pp.
4. C. Löh, *Geometric group theory – an introduction*. Universitext. Springer, 2017. xi+389 pp.
5. J. Meier, *Groups, Graphs and Trees: An Introduction to the Geometry of Infinite Groups*. London Mathematical Society Student Texts, 73. Cambridge University Press, Cambridge, 2008. xii+231 pp.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Simplicial Homotopy Theory (L24)

Professor J. Hahn

Simplicial sets simultaneously serve as combinatorial models for topological spaces up to homotopy equivalence (through the theory of Kan complexes) and as the foundational objects in the quasi-categorical approach to higher category theory. In this course we develop the basics of homotopy theory using the language of Kan complexes. We then introduce infinity categories, defining and exploring the infinity category of Kan complexes. Topics will include:

- Homotopy groups
- Fibrations and cofibrations
- The Whitehead and Hurewicz theorems
- Postnikov towers
- Quasi-categories, including the ∞ -categories of Kan complexes and chain complexes
- Homotopy (co)limits
- The Serre spectral sequence
- Brown representability.

Prerequisites

While there will be some review, the course assumes a working knowledge of both 1-category theory (colimits, limits, adjunctions, and the Yoneda embedding) and the singular homology of topological spaces.

Literature

The readings below may prove useful, but in total cover far more than the course content:

1. Lurie, J. *Kerodon*. Available at <https://kerodon.net/>.
2. Goerss, P. and Jardine, R. *Simplicial Homotopy theory*. Birkhäuser, 2009.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Symplectic Topology (L16)

Non-Examinable (Part III Level)

Dr A. Ward

This course is intended as a graduate level introduction to modern symplectic topology with a view towards Floer-theoretic techniques. Topics will include:

- Basics of symplectic topology: symplectic linear algebra, neighborhood theorems, Lagrangian submanifolds, symplectomorphisms.
- Constructions of symplectic manifolds: symplectic blow-ups, Lefschetz pencils.
- Moduli spaces of pseudo-holomorphic curves, Gromov-Witten invariants.
- Quantum cohomology, Hamiltonian Floer theory, Lagrangian Floer theory.
- Construction of the Fukaya category.

Prerequisites

Knowledge of algebraic topology and differential geometry will be essential.

Literature

1. Auroux, D. *A Beginner's Introduction to Fukaya Categories*. Available at <https://arxiv.org/pdf/1301.7056>.
2. McDuff, D and Salamon, D. *Introduction to Symplectic Topology (3rd Edition)*. Oxford Univ. Press, 2017.
3. McDuff, D and Salamon, D. *J-holomorphic Curves and Symplectic Topology*, AMS Colloquium Publ. 52, AMS, 2004.
4. Smith, I. *A Symplectic Prolegomenon*. Available at <https://arxiv.org/pdf/1401.0269>.

Complex Manifolds (L24)

Dr. Maxim Jeffs

This course will be a first introduction to complex manifolds: smooth manifolds that are equipped with an atlas of holomorphic charts. Such manifolds arise naturally in algebraic geometry, representation theory, and certain aspects of

mathematical physics. When equipped with a kind of compatible Riemannian metric, called a *Kähler metric*, complex manifolds have a rich geometric structure. This ‘Kähler package’, which shall be our focus for the second half of the course, has inspired significant developments over many different fields of mathematics.

The course will cover a selection of topics including:

- Analysis of several complex variables;
- Divisors, line bundles, and Chern classes;
- Complex algebraic curves and projective varieties;
- Kähler metrics and Hodge theory;
- The Ricci form and Calabi-Yau manifolds.

Prerequisites

Familiarity with complex analysis, smooth manifolds, and de Rham cohomology will be essential. Differential geometry and algebraic topology from Part III (or equivalent) are strongly recommended.

Literature

1. D. Huybrechts. *Complex geometry: An introduction*, Springer 2005.
2. D. Arapura. *Algebraic Geometry Over the Complex Numbers*, Springer Universitext, 2012.
3. C. Voisin. *Hodge Theory and Complex Algebraic Geometry I*, Cambridge University Press, 2002.
4. P. Griffiths and J. Harris. *Principles of algebraic geometry*, Wiley 1978.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

10 Foundations

Category Theory (M24)

Peter Johnstone

Category theory begins with the observation (Eilenberg–Mac Lane 1942) that the collection of all mathematical structures of a given type, together with all the maps between them, is itself an instance of a nontrivial structure which can be studied in its own right. In keeping with this idea, the real objects of study are not so much categories themselves as the maps between them—functors, natural transformations and (perhaps most important of all) adjunctions. Category theory has had considerable success in unifying ideas from different areas of mathematics; it is now an indispensable tool for anyone doing research in topology, abstract algebra, mathematical logic or theoretical computer science (to name just a few areas). This course aims to give a general introduction to the basic grammar of category theory, without any (intentional!) bias in the direction of any particular application. It should therefore be of interest to a large proportion of pure Part III students.

The following topics will be covered in the first three-quarters of the course:

- **Categories, functors and natural transformations.** Examples drawn from different areas of mathematics. Faithful and full functors, equivalence of categories, skeletons. [4 lectures]
- **Locally small categories.** The Yoneda lemma. Structure of set-valued functor categories: generating sets, projective and injective objects. [2 lectures]
- **Adjunctions.** Description in terms of comma categories, and by triangular identities. Uniqueness of adjoints. Reflections and coreflections. [3 lectures]
- **Limits.** Construction of limits from products and equalizers. Preservation and creation of limits. The Adjoint Functor Theorems. [4 lectures]
- **Monads.** The monad induced by an adjunction. The Eilenberg–Moore and Kleisli categories, and their universal properties. Monadic adjunctions; Beck’s Theorem. [4 lectures]

The remaining seven lectures will be devoted to topics chosen by the lecturer, probably from among the following:

- **Filtered colimits.** Finitary functors, finitely-presentable objects. Applications to universal algebra.

- **Regular categories.** Embedding theorems. Categories of relations, introduction to allegories.
- **Abelian categories.** Exact sequences, projective resolutions, derived functors. Introduction to homological algebra.
- **Monoidal categories.** Coherence theorems, monoidal closed categories, enriched categories. Weighted limits.
- **Fibrations.** Indexed categories, internal categories, definability. The indexed adjoint functor theorem.

Prerequisites

There are no specific prerequisites other than some familiarity with undergraduate-level abstract algebra, although a first course in logic would be helpful. Some of the examples discussed will involve more detailed knowledge of particular topics in algebra or topology, but the aim will be to provide enough examples for everyone to understand at least some of them.

Literature

1. S. Mac Lane *Categories for the Working Mathematician*. Springer 1971 (second edition 1998). Still the best one-volume book on the subject, written by one of its founders.
2. S. Awodey *Category Theory*. Oxford U.P. 2006. A more recent treatment very much in the spirit of Mac Lane's classic (Awodey was Mac Lane's last PhD student), but rather more gently paced.
3. T. Leinster *Basic Category Theory*. Cambridge U.P. 2014. Another gently-paced alternative to Mac Lane: easy to read, but it doesn't cover the whole course.
4. E. Riehl *Category Theory in Context*. Dover Publications 2016. A recent account of the subject by someone who first encountered it as a Part III student fifteen years ago.
5. F. Borceux *Handbook of Categorical Algebra*. Cambridge U.P. 1994. Three volumes which together provide the best modern account of everything an educated mathematician should know about categories: volume 1 covers most but not all of the Part III course.

Additional support

A full set of written notes for the course will be provided; these contain exercises for the reader, and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Forcing & the Continuum Hypothesis (L16)

Professor Benedikt Löwe

The method of forcing is one of the most important model constructions in set theory and its versatility is the reason for the plethora of independence results in set theory. It was developed to solve one of the most important foundational problems of the 20th century: determining the cardinality of the set of real numbers. Cantor's *Continuum Hypothesis* asserts that the cardinality of the set of real numbers has the smallest possible value:

Every infinite set of reals is either equinumerous with the set of $\mathbb{N}$ -natural numbers or equinumerous with the set of all real numbers.⁽¹⁾

Equivalently, $2^{\aleph_0} = \aleph_1$.

When he presented his list of twenty-three problems for the 20th century at the *International Congress of Mathematicians* in Paris in 1900, David Hilbert listed the question whether the Continuum Hypothesis is true as the very first problem on his list. It turned out that this question cannot be solved on the basis of ZFC: in 1938, Kurt Gödel showed that CH cannot be disproved in ZFC; in 1963, Paul Cohen invented the method of forcing to show that CH cannot be proved in ZFC.

In this course we shall study the basics of the method of forcing in order to present Cohen's proof. The course will discuss:

1. models of set theory (in particular, transitive models, the Lévy reflection theorem, and the theory of absoluteness);
2. the construction of the generic extension of a countable transitive model (in particular, the syntactic and semantic forcing relation, the forcing theorem, and the generic model theorem);
3. showing relative consistency proofs with the method of forcing;
4. applications to cardinal arithmetic (in particular, preservation results and the consistency of $\neg\text{CH}$ and related results).

Prerequisites

The Part II course *Logic & Set Theory* or an equivalent course is essential. This course uses some content from the Part III course *Model Theory* (Lent 2024). It is recommended (but not strictly necessary) to take *Model Theory* to supplement the discussion of models of set theory in this course.

Literature

Kenneth Kunen. *Set Theory. An Introduction to Independence Proofs*. Elsevier 1980 [Studies in Logic and the Foundations of Mathematics, Vol. 102].

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Logic and Computability (M24)

J. Siqueira

This course builds upon the foundations laid out by the Part II courses *Logic and Set Theory* and *Automata and Formal Languages*, introducing a new notion of computability and a non-classical logic, as well as discussing the relationship between these subjects.

In Part II, there was an exposition of two general models of computation: register machines and recursive functions. There is, however, another alternative which has certain practical advantages and is at the core of functional programming languages like Haskell: the (untyped) λ -calculus. We will explore its properties and see that a version of this calculus has a strong connection to intuitionistic logical proofs.

Intuitionistic logic departs from classical logic by adopting a new interpretation for the logical connectives and quantifiers that makes it constructive and, therefore, carry computational content. We will discuss its syntax and semantics, the accompanying completeness theorems, and how it compares to classical logic.

Finally, there are interesting points in which logic and computability intersect. Examples include the incompleteness phenomenon and the arithmetic hierarchy (which generalises the characterisation of recursively enumerable sets as Σ_1 -definable sets). We will explore some of these ideas and see what computability theory has to say about models of arithmetic.

The intent is to cover the following topics:

Computability

Recursive functions and computability; Recursive and recursively enumerable sets; The untyped λ -calculus; Reduction and the Church-Rosser Theorem; Church numerals; Fixed point combinators and λ -computability.

Logic

Nonstandard models of arithmetic; Natural deduction and intuitionistic logic; Diaconescu's Theorem; Completeness of the Heyting and Kripke semantics; Negative translations.

Computability in logic

Craig's Theorem; Gödel numberings and Incompleteness, Tennenbaum's Theorem; The simply typed λ -calculus and the Curry-Howard isomorphism; Weak and strong normalisation for the simply typed λ -calculus.

If time permits, we may also discuss either the Friedberg-Muchnik Theorem or Priestley Duality.

The course will appeal to those wishing to pursue research in the Foundations of Mathematics, Computer Science (particularly in semantics of programming languages), and Proof Formalisation. Those interested in topos theory will find the material on intuitionistic logic particularly useful.

Prerequisites

Familiarity with the contents of the Part II courses Logic and Set Theory and Automata and Formal Languages (or equivalent) is essential. Some of the relevant facts and definitions pertaining to computability, however, will be revised at a fast pace.

Literature

A primary source for intuitionistic logic and its relation to the λ -calculus is '*Proofs and Types*' [1]. It doesn't address the Heyting and Kripke semantics, however, and thus has to be complemented in other ways. A categorically minded student will enjoy Ghilardi's '*A First Introduction to the Algebra of Sentences*' [2], but everyone can appreciate Palmgren's notes '*Semantics of intuitionistic propositional logic*' [3] — they were both made available online. This is also the case for the very good and straightforward notes on the untyped λ -calculus by Barendregt and Barendsen '[4]'.

Kaye's '*Models of Peano Arithmetic*' [5] is the go-to reference on anything related to the formal theory of arithmetic.

If we end up pursuing the extra topics, Soare's book [6] has the best exposition of Turing degrees and finite priority injury methods I am aware of. Discussions

on duality will benefit from Priestley’s own book [7], as well as the more modern exposition by Gehrke and van Gool [9].

References

- [1] J. Girard, Y. Lafont, P. Taylor. *Proofs and Types*, Cambridge University Press, 1989.
- [2] S. Ghilardi. *A First Introduction to the Algebra of Sentences*, Lecture Notes, Dipartimento di Scienze dell’Informazione, Università degli Studi di Milano, 2000, <http://users.mat.unimi.it/users/ghilardi/allegati/dispcesena.pdf>.
- [3] E. Palmgren. *Semantics of intuitionistic propositional logic*, Lecture Notes, Department of Mathematics, Uppsala University, 2009, <http://www2.math.uu.se/palmgren/tillog/heyting3.pdf>.
- [4] H. Barendregt, E. Barendsen. *Introduction to Lambda Calculus*, Revised edition, 2000, <https://www.cse.chalmers.se/research/group/logic/TypesSS05/Extra/geuvers.pdf>.
- [5] R. Kaye. *Models Of Peano Arithmetic*, Oxford Logic Guides **15**, Oxford University Press, 1991.
- [6] R. I. Soare. *Turing Computability: Theory and Applications*, Theory and Applications of Computability Series, Springer, 2016.
- [7] B. A. Davey, H. A. Priestley. *Introduction to Lattices and Order*, 2nd Edition, Cambridge University Press, 2002.
- [8] A. S. Troelstra, D. van Dalen. *Constructivism in Mathematics: an introduction*, Vol. 1, Elsevier, 1988.
- [9] M. Gehrke, S. van Gool. *Topological Duality for Distributive Lattices: Theory and Applications*, Cambridge University Press (Cambridge Tracts in Theoretical Computer Science), 2024.
- [10] H. Rogers Jr. *Theory of Recursive Functions and Effective Computability*, Higher Mathematics Series, McGraw-Hill, 1967.

Additional support

Four Example Sheets (and accompanying Example Classes) are planned for this course. There will also be a one-hour revision class in Easter Term.

Model Theory (L16)

Dr C. Kestner

Model theory is a branch of mathematical logic. Initially, the focus was on how far a set of sentences in first-order language determines the class of structures it describes (e.g. are two models of the same cardinality of a particular set of sentences isomorphic). Later, the subject evolved in several directions. For example, part of the attention shifted to contexts where the subsets of a structure that are definable are also mathematically meaningful (for instance, the definable sets in an algebraically closed field are the constructible sets). This is one of several ways in which model theory interacts with other areas of mathematics. Connections have emerged to real and algebraic geometry, number theory and, more recently, combinatorics. This course introduces some basic model-theoretic tools and ideas up to initial notions in stability theory.

We will cover a selection of the following topics:

- Preliminaries: structures, theories, elementarity (including elementary substructures, Tarski- Vaught test, downward Löwenheim-Skolem theorem).
- Examples of structures (relational structures, basic algebraic structures).
- Back and forth and Quantifier Elimination. Types, type spaces.
- Ultraproducts, Ultrapowers and Łoś's theorem.
- The compactness theorem for theories and for types.
- Prime models. Omitting types theorem, possibly including ω -categorical theories.
- Strongly Minimal Structures.
- Stability theory.
- O-minimal structures.

Prerequisites

The Part II course *Set Theory and Logic*, or an equivalent course, is essential.

Literature

1. D. Marker *Model Theory: An Introduction* Vol. 217. Springer Science & Business Media, 2006.

2. Katrin Tent and Martin Ziegler, *A Course in Model Theory* Lecture Notes in Logic 40. Cambridge University Press, 2012.
3. W. Hodges, *A shorter model theory*. Cambridge University Press, Cambridge, 1997.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

11 Information and Finance

Information Theory (M16)

Ioannis Kontoyiannis

Information theory is the mathematical foundation of the ideas and tools required for the quantitative description and analysis of the notion of information. The origin of the theory in Claude Shannon's landmark 1948 paper is motivated by questions in communications engineering, but since then information theory has forged deep connections with many areas of mathematics, most notably with probability and statistics. This will be an introductory course to the main information-theoretic ideas, results, and techniques. We will discuss entropy as a measure of information, relative entropy as a natural distance between probability distributions, and mutual information as a universal dependence measure between random variables. Their properties will be established (monotonicity, chain rules, data processing inequalities, asymptotic equipartition) and different versions of Shannon's source coding theorems will be proved, including their extensions to universal data compression by Shtarkov and Rissanen. Connections with large deviations theory and with additive combinatorics will also be developed.

Pre-requisites

The only pre-requisite is knowledge of basic probability, although a certain level of maturity and familiarity with the use of probabilistic techniques will be helpful. Knowledge of advanced (including measure-theoretic) probability is not necessary.

Literature

1. Mainstream introduction to information theory:
T.M. Cover and J. Thomas. *Elements of Information Theory*. 2nd edition. Wiley-Interscience, 2006.
2. A more applied perspective:
D. MacKay. *Information Theory, Inference, and Learning Algorithms*. Cambridge University Press, 2003. Available free online at: <https://www.inference.org.uk/itprnn/book.pdf>.
3. Theoretical connections with ergodic theory and probability...
P. Billingsley. *Ergodic Theory and Information*. J. Wiley, 1965.
4. ... and: P. Shields. *The Ergodic Theory of Discrete Sample Paths*. American Mathematical Society, 1996.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Stochastic Calculus and Applications to Finance (L24)

M. Tehranchi

This course is an introduction to the theory of continuous-time stochastic processes, including its application to financial theory. It complements the material in Advanced Probability.

- *Stochastic integration*. Finite variation processes. Martingales, local martingales and semi-martingales. Quadratic variation and co-variation. Itô's isometry and definition of stochastic integral. Kunita–Watanabe's theorem. Itô's formula.
- *Applications to Brownian motion*. Lévy's characterization of Brownian motion. Dambis–Dubins–Schwartz theorem. Girsanov's theorem. Martingale representation theorems.
- *Stochastic differential equations*. Notions of existence and uniqueness of solutions. Yamada–Watanabe theorem. Kolmogorov, Fokker–Planck and Feynmann–Kac partial differential equations.
- *Financial applications*. Arbitrage. Pricing and hedging contingent claims. Optimal investment.

Pre-requisites

Knowledge of measure theoretic probability at the level of Part III Advanced Probability will be assumed, especially familiarity with discrete-time martingales and basic properties of Brownian motion.

Literature

1. M. Musiela and M. Rutkowski. *Martingale Methods in Financial Modelling*. Springer. 2005
2. D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*. Springer. 2001
3. L.C. Rogers and D. Williams. *Diffusions, Markov Processes and Martingales. Vol.1 and 2*. Cambridge University Press. 2002

Additional support

Four sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Concentration Inequalities (L16)

Dr L. Gavalakis

Informally speaking, the *concentration of measure phenomenon* asserts that if $X_1, \dots, X_n$ are independent random variables, then $f(X_1, \dots, X_n)$ does not deviate much from its mean provided that $f(x_1, \dots, x_n)$ is not too sensitive in any of its coordinates x_i . The topic of study of this course is the set of the various tools used to quantify this phenomenon via *non-asymptotic inequalities*. These inequalities find applications in many domains including statistics, learning theory, discrete mathematics, random matrix theory, information theory, and high-dimensional geometry.

The course will cover a selection of topics including:

- Markov's inequality and Chernoff bound; inequalities for sub-Gaussian random variables; Hoeffding's inequality and related inequalities
- Efron-Stein inequality; log-Sobolev and Poincaré inequality; basic information-theoretic inequalities; the entropy method
- Talagrand's inequality; the transport method; Marton's transport cost inequalities

Applications of the above inequalities to random matrices, Boolean analysis and combinatorics will be considered throughout the course.

Prerequisites

The only pre-requisite is knowledge of basic probability, although a certain level of maturity and familiarity with the use of probabilistic techniques will be helpful. Knowledge of advanced (including measure-theoretic) probability is not necessary, but would be useful.

Literature

1. S. Boucheron, G. Lugosi and P. Massart, *Concentration Inequalities: A Non-Asymptotic Theory of Independence*, Oxford University Press, 2013.
2. R. van Handel, *Probability in High Dimensions*. Available at <https://web.math.princeton.edu/~rvan/APC/>
3. R. Vershynin, *High Dimensional Probability*. Cambridge University Press, 2018, also available at <https://www.math.uci.edu/~rvershyn/papers/HDP-book/HDP-book.pdf>.
4. M. Ledoux, *The Concentration of Measure Phenomenon*, Mathematical Surveys and Monographs, 2001.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

12 Number Theory

Analytic Number Theory (L24)

Dr. J. Teräväinen

In this course we will cover results on the distribution of prime numbers, in particular the celebrated prime number theorem. This is done by studying analytic properties of the Riemann zeta function and connections between the location of its zeros and results on prime numbers. We will also cover results from the theory of multiplicative functions, such as mean value results for these. This is a topic that has seen a lot of recent activity.

Topics likely to be covered include:

- Arithmetic functions and Dirichlet series
- Properties of the Riemann zeta function
- The prime number theorem
- The Riemann hypothesis and its connections with the distribution of primes
- Mean values of multiplicative functions

Prerequisites

Elementary number theory (as covered in Part II *Number Theory*), Basic complex analysis (as covered in Part IB *Complex Analysis*).

Literature

1. T. Apostol, *Introduction to Analytic Number Theory*, Springer, 1976.
2. H. Davenport, *Multiplicative number theory*, Springer, 2000.
3. H.L. Montgomery, R.C. Vaughan, *Multiplicative number theory. I. Classical theory*, CUP, 2007.
4. G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, AMS, 2015.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Diophantine Analysis (M24)

Professor P. P. Varjú

This course will discuss two classical methods in Diophantine analysis. The first one can be traced back to Thue, who proved the following result in Diophantine approximation. Let α be an algebraic number of degree d . Then for any

$$\kappa > \frac{d}{2} + 1,$$

there is a constant c such that

$$\left| \alpha - \frac{r}{s} \right| > \frac{c}{s^\kappa}$$

for all $r, s \in \mathbf{Z}$. This result is a significant improvement of Liouville's bound, in which the exponent is $\kappa = d$. Thue used his result to prove that certain Diophantine equations have only finitely many solutions. Thue's result has been subsequently improved by Siegel, Dyson and finally Roth, who achieved the optimal exponent $\kappa = 2 + \varepsilon$. Schmidt generalized Roth's result to the setting of a system of inequalities for linear forms, which is known as the Subspace Theorem.

The second method that will be discussed in the course goes back to the work of Baker. Gelfond and Schneider independently proved that α^β is transcendental whenever $\alpha \neq \{0, 1\}$ and β is an irrational algebraic number, which was Hilbert's seventh problem. This can be reformulated as

$$\beta_1 \log \alpha_1 + \beta_2 \log \alpha_2 \neq 0$$

for any non-zero algebraic numbers $\alpha_1, \alpha_2, \beta_1, \beta_2$, provided $\log \alpha_1$ and $\log \alpha_2$ are linearly independent over the rationals. Baker generalized this result to linear forms in arbitrarily many logarithms, and, moreover, he gave lower bounds for the absolute value of such a form. These estimates have been revisited and improved on by many authors.

Both methods have been utilized by many authors for a wide range of applications in number theory and beyond. In the course, we will discuss both methods, and we will sample from their applications.

Prerequisites

Some knowledge of Galois theory, number fields and complex analysis will be assumed.

Literature

The course will not follow any particular source, but there are many excellent books that discuss some of the course material including the following.

1. A. Baker, *Transcendental number theory*. Cambridge University Press, Cambridge, 1990.
2. E. Bombieri and W. Gubler, *Heights in Diophantine geometry*. Cambridge University Press, Cambridge, 2006.
3. Y. Bugeaud, *Linear forms in logarithms and applications*. European Mathematical Society (EMS), Zürich, 2018.
4. J. W. S. Cassels, *An introduction to Diophantine approximation*. Hafner Publishing Co., New York, 1972.

5. D. Masser, *Auxiliary polynomials in number theory*. Cambridge University Press, Cambridge, 2016.
6. M. Waldschmidt, *Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables*. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 326. Springer-Verlag, Berlin, 2000.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Elliptic Curves (L24)

Professor T. A. Fisher

Elliptic curves are the first non-trivial curves, and it is a remarkable fact that they have continuously been at the centre stage of mathematical research for centuries. This will be an introductory course on the arithmetic of elliptic curves, concentrating on the study of the group of rational points.

The following topics will be covered, and possibly others if time is available.

- **Weierstrass equations and the group law.** Methods for putting an elliptic curve in Weierstrass form. Definition of the group law in terms of the chord and tangent process.
- **Isogenies.** The degree of an isogeny is a quadratic form. The invariant differential and separability. The torsion subgroup over an algebraically closed field.
- **Elliptic curves over finite fields.** Hasse's theorem and zeta functions.
- **Elliptic curves over local fields.** Formal groups and their classification over fields of characteristic 0. Minimal models, reduction mod p , and the formal group of an elliptic curve. Singular Weierstrass equations.
- **Elliptic curves over number fields.** The torsion subgroup. The Lutz-Nagell theorem. The weak Mordell-Weil theorem via Kummer theory. Heights. The Mordell-Weil theorem. Galois cohomology and Selmer groups. Descent by 2-isogeny. Numerical examples.

Prerequisites

Familiarity with the main ideas in the Part II courses *Galois Theory* and *Number Fields* will be assumed. The first few lectures will include a brief review of the necessary geometric background, but this will assume some previous knowledge of algebraic curves, at the level of the Part II course *Algebraic Geometry* or the first two chapters of [3]. Later in the course, some basic knowledge of the Part III course *Local Fields* will be assumed.

Preliminary Reading

1. J.H. Silverman and J. Tate, *Rational Points on Elliptic Curves*, Springer, 1992.

Literature

2. J.W.S. Cassels, *Lectures on Elliptic Curves*, CUP, 1991.
3. J.H. Silverman, *The Arithmetic of Elliptic Curves*, Springer, 1986.

Additional support

Four example sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Local Fields (M24)

Rong Zhou

The p -adic numbers $\mathbb{Q}_p$ were introduced by Hensel at the end of the 19th century and are now a ubiquitous tool in modern number theory as well as many other fields including algebraic topology, representation theory and algebraic geometry. The idea is to consider the completion of $\mathbb{Q}$ with respect to the absolute value defined by $|x|_p = p^{-n}$ for non-zero $x \in \mathbb{Q}$ where $x = p^n \frac{a}{b}$ with $a, b, n \in \mathbb{Z}$ and a, b coprime to p . The resulting field $\mathbb{Q}_p$ gives a neat way of packaging the information of congruences modulo p^n for all n and is the basic example of a local field. From this point of view, local fields are objects lying on the interface between algebra and analysis and the techniques used to study them involve an interesting mix of the two subjects.

This course will cover the basic theory of local fields, as well as some more advanced topics such as local class field theory, and is likely to be useful for students interested in studying other Part III courses on number theory. Topics likely to be covered include:

Absolute values on fields; valuations; Hensel's Lemma; structure of local fields;
 Extensions of complete fields; Dedekind domains; Decomposition groups;
 Ramification theory; higher ramification groups; Weil group;
 Statements of local class field theory; formal groups; Lubin–Tate theory.

Prerequisites

Basic algebra up to and including Part II Galois theory as well as knowledge of concepts in point set topology and metric spaces are essential pre-requisites. It will be assumed that students have attended a first course in algebraic number fields.

Literature

1. J.W.S. Cassels, *Local fields*, Cambridge University Press, 1986. .
2. J.W.S. Cassels; A. Fröhlich, *Algebraic Number Theory*, Academic Press, 1967
3. J. Neukirch, *Algebraic Number Theory*, Springer–Verlag, 1999.
4. J. P. Serre, *Local fields*, Graduate Texts in Mathematics, 67. Springer–Verlag, 1979.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quadratic Forms (L16)

Non-Examinable (Part III Level)

Professor J. A. Thorne

A quadratic form over a ring R is a homogeneous polynomial of degree 2 in some number of variables $x_1, \dots, x_n$, with coefficients in R . The study of quadratic forms over arithmetic rings R (such as number fields or their completions, or their rings of integers) is a beautiful and classical topic in its own right, but also serves as an entry point into many important parts of number theory, including the theory of algebraic and arithmetic groups and the theory of automorphic

forms. In this course we will develop the theory of quadratic forms from scratch, focusing on the classification of quadratic forms over the rings $\mathbf{Q}$ and $\mathbf{Z}$.

Topics to be covered include:

- Classification of quadratic forms over $\mathbf{Q}_p$, $\mathbf{R}$, and $\mathbf{Q}$.
- Theorem of Hasse–Minkowski.
- Theory over $\mathbf{Z}$ and $\mathbf{Z}_p$: lattices, genus theory.
- Reduction theory and finiteness of the class number.

Prerequisites

I will assume familiarity with the theory of symmetric bilinear forms (at the level of IB Linear Algebra) and p -adic numbers (at the level of Part III Local Fields).

Literature

1. Serre, J.-P., *A course in arithmetic*. Graduate Texts in Mathematics, No. 7. Springer-Verlag, 1973.
2. Cassels, J. W. S., *Rational quadratic forms*. London Mathematical Society Monographs, 13, Academic Press, Inc., 1978.
3. O'Meara, O. T., *Introduction to quadratic forms*. Classics in Mathematics. Springer-Verlag, 2000.

13 Particle Physics and Quantum Fields

Applications of Quantum Field Theory (E16)

Sean Hartnoll

This course will explore quantum field theories that emerge as the long distance, continuum, description of quantum many-body systems. These quantum field theories share important conceptual tools with those encountered in particle physics, such as renormalisation group flow, spontaneous symmetry breaking and diagrammatic perturbation theory. However, due to the immense diversity of many-body systems, which can have 1, 2 or 3 spatial dimensions, are not Lorentz invariant, and admit a variety of different collective modes, the quantum field theories to be considered in this course exhibit a range of rich and novel

behaviours. Furthermore, while collider physics occurs in the vacuum, many-body physics takes place in states with a large number of particles and often at nonzero temperature. It becomes crucial to understand the interplay between quantum and thermal fluctuations.

Some subset of the following topics will be covered, as time allows:

- Spontaneous symmetry breaking in quantum many-body physics. Effective field theories of superfluids and magnets. Nonlinear sigma models. Coherent state path integral. Spin chains, topological terms and large S expansion.
- Nonzero temperature. Periodicity in imaginary time. Matsubara frequencies. Spectral functions. Fluctuation-dissipation theorem.
- Quantum Criticality. Quantum and thermal fluctuations. Large N expansion.
- Effective field theory of Fermi surfaces. Renormalisation group picture of Fermi liquids. Superconducting instability of Fermi surfaces. Non-Fermi liquids.
- Dimerized and valence-bond solid phases. Spin liquids, spinons and emergent gauge fields. Deconfined quantum criticality and monopoles.

Prerequisites

Understanding of quantum field theory at the level of the Advanced Quantum Field Theory course is required. In particular, familiarity with path integrals and Wilsonian renormalisation will be assumed. Exposure to material in the Statistical Field Theory course (in particular, classical Landau-Ginzburg theory) and The Standard Model course (in particular, spontaneous symmetry breaking in quantum field theory) may be useful but will not be assumed. No prior exposure to solid state physics will be assumed.

Literature

1. S. Sachdev, *Quantum Phase Transitions*, 2nd Edition, CUP 2011.
2. X-G. Wen, *Quantum Field Theory of Many-Body Systems*, OUP 2004.
3. A. Zee, *Quantum Field Theory in a Nutshell*, PUP, 2003.
4. A. Altland and B. Simons, *Condensed Matter Field Theory*, 2nd Edition, CUP 2010.
5. S. Sachdev, *Quantum Phases of Matter*, CUP 2023.

Additional support

Two examples sheets will be provided and associated examples classes will be given. There will be a revision session before the exam.

Advanced Quantum Field Theory (L24)

Dr R. A. Reid-Edwards

Quantum Field Theory (QFT) provides the most profound description of Nature we currently possess and is the basic theoretical framework for describing elementary particles and their interactions (excluding gravity). QFT also plays a major role in areas of physics and mathematics as diverse as string theory, condensed matter physics, topology and geometry, astrophysics and cosmology.

This course introduces techniques of path integrals and functional methods to study QFT including gauge theories. Gauge Theories are a generalization of electrodynamics and form the backbone of the Standard Model – our best theory encompassing all particle physics. In a gauge theory, fields have an infinitely redundant description; we can transform the fields by a different element of a Lie group at every point in space-time and yet still describe the same physics. Quantizing a gauge theory requires us to eliminate this infinite redundancy. In the path integral approach, this is done using tools such as ghost fields and BRST symmetry.

A further major component of the course is to study renormalization. Wilson's picture of renormalization is one of the deepest insights into QFT – it explains why we can do physics at all! The essential point is that the physics we see depends on the scale at which we look. In QFT, this dependence is governed by evolution along the renormalization group (RG) flow. The course explores renormalization systematically using dimensional regularization in perturbative loop integrals. We discuss the various possible behaviours of a QFT under RG flow, showing in particular that the coupling constant of a nonabelian gauge theory can effectively become small at high energies. Known as *asymptotic freedom*, this phenomenon revolutionized our understanding of the strong interactions. We introduce the notion of an Effective Field Theory that describes the low energy limit of a more fundamental theory and helps parametrize possible departures from this low energy approximation. From a modern perspective, the Standard Model itself appears to be but an effective field theory.

Prerequisites

Knowledge of the Michaelmas term Quantum Field Theory course will be assumed. Familiarity with the course Symmetries, Fields and Particles would be

helpful for the section on nonabelian gauge theory. The section on the renormalization group has some overlap with Statistical Field Theory but no knowledge of this course will be assumed.

Literature

1. Peskin, M. and Schroeder, D., *An Introduction to Quantum Field Theory*, Perseus Books (1995).
2. Srednicki, M., *Quantum Field Theory*, CUP (2007) A pre-publication draft is available at <http://web.physics.ucsb.edu/~mark/qft.html>.
3. Weinberg, S., *The Quantum Theory of Fields*, vols. 1 & 2, CUP (1996).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Field Theory (M24)

Alejandra Castro

Quantum Field Theory is a marriage of quantum mechanics and special relativity which provides the mathematical framework for describing the interactions of elementary particles. This first Quantum Field Theory course introduces the basic types of fields which play an important role in high energy physics: scalars, (Dirac) spinors, and gauge fields. The relativistic covariance and symmetry properties of these fields are discussed using the language of Lagrangians and Noether's theorem. The quantisation of free fields is developed using Hamiltonian methods in terms of operators which create and annihilate particles and anti-particles. The Fock space of quantum physical states and the associated realisation of particle statistics are described.

Interacting field theory is developed next using the LSZ formula for the scattering amplitude. Perturbation theory is formulated by applying the Schwinger-Dyson equation and Wick's theorem; Feynman diagrams are then introduced as an efficient way of organising the resulting calculations. Physically measurable quantities such as decay rates and cross-sections are defined. Spinors and the Dirac equation are explored in detail, along with parity and chirality. Fermionic quantisation is developed, along with Feynman rules and Feynman propagators for fermions.

We review the relativistic formulation of Maxwell's equations formulating a corresponding action principle. The significance of gauge invariance is discussed.

Lorentz gauge quantisation of the electromagnetic field is accomplished by imposing the Gupta-Bleuler condition. Finally, quantum electrodynamics (QED) is developed. Interactions between photons and charged matter are introduced via the principle of minimal coupling. We discuss the calculation of tree-level scattering amplitudes in QED.

Prerequisites

You will need to be comfortable with the Lagrangian and Hamiltonian formulations of classical mechanics and with special relativity. You will also need to have taken an advanced course on quantum mechanics including Dirac's "bra/ket" notation. A basic knowledge of group theory is also very useful for this course.

Literature

1. David Tong, *Lectures on Quantum Field Theory*, <http://www.damtp.cam.ac.uk/user/tong/qft.html>**link**.
2. M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, 2014.
3. M. Peskin and D. Schroeder, *An Introduction to Quantum Field Theory*, Harper Collins Publishers, 1995.
4. T. Zee, *Quantum Field Theory in a Nutshell*, Princeton University Press, 2003.
5. M. Srednicki, *Quantum Field Theory*, Cambridge University Press, 2007. <http://web.physics.ucsb.edu/~mark/qft.html>**link**.
6. P. Ramond, *Field Theory: A Modern Primer*, Benjamin/Cummings Pub. Co., 1981.

Quantum field theory is an extremely vast subject. The course will mainly follow David Tong's lecture notes, and specific chapters in Schwartz's book listed above. Every book listed here has strengths and weaknesses, so it is useful to be familiar with several. For instance, Zee's book is very good for a conceptual and intuitive introduction, but lacks details on how to do computations. Srednicki's book on the other hand is very explicit on how to carry out computations, but there are few explanations and motivations. Schwartz's book gives a modern understanding of effective field theories and balances well formal aspects, explicit derivations and explanations. Ramond's book is short but has wonderful explanations which I personally find useful.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

The Standard Model (L24)

David Tong

The Standard Model is, by any measure, the most successful scientific theory of all time. The culmination of 350 years of scientific enquiry, it correctly explains every experiment that we've ever done (at least if you're willing to turn a blind eye to gravity for now). The purpose of this course is to explain the ingredients that go into the Standard Model, and the way in which they fit together.

In many ways, this course can be thought of as “QFT 3”. We will cover many concepts that are widely applicable to any quantum field theory. In particular, there will be a focus on symmetries, including global symmetries, gauge symmetries, Poincaré symmetries, and discrete symmetries such as C,P and T. Associated to many of these symmetries is the idea of “symmetry breaking” and, to this end, we will prove Goldstone's theorem and explain the Higgs mechanism.

There will also be a focus on mathematical consistency. A phenomenon known as “anomaly cancellation” means that certain features of the Standard Model cannot be any other way: they are rigid and fixed. So you cannot, for example, have an electron-type particle without also having quarks and neutrinos. Other parts of the Standard Model appear to be much less rigid, especially those concerning the various interactions with the Higgs boson, usually referred to as “flavour”, and these too will be described in detail.

Prerequisites

You should be comfortable with the material from the Michaelmas term courses on QFT and Symmetries. It will also be useful to attend Advanced Quantum Field Theory in Lent.

Literature

Lecture notes for this course are available at

<https://www.damtp.cam.ac.uk/user/tong/standardmodel.html><https://www.damtp.cam.ac.uk/user/tong/standardmodel.html>

In addition, I previously wrote lecture notes for the CERN summer school which cover the material using only high school maths:

<http://www.damtp.cam.ac.uk/user/tong/particle.html><http://www.damtp.cam.ac.uk/user/tong/particle.html>

Some books that may be worth consulting include

1. David Griffiths, “*Introduction to Elementary Particles*”, Wiley 2008
2. Matt Schwartz, “*Quantum Field Theory and the Standard Model*”. Cambridge University Press, 2014.
3. Cliff Burgess and Guy Moore, “*The Standard Model: A Primer*”, Cambridge University Press, 2007.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistical Field Theory (M16)

Prof. H.S. Reall

This course introduces the renormalization group, focusing on statistical systems such as spin models with further connections to quantum field theory.

After presenting the Ising Model, Landau’s mean field theory is introduced and used to describe phase transitions. The extension to the Landau-Ginzburg theory reveals broader aspects of fluctuations whilst consolidating connections to quantum field theory. At second order phase transitions, also known as ‘critical points’, renormalisation group methods play a starring role. Ideas such as scaling, critical exponents and anomalous dimensions are developed and applied to a number of different systems, including those with continuous symmetries.

Prerequisites

Background knowledge of statistical mechanics at an undergraduate level is essential. This course complements the Quantum Field Theory and Advanced Quantum Field Theory courses.

Literature

1. N Goldenfeld, *Lectures on Phase Transitions and the Renormalization Group*, Addison-Wesley (1992).
2. M Kardar, *Statistical Physics of Fields*, Cambridge University Press (2007).
3. J Cardy, *Scaling and Renormalisation in Statistical Physics*, Cambridge University Press (1996).

4. J M Yeomans, *Statistical Mechanics of Phase Transitions*, Clarendon Press (1992).
5. M Le Bellac, *Quantum and Statistical Field Theory*, Oxford University Press (1991).
6. J J Binney, N J Dowrick, A J Fisher, and M E J Newman, *The Theory of Critical Phenomena*, Oxford University Press (1992).
7. D Amit and V Martin-Mayor, *Field Theory, the Renormalization Group, and Critical Phenomena*, 3rd edition, World Scientific (2005).
8. L D Landau and E M Lifshitz, *Statistical Physics*, Pergamon Press (1996).

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a revision class in Easter Term.

String Theory (L24)

Professor N. Dorey

String theory is the quantum theory of interacting one-dimensional extended objects (strings). What makes it so appealing is that it's a quantum theory that includes gravity, thus providing the first tentative steps towards a full quantum theory of gravity. It has become clear that string theory is also much more than this. It is a framework in which to reinterpret problems in quantum field theory in terms of geometry, and acts as a crucible for new ideas in mathematics. This course provides an introduction to String Theory. We begin by generalising the worldline of a particle to the two-dimensional surface swept out by a string. The quantum theory of the embedding of these surfaces in space-time is governed by a two-dimensional quantum field theory and we shall study the simplest example – the bosonic string – in detail. An introduction to relevant ideas in Conformal Field Theory (CFT) will be given. The quantisation of the string will be studied, its spectrum obtained, and the relationship between states on the two dimensional CFT and fields in space-time will be discussed. We will see the necessity of the critical dimension of space-time. The path integral approach to the theory will be discussed. Fadeev-Popov methods will be introduced to deal with the redundancies that appear in the theory. Vertex operators will be introduced and scattering amplitudes will be computed at tree level. Perturbation theory at higher loops and the role played by moduli space of Riemann surfaces will be sketched. The course will focus on closed strings, but time permitting, open strings and the role of D- branes may be discussed. There may also be some discussion of more stringy phenomena such as symmetry enhancement and duality.

Prerequisites

Knowledge of the Quantum Field Theory course in Michaelmas term is assumed. Advanced Quantum Field Theory will complement this course but will not be assumed.

Literature

1. M. Green, J. Schwarz and E. Witten, *Superstring Theory, Vol. 1: Introduction*, CUP 1987.
2. J. Polchinski, *String Theory, Vol. 1: An Introduction to the Bosonic String*, CUP 1998.
3. D. Tong, *Lectures on String Theory*, <https://arxiv.org/abs/0908.0333>

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Supersymmetry (L16)

Ben Allanach

This course provides an introduction to the use of supersymmetry in quantum field theory. Supersymmetry combines commuting and anti-commuting dynamical variables, thus relating fermions to bosons.

Physical and theoretical motivations for supersymmetry are provided. The supersymmetry algebra and representations are then introduced, followed by superfields and superspace. 4-dimensional supersymmetric Lagrangians are then discussed, along with the basics of supersymmetry breaking. The minimal supersymmetric standard model will be introduced.

Three examples sheets and examples classes will complement the course.

Desirable Previous Knowledge

It is necessary to have attended the Quantum Field Theory and the Symmetries, Particles and Fields courses, or be familiar with the material covered in them.

Introductory Reading

1. The first chapters of <http://arxiv.org/abs/hep-ph/0505105>

Reading to complement course material

For more advanced topics later in the course, it will be helpful to have a knowledge of renormalisation, as provided by the Advanced Quantum Field Theory course. It may also be helpful (but not essential) to be familiar with the structure of The Standard Model in order to understand the final lecture on the minimal supersymmetric standard model.

Beware: most of the supersymmetry references contain errors in minus signs.

1. Supersymmetric Gauge Field Theory and String Theory, Bailin and Love, IoP Publishing (1994) has nice explanations of the physics.
2. Introduction to supersymmetry, J.D. Lykken, [hep-th/9612114](#). This introduction is good for extended supersymmetry and more mathematical aspects.
3. Supersymmetry and Supergravity, Wess and Bagger, Princeton University Press (1992). Note that this terse and more mathematical book has the opposite sign of metric to the course.
4. A supersymmetry primer, S.P. Martin, [hep-ph/9709256](#) is detailed on phenomenological aspects, although it has the opposite sign metric to the course.
5. From Spinors to Supersymmetry, Dreiner, Haber and Martin, CUP publishing (2023) is an extensive (over 1000 pages) and definitive supersymmetry text.

Symmetries, Particles, and Fields (M24)

M B Wingate

Lie groups and Lie algebras are important in the construction of quantum field theories which describe interactions between particles. Particle states and quantum fields are described in terms of irreducible representations of the Poincaré group, a Lie group. Gauge theories, which describe interactions in the Standard Model of particle physics, rely on Lie groups.

This course will introduce the mathematics necessary for employing the Lie groups and algebras in high energy physics and beyond. The focus will be on the mathematical concepts and key applications. Mathematical proofs will be given where straightforward or useful for a deeper understanding; however, proof of some facts will remain beyond the scope of this course.

After preliminary remarks, we introduce Lie matrix groups, which depend on continuous parameters. Differentially, these act as a Lie algebra. The exponential map connects the Lie algebra to the Lie group. We then introduce group and algebra representations as linear maps on vector spaces.

The group of rotations in 3-dimensional space, $SO(3)$, is examined, along with $SU(2)$ and the connection to angular momentum states in quantum theory. Representations of these are covered and used later in the course. The relativistic symmetries (Lorentz group and Poincaré group in four dimensions) are studied from the point of view of their group elements and Lie algebras. We shall cover irreducible induced representations of the Poincaré group, which are used to describe one-particle states and fields.

Analysis of simple Lie algebras and their finite representations comes from mapping them to a geometrical picture involving roots and weights via the Cartan matrix. An overview of the results of the Cartan classification of simple Lie algebras is included.

An application in terms of representations of a global $SU(3)_F$ flavour symmetry explains some features of the spectrum of hadronic particles. Further properties of the spectrum lead one to introduce an additional local $SU(3)_c$ colour symmetry leading a particular gauge theory called quantum chromodynamics. We shall close by covering gauge theory for general Lie groups and different representations of matter.

Prerequisites

Linear algebra, including direct sums and tensor products of vector spaces. Special relativity and quantum theory, including angular momentum theory and Pauli spin matrices. Brief reminders of some definitions are given; however familiarity with undergraduate quantum mechanics will be important for fully appreciating some of the physical applications of the mathematics in this course.

Literature

1. For a more basic introduction to the use of groups in physics, *Group Theory in a Nutshell for Physicists*, A. Zee (Princeton University Press, 2016).
2. *Lie Algebras in Particle Physics* by H. Georgi (Westview Press, 1999) gives an introduction aimed at physicists.
3. *Groups, Representations, and Physics*, H. F. Jones (Taylor and Francis, 1998). The first half covers discrete groups; later chapters cover some material in this course.

4. *Lie Groups, Lie Algebras, and Representations: An Elementary Introduction*, B. C. Hall (Springer, 2015). A more mathematical text, useful reference for detail beyond this course.
5. *Symmetries and Group Theory in Particle Physics*, G. Costa and G. Fogli (Springer, 2012) has nice geometrical aspects.

Additional support

Four Examples Sheets will be provided and four associated Examples Classes will be given (the final one in Lent term). There will be a Revision Class in the Easter Term.

14 Probability

Advanced Probability (M24)

Professor J. Miller

The aim of the course is to introduce students to advanced topics in modern probability theory. The emphasis is on tools required in the rigorous analysis of stochastic processes, such as Brownian motion, and in applications where probability theory plays an important role.

- **Review of measure and integration:** sigma-algebras, measures and filtrations; integrals and expectation; convergence theorems; product measures, independence and Fubini's theorem.
- **Conditional expectation:** Discrete case, Gaussian case, conditional density functions; existence and uniqueness; basic properties.
- **Martingales:** Martingales and submartingales in discrete time; optional stopping; Doob's inequalities, upcrossings, martingale convergence theorems; applications of martingale techniques.
- **Stochastic processes in continuous time:** Kolmogorov's criterion, regularization of paths; martingales in continuous time.
- **Weak convergence:** Definitions and characterizations; convergence in distribution, tightness, Prokhorov's theorem; characteristic functions, Lévy's continuity theorem.
- **Sums of independent random variables:** Strong laws of large numbers; central limit theorem; Cramér's theory of large deviations.

- **Brownian motion:** Wiener's existence theorem, scaling and symmetry properties; martingales associated with Brownian motion, the strong Markov property, hitting times; properties of sample paths, recurrence and transience; Brownian motion and the Dirichlet problem; Donsker's invariance principle.
- **Poisson random measures:** Construction and properties; integrals.
- **Lévy processes:** Lévy-Khinchin theorem.

Pre-requisites

A basic familiarity with measure theory and the measure-theoretic formulation of probability theory is very helpful. These foundational topics will be reviewed at the beginning of the course, but students unfamiliar with them are expected to consult the literature (for instance, Williams' book) to strengthen their understanding.

Literature

- Lecture notes online
- D. Applebaum, Lévy processes (2nd ed.), Cambridge University Press 2009.
- R. Durrett, Probability: Theory and Examples (4th ed.), CUP 2010.
- O. Kallenberg, Foundations of Modern Probability, Springer-Verlag, 1997.
- D. Williams, Probability with martingales, CUP 1991.

Additional support

Four example sheets will be provided along with supervisions or examples classes. There will be a revision class in Easter term.

Mixing times of Markov chains (M24)

Andjela Šarković

An ergodic Markov chain is known to converge to its equilibrium distribution as time goes to infinity. But how long should one wait until the distribution of the chain is close to the invariant one? How many times should one shuffle a deck of cards until the order becomes uniform? This question lies at the heart

of the modern theory of mixing times for Markov chains. The classical theory of Markov chains studied fixed chains, focusing on large-time asymptotics of their distribution. Recently, the need to analyse large spaces has increased, shifting the focus to studying asymptotics of the *mixing time* (the first time distribution of the chain gets close enough to invariant one) as the size of the state space tends to infinity.

In this course, we will develop the basic theory and some of the main techniques and tools from probability and spectral theory used to estimate mixing times. We will apply them to study the mixing time of several chains of interest. We shall also discuss the *cutoff phenomenon*, first discovered by Diaconis in the context of card shuffling, which says that a Markov chain converges to equilibrium abruptly. This phenomenon seems to be widespread but it remains a challenging question to obtain criteria for cutoff for general classes of chains.

Prerequisites

This course assumes almost no background, except for prior exposure to Markov chains at an elementary level. Parallel attendance of the first part of Advanced Probability, covering martingales, is encouraged but not necessary.

Literature

1. D. Levin and Y. Peres and E. Wilmer *Markov chains and Mixing Times*. American Mathematical Society, 2008.
2. D. Aldous and J. Fill, Reversible Markov Chains and Random Walks on Graphs. book in preparation available online at <https://www.stat.berkeley.edu/~aldous/RWG/book.html>.
3. R. Montenegro and P. Tetali, *Mathematical aspects of mixing times in Markov chains*. *Foundations and Trends in Theoretical Computer Science: Vol. 1: No. 3*, pp 237-354, 2006.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Random structures in finite-dimensional space (L24)

Wendelin Werner

As opposed to the cases where randomness is naturally structured/created and read/collected along a one-dimensional time-line (such as for Markov chains, martingales, or stochastic calculus), the present course will deal with random structures defined in d -dimensional space (discrete, such as $\mathbf{Z}^d$ or other lattices, or in the continuum such as $\mathbf{R}^d$) for $d \geq 2$. In the discrete cases, these will typically be random functions defined on subsets of $\mathbf{Z}^d$, with values in $\{0, 1\}$ or $\mathbf{R}$.

Examples include percolation, the Ising model, uniform spanning trees and forests, the Gaussian Free Field and others. Most of these models are motivated from physics. They are all simple to define, yet they often have very rich behaviour, and a number of fundamental and easy-to-state questions remain open.

This course will present a selection of ideas and results on this topic. More specifically, we first plan to focus on discrete models defined on lattices and discuss:

- The phase transition for percolation and for the Ising model.
- The asymptotic conformal invariance of critical percolation on the triangular planar lattice (“Smirnov’s Theorem”).
- Wilson’s algorithm to construct uniform spanning trees.
- Some properties of the discrete Gaussian Free Field.

In the final part of the course, we will discuss some features of random structures defined in the continuum:

- The definition of the Gaussian Free Field as a random generalized function and some of its properties.
- How conformal invariance enables to characterize and construct natural random planar fractal structures.

Prerequisites

This course assumes familiarity with probability, measure theory, and analysis at Part II level, as well as the content of the Advanced Probability course. It is probably useful but not absolutely necessary to follow the stochastic calculus course in parallel.

Literature

Lecture notes will be provided. Some additional helpful references are:

1. H. Duminil-Copin, *Introduction to percolation theory*. (available here: <https://www.unige.ch/~duminil/publi/2017percolation.pdf>)
2. G. Grimmett, *Probability on Graphs*, Cambridge University Press (available here: <http://www.statslab.cam.ac.uk/~grg/books/pgs.html>)
3. W. Werner and E. Powell, Lecture notes on the Gaussian Free Field, Cours Spécialisés S.M.F. (available here: <https://arxiv.org/abs/2004.04720>)

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

15 Quantum Computation, Information and Foundations

Quantum Entanglement in Many-body Physics (LT16)

F. Verstraete

The theory of quantum entanglement provides a new language for describing correlations in quantum many-body systems, the central problem in theoretical physics. One of the big challenges is to develop the grammar and semantics of that new language. This course will provide an introduction to this novel but striving field of research which spans a wide array of fields such as quantum computation, condensed matter physics, quantum field theory and quantum chemistry.

The course will cover a selection of topics including:

- Introduction to the theory of entanglement
- The relevant physical corner of Hilbert space
- Lieb-Robinson bounds and the spreading of entanglement
- Area laws of the entanglement entropy

- Quantum tensor networks
- Entanglement-based Variational methods for strongly correlated systems
- Topological quantum order and entanglement patterns
- Notions of entanglement in condensed matter physics, quantum field theory and quantum chemistry.

Prerequisites

It will be assumed that you have taken an advanced quantum mechanics course, similar to Part II Principles of Quantum Mechanics. Familiarity with quantum computation (e.g., from Part II and Part III Quantum Information and Computation) will be beneficial.

Literature

1. B Zeng, X Chen, DL Zhou, XG Wen, *Quantum information meets quantum matter*, Springer.
2. R. Horodecki, P. Horodecki, M. Horodecki, K. Horodecki, *Quantum entanglement*, Reviews of modern physics, 17;81(2):865 (2009).
3. M.B. Hastings, *Locality in Quantum Systems*, arXiv:1008.5137.
4. J.I. Cirac, D. Perez-Garcia, N. Schuch, F. Verstraete, *Matrix product states and projected entangled pair states: Concepts, symmetries, theorems*, Reviews of Modern Physics 93 (4), 045003 (2021).

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Information Theory (L16)

Dr. Ángela Capel Cuevas

Quantum Information Theory lies at the intersection of Mathematics, Physics and Computer Science. It was born out of Classical Information Theory, which is the mathematical theory of acquisition, storage, transmission and processing of information.

This course will provide an introduction to how these tasks can be accomplished using quantum-mechanical systems. The underlying quantum mechanics leads to some distinctively new features which have no classical counterparts. These new features can be exploited both to improve the performance of certain information-processing tasks and to accomplish tasks which are impossible or intractable in the classical realm.

The course will cover a selection of topics including:

- Basic concepts in quantum theory and quantum information: pure and mixed states, density matrix, the Schmidt decomposition, quantum operations.
- Study of open systems: Generalization of the postulates of quantum mechanics, generalized measurements, POVMs, Kraus representation, the Choi-Jamilkowski isomorphism, etc.
- Quantum channels: Application to coherent communication and purification, quantum data compression limit, and random coding arguments. Quantum channel capacities and Holevo bound on the accessible information.
- Trace distance, fidelity and entropy measures. Quantum relative entropy and entropy inequalities. Applications to hypothesis testing, monotonicity and recoverability.

Prerequisites

For this course, familiarity with an undergraduate level quantum mechanics course, such as Part II Quantum Mechanics or Part II Quantum Information and Computation, is highly beneficial. In particular, knowledge of basic quantum mechanics will be assumed.

Elementary knowledge of probability theory, vector spaces and linear algebra will be very useful.

Literature

1. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*; Cambridge University Press, 2000.
2. M. M. Wilde, *From Classical to Quantum Shannon Theory*; Cambridge University Press, 2017, <http://arxiv.org/abs/1106.1445>.
3. J. Watrous, *The theory of quantum information*; Cambridge University Press, 2018, <https://cs.uwaterloo.ca/~watrous/TQI/>.

4. E. Carlen, *Trace inequalities and quantum entropy: An introductory course*; lecture notes, 2009, <http://www.ueltschi.org/AZschool/notes/EricCarlen.pdf>.
5. M. M. Wolf, *Quantum Channels and Operations. Guided Tour*; lecture notes, 2012, <https://mediatum.ub.tum.de/doc/1701036/1701036.pdf>.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Information, Foundations and Gravity (L16)

Prof. Adrian Kent

The course gives an introduction to foundational questions in quantum theory and quantum information, with a focus on the relationship between quantum theory and gravity. We will examine arguments that the gravitational field must necessarily be quantized, together with counter-arguments, review old ideas of semi-classical theories of gravity and their problems. Then we will discuss recently proposed mass interferometry experiments that aim to use the properties of quantum information to test whether gravity is indeed mediated by quantum particle exchange.

Topics covered in the course will include:

- Basic concepts in quantum theory and quantum information: pure and mixed states, density matrix, the Schmidt decomposition, entanglement, the no-cloning theorem.
- Relationship between quantum theory and relativity: the no-signalling principle; quantum state summoning and the no-summoning theorem; possible and impossible summoning tasks.
- Bell's theorem and Bell nonlocality.
- Entanglement measures and entanglement witnesses.
- Brief review of Everettian quantum theory and collapse-based versions of quantum theory.
- Basic notions of semi-classical gravity; the Page-Geilker experiment.
- Quantum theory and gravity: the Colella, Overhauser and Werner experiment; the Eppley-Hannah argument.

- Non-linear extensions of quantum theory; refutation of the Eppley-Hannah argument
- The proposed experiments of Bose et al. and Marletto-Vedral (BMV) testing the generation of entanglement between separated mesoscopic quantum systems in superposition.
- Analysis of the implications of possible outcomes of BMV experiments.

If time permits we may also discuss other recent experimental proposals.

Prerequisites

Familiarity with undergraduate level quantum mechanics is essential. Familiarity with a first course in quantum information theory, such as the Cambridge Part II Quantum Information and Computation course, would be highly advantageous.

Some relevant papers

1. Don N Page and CD Geilker. Indirect evidence for quantum gravity. *Physical Review Letters*, 47(14):979, 1981.
2. Adrian Kent, Simple Refutation of the Eppley-Hannah argument. *Classical and Quantum Gravity* 35 (24) 245008 (2018)
3. Sougato Bose, Anupam Mazumdar, Gavin W. Morley, Hendrik Ulbricht, Marko Toros, Mauro Paternostro, Andrew A Geraci, Peter F Barker, MS Kim, and Gerard Milburn. Spin entanglement witness for quantum gravity. *Physical Review Letters*, 119(24):240401, 2017.
4. Chiara Marletto and Vlatko Vedral. Gravitationally induced entanglement between two massive particles is sufficient evidence of quantum effects in gravity. *Physical Review Letters*, 119(24):240402, 2017.
5. Daniel Carney, Philip CE Stamp, and Jacob M Taylor. Tabletop experiments for quantum gravity: a user's manual. *Classical and Quantum Gravity*, 36(3):034001, 2019.

Additional support

Three examples sheets will be provided and three associated one-hour examples classes will be given. There will be a one-hour revision class in the Easter Term.

Topological Quantum Matter (L16)

Prof. B. Béri

Topologically ordered systems display quantum mechanical features that are insensitive to the local details of the system but are robustly set by its topology. Such features include “anyonic” particles beyond the boson/fermion dichotomy, a “holographic” correspondence between the system’s bulk and its boundary, or a topology-dependent ground-state degeneracy. These remarkable properties, fundamentally linked to certain gauge theories, have intriguing potential applications, including the use of topological ground-state degeneracy in quantum error correction or employing so-called non-Abelian anyons in robust schemes for quantum computation.

This course is an introduction to topological order and its quantum computing applications. We begin with some basics of conventional (i.e., symmetry-breaking) order, illustrated using the quantum Ising chain. This will serve as a reference with which to compare topological order.

We then embark on our study of topological order. Concepts we shall aim to discuss include: ground-state degeneracy, anyons, the robustness of topological order, bulk-boundary correspondence, Majorana zero modes, non-Abelian anyons. We shall use the toric code and the Kitaev chain – a boson-fermion dual of the quantum Ising chain – as illustrative examples.

Of the quantum computing applications of topological order, we shall aim to discuss quantum error correction with the toric code, and topological quantum computation using non-Abelian anyons, in particular those based on Majorana zero modes.

Some of our discussions will use second quantisation, the basics of path integrals, and of quantum error correction. These shall be reviewed or introduced at the appropriate points of the course.

Prerequisites

It will be assumed that you have taken an advanced quantum mechanics course, similar to Part II Principles of Quantum Mechanics. Familiarity with the basics of band theory (e.g., from Part II Applications of Quantum Mechanics) and of quantum computation (e.g., from Part II Quantum Information and Computation) will also be useful.

Literature

1. S. H. Simon, *Topological Quantum: Lecture Notes and Proto-Book*, available at Steve Simon’s <http://www-thphys.physics.ox.ac.uk/people/SteveSimon/website>. Chapter 3 and Parts II, V are especially useful for this course.

2. S. Sachdev, *Quantum Phase Transitions*, CUP (2011), parts of chapters 1, 5, 10.
3. B. A. Bernevig with T. L. Hughes, *Topological Insulators and Topological Superconductors*, Princeton University Press (2013), chapter 16.
4. S. Das Sarma, M. Freedman, C. Nayak, *Majorana zero modes and topological quantum computation* <https://www.nature.com/articles/npjqi20151npj> Quantum Inf. **1**, 15001, (2015).
5. A. Altland and B. Simons, *Condensed Matter Field Theory*, CUP (2010), chapters 2, 3 for background on second quantisation and path integrals.

Additional support

Three example sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Quantum Computation (M16)

Sathyawageeswar Subramanian

Quantum mechanical processes can be exploited to design new modes of information processing that are beyond the capabilities of any classical computer. This leads to remarkable new kinds of algorithms (so-called quantum algorithms) that can offer a dramatically increased efficiency for the execution of some computational tasks. In addition to such potential practical benefits, the study of quantum computation has great theoretical interest, combining concepts from computational complexity theory and quantum physics to provide striking fundamental insights into the nature of both disciplines.

This course will be a ‘second’ course in the subject, following the Part II course Quantum Information and Computation (see below in prerequisites) that was introduced in 2017-2018.

In this course we will aim to cover the following topics:

- The hidden subgroup problem and quantum Fourier transform on a group;
- The quantum phase estimation algorithm and applications;
- Amplitude amplification and applications;
- Quantum simulation of time evolution under local hamiltonians.

Depending on the availability of time, we will then cover a selection of the following topics:

- The Harrow-Hassidim-Lloyd algorithm for solving a system of linear equations;
- The Quantum Singular Value Transformations toolkit and applications;
- Introduction to Clifford operations and Stabilizer formalism; classical simulation properties of Clifford circuits (Gottesman-Knill theorem).

Prerequisites

This course will assume a prior basic acquaintance with quantum computing, to the extent presented in the course notes for the Cambridge Part II course Quantum Information and Computation available at <http://www.qi.damtp.cam.ac.uk/part-iii-quantum-computation>

In particular you should be familiar with Dirac notation and principles of quantum mechanics, as presented in the course notes sections 2.1, 2.2 and 2.3. You should also have a basic acquaintance with quantum computation to the extent of the second half of the course notes, pages 47 to 88 (Chapters 6-11). It would be desirable for you to look through this material before the start of the course.

Literature

1. Michael Nielsen and Isaac Chuang, *Quantum Computation and Quantum Information*. CUP, 2000.
2. John Preskill, <http://theory.caltech.edu/~preskill/ph229/> *Lecture Notes on Quantum Information Theory* (especially Chapter 6).
3. Ronald de Wolf, <https://arxiv.org/pdf/1907.09415.pdf> *Quantum Computing: Lecture Notes*.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

16 Relativity and Cosmology

Cosmology (M24)

Blake Sherwin

This course discusses what we know (and don't know) about the evolution of our universe, from inflationary quantum fluctuations in the first fraction of a second

to the formation of galaxies and structures today. It also seeks to illustrate how cosmology can serve as a uniquely powerful laboratory for understanding fundamental physics.

In detail, the course will cover the following topics:

- Geometry and dynamics of our Universe
- Inflation
- Thermal history
- Cosmological perturbation theory
- Structure formation
- Cosmic microwave background basics
- Initial conditions from inflation

Pre-requisites

Although the course is intended to be as self contained as possible, knowledge of relativity, quantum mechanics and statistical mechanics will be very useful. General relativity and quantum field theory courses may allow a deeper understanding of some of the material covered.

Literature

1. D. Baumann, *Cosmology*
2. S. Dodelson, *Modern Cosmology*
3. E. Kolb and M. Turner, *The Early Universe*
4. S. Weinberg, *Cosmology*

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Black Holes (L24)

J. E. Santos

A black hole is a region of spacetime that is causally disconnected from the rest of the Universe. These objects appear to be pervasive in Nature, and their properties have direct implications for advances in gravitational wave astronomy. Besides being astrophysically relevant, black holes also play a fundamental role in quantum theory and are a natural arena to study and test any consistent quantum theory of gravity.

Topics to be discussed will include:

- Upper mass limit for relativistic stars. Schwarzschild black hole. Gravitational collapse.
- The initial value problem, strong cosmic censorship.
- Causal structure, null geodesic congruences, Penrose singularity theorem.
- Penrose diagrams, asymptotic flatness, weak cosmic censorship.
- Reissner-Nordström and Kerr black holes.
- Energy, angular momentum and charge in curved spacetime.
- The laws of black hole mechanics. The analogy with laws of thermodynamics.
- Quantum field theory in curved spacetime. The Hawking effect and its implications.

Prerequisites

Familiarity with the Michaelmas term courses *General Relativity* and *Quantum Field Theory* is essential.

Literature

1. H. S. Reall, *Part 3 Black Holes: lecture notes* available at www.damtp.cam.ac.uk/user/hsr1000.
2. R. M. Wald, *General relativity*, University of Chicago Press, 1984.
3. S. W. Hawking and G.F.R. Ellis, *The large scale structure of space-time*, Cambridge University Press, 1973.
4. V. P. Frolov and I.D. Novikov, *Black holes physics*, Kluwer, 1998.

5. N. D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. R. M. Wald, *Quantum field theory in curved spacetime and black hole thermodynamics*, University of Chicago Press, 1994.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Canonical Gravity (L16)

Non-Examinable (Part III Level)

Professor M.J. Perry

General relativity is often presented in a way that unifies space and time into a complete whole, spacetime. This approach is encapsulated in the Einstein equation

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi GT_{ab} \quad (2)$$

where there is no separation into space and time. Nevertheless, physics is often formulated in terms of the initial value problem where the initial configuration of spatial geometry, fields and particles together with their time derivatives are specified. One can then compute the time evolution of the system. We apply these ideas to general relativity. Firstly, we need to understand the general picture of constrained Hamiltonian systems and why the idea of constraints is related to gauge degrees of freedom. We will practice these ideas for particle mechanics, scalar field theory and then electromagnetism. Finally we apply these methods to general relativity to provide a canonical formulation along the lines first found by Dirac and by Arnowitt, Deser and Misner. More generally, we will then examine the evolution of a spacetime in terms of its trajectories in superspace, the space of all spatial geometries. We then will apply these ideas to a simple cosmological model and subsequently look how these ideas impact the problem of singularities in spacetime and the chaotic behaviour associated with them.

Quantum mechanics is usually presented starting from a Hamiltonian and then developed as a theory in which classical phase space is deformed using the Heisenberg algebra in which classical quantities are replaced by operators. We apply this technique to general relativity and so develop the Wheeler-DeWitt equation. Then we can use this to see how a classical spacetime can emerge from the quantum picture by use of the WKB approximation. Finally, if time permits, we will look at the relationship between the Wheeler-DeWitt equation

and the path integral, what happens when the Wheeler-DeWitt equation is used to examine spacetime singularities, and lastly the role of observers to extract measurements in quantum gravity.

Prerequisites

Basic general relativity. Quantum Mechanics.

Additional support

There will be no examples sheets. An office hour will be arranged but perhaps some kind of discussion session can be organised.

Field theory in cosmology (L24)

Professor E. Pajer

This course discusses applications of classical, statistical and quantum field theory to cosmology. The course comprises three interconnected topics:

- Cosmological inflation and primordial quantum perturbations (QFT in curved spacetime)
- The matter and galaxy distribution in the Large Scale Structure of the Universe (statistical field theory)
- The physics of the Cosmic Microwave Background (classical and statistical field theory)

More specifically, the following topics will be discussed:

- Inflation: A brief review of background cosmology and inflation
- Inflation: Free fields on de Sitter spacetime
- Inflation: Interacting fields, the in-in formalism and correlators in $P(X, \phi)$ theories
- Inflation: Quantizing pure gravity, gravity plus matter, symmetries and phenomenology
- Large Scale Structures: Newtonian dynamics
- Large Scale Structures: Standard perturbation theory

- Large Scale Structures: the effective field theory approach
- Cosmic Microwave background: the Boltzmann equation
- Cosmic Microwave background: anisotropies and the angular power spectrum

Prerequisites

Some familiarity with introductory quantum field theory and general relativity, as for example provided by the respective Michaelmas courses, is highly recommended. Basic knowledge of introductory cosmology is very helpful. Students who did not attend the Michaelmas course on cosmology may still follow this course, and may want to read Section 1.1 and 1.2 of the notes before the course starts.

Literature

The lecture notes for this course can be found on E. Pajer’s website at <http://www.damtp.cam.ac.uk/user/ep551>

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Gauge-Gravity Duality (E16)

Professor A. Wall

Gauge/Gravity duality (also known as AdS/CFT) is an amazing duality that relates theories of quantum gravity (with a negative cosmological constant) to certain quantum field theories living in a smaller dimensional spacetime. This is the most precise known realization of the holographic principle, the idea that all information in the universe is encoded somehow at the boundary of the universe. These lectures will describe in detail the “dictionary” used to relate observables on the bulk side to observables on the boundary side.

Topics covered: Anti-de Sitter spacetime; conformal field theory; wave equations in AdS, and their relationship to CFT operators and sources; the duality between black holes and thermal states; holographic entanglement entropy. If time permits: recent developments concerning bulk reconstruction, and the black hole information puzzle.

Prerequisites

Required: General Relativity, Black Holes, Advanced Quantum Field Theory

Helpful: Some basic aspects of quantum information theory and conformal symmetry will play an important role in this course, but the relevant aspects will be reviewed in a self-contained manner.

Not Required: String Theory, Supersymmetry. Although most of the specific known examples of AdS/CFT come from superstring theories, these aspects will not be emphasized in these lectures. (There will be one or two lectures on Maldacena's original derivation of black hole entropy in string theory, but this topic will not be examinable.)

Literature

Lecture notes for the course will be provided in pdf format. Additional information about the course, including a list of relevant review articles, is available here:

<http://www.damtp.cam.ac.uk/user/aw846/AdSCFT.html>

Additional support

Two examples sheets will be provided, and two associated examples classes will be given. There will be a one-hour revision class in the Easter Term, subsequent to the lectures.

General Relativity (M24)

Claude Warnick

General Relativity is the theory of space-time and gravitation proposed by Einstein in 1915. It remains at the centre of theoretical physics research, with applications ranging from astrophysics to string theory. This course will introduce the theory using a modern, geometric, approach.

This is a second course on General Relativity, albeit one that could just about be followed without prior exposure to the subject. The first half of the course will give an introduction to differential geometry, the mathematics that underlies curved spacetime. The second half of the course will discuss the physics of gravity.

Prerequisites

Familiarity with Newtonian gravity, special relativity, finite-dimensional vector spaces and the Euler-Lagrange equations is essential. Knowledge of the relativistic formulation of Maxwell's equations is highly desirable.

Most students attending this course have already taken an introductory course in General Relativity (e.g. the Part II course). If you have not studied GR before then you should read an introductory book (e.g. Hartle or Rindler) before attending this course. Certain topics usually covered in a first course, e.g. the solar system tests of GR, will not be covered in this course.

Literature

1. *Gravity: An introduction to Einstein's General Relativity*, J.B. Hartle, Addison-Wesley, 2003.
2. *Relativity: Special, General, and Cosmological*, 2nd ed., W. Rindler, OUP, 2006.
3. *General Relativity*, R.M. Wald, Chicago UP, 1984.
4. *Gravitation*, C.W. Misner, K.S. Thorne and J.A. Wheeler, W.H. Freeman, 1973.
5. *Spacetime and geometry: an introduction to General Relativity*, S.M. Carroll, Addison-Wesley, 2004.
6. *Advanced General Relativity*, J.M. Stewart, CUP, 1993.
7. *Gravitation and Cosmology*, S. Weinberg, Wiley, 1972.

The course will follow a similar approach to Wald and Misner, Thorne and Wheeler. Carroll's book is a very readable introduction at about the same level. Stewart's book is based on a previous version of this course. Weinberg's book gives a good discussion of the Equivalence Principle.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Gravitational Waves and Numerical Relativity (E16)

Ulrich Sperhake

The direct detection of a gravitational-wave signal from the black-hole binary merger GW150914, detected by LIGO in September 2015, has promoted gravitational-wave physics to a rich field combining theory with observation. With about 100 further events observed over the ensuing years, a vast range of physical and astrophysical phenomena are now accessible to a qualitatively new channel of observations. New gravitational-wave observatories on Earth and in Space will greatly enhance the sensitivity and frequency range in future decades. Gravitational-wave observations heavily rely on accurate theoretical predictions that require source modelling in the framework of Einstein's theory of relativity. Due to the complexity of Einstein's field equations and the highly dynamical character of the physical systems under consideration, the calculation of such templates is a formidable challenge that requires the combination of analytic and numerical techniques. In this lecture course, we will discuss the theoretical foundations of gravitational waves (at linear and non-linear level), formulations of the Einstein field equations amenable for a numerical treatment, initial data, gauge conditions and diagnostic tools.

More specifically, the course will cover the following topics.

- Characteristic surfaces and the classification of partial differential equations.
- The structure of the Einstein field equations.
- Gravitational waves in the characteristic (aka *Bondi-Sachs*) formalism of general relativity.
- The space-time split of the Einstein equations and well-posed formulations of the field equations.
- The construction of constraint satisfying initial data representing realistic snapshots of systems of multiple black holes.
- Singularity avoiding coordinate choices employed in contemporary numerical relativity simulations.
- Diagnostic tools to extract the gravitational-wave signal from numerically generated solutions.

Prerequisites

This course assumes knowledge of Part III General Relativity. Some knowledge of Part III Black Holes will be helpful but not essential for following this course.

Literature

1. T. W. Baumgarte and S. L. Shapiro *Numerical Relativity*. Cambridge University Press, 2010.
2. T. W. Baumgarte and S. L. Shapiro *Numerical Relativity: Starting from Scratch*. Cambridge University Press, 2021.
3. M. Alcubierre, *Introduction to 3+1 Numerical Relativity*, Oxford University Press, 2008.
4. E.ourgoulhon, *3+1 Formalism and Bases of Numerical Relativity*, Springer, New York, 2012. See also <https://arxiv.org/abs/gr-qc/0703035>.
5. M. Maggiore, *Gravitational Waves*. Vols. 1 (Theory and Experiments) and 2 (Astrophysics and Cosmology), Oxford University Press, 2018.
6. R. d’Inverno, *Introducing Einstein’s Relativity*, Oxford Clarendon Press, 1992.

Additional support

Three examples sheets will be provided and three associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Solitons, Instantons, and Geometry (L16)

Professor Maciej Dunajski

Solitons are solutions of classical field equations with particle-like properties. They are localised in space, have finite energy and are stable against decay into radiation. The stability usually has a topological explanation. Instantons are non-singular solutions of classical field equations in Euclidean space whose action is finite. Instantons describe quantum-mechanical tunnelling between the classical minima of the potential in Minkowski space. Solitons and instantons are globally characterised by some topological charge, related to the behaviour of solutions at spatial infinity. Its understanding requires ideas from topology, and geometry.

The course will cover a selection of topics including:

- Solitons (kinks in 1D, vortices in 2D, monopoles in 3D)
- Yang Mills instantons, and their integrability.
- Fibre bundles and instantons: connection, curvature and characteristic classes.

Prerequisites

Basic General Relativity (Part II level) or some introductory Differential Geometry course (e.g. Part II differential geometry) is essential. Part III General Relativity is desirable.

Literature

1. Dunajski, M. Solitons, Instantons, and Twistors. (2nd Edition) Oxford Graduate Texts in Mathematics **31**, Oxford University Press, 2024.
2. Eguchi, T., Gilkey, P. and Hanson. A. J. Physics Reports **66** (1980) 213-393
3. Manton, N. and Sutcliffe, P. Topological Solitons. Cambridge University Press, 2004.

Additional support

Three examples sheets will be provided and three examples classes will be given. There will be a one-hour revision class in the Easter Term.

17 Statistics

Causal Inference (L16)

Dr. J. Shi

From its onset, modern statistics engages in the problem of inferring causality from data. A common belief is that causal inference is only possible using randomized experiments, but recent developments in statistics and related fields have shown that this view is too simplistic. We now have a much better understanding of the assumptions and methodologies that enable causal inference from observational data. This course aims to cover the most fundamental ideas in causal inference, a vibrant research area where statistical theory meets scientific practice.

1. Motivation and basic methods:

- Principles of causal inference: motivations; historical perspectives; basic concepts.
- Randomized experiments: randomization tests, regression adjustment and its asymptotic inference.

- Path analysis and linear structural equation models (SEMs).

2. Causal graphical models and identification:

- Directed acyclic graphical (DAG) models and extensions: Markov properties, d-separation, structure discovery.
- Counterfactual causal models: nonparametric SEMs; single-world intervention graphs; g-computation formula.
- Causal identification: back-door criterion, front-door criterion; fixing operator and potential outcomes calculus; examples.

3. Design and statistical methods:

- Observed confounders: matching, randomization inference, influence functions and semiparametric inference.
- Instrumental variables (IV): core IV assumptions; generalised method of moments; principal stratification.
- Additional selected topics (time permitting): mediation analysis; regression discontinuity design; difference in differences and negative control methods; longitudinal data and time-varying treatments; meta-analysis and evidence synthesis.

Pre-requisites

This course assumes a good understanding of undergraduate-level probability and statistics. Familiarity with asymptotic statistics and applied statistics may be useful.

Literature

1. Hernán M. A. and Robins, J. M. (2020) *Causal Inference: What If*. Chapman & Hall.
2. Imbens, G. W. and Rubin, D. B. (2015) *Causal Inference in Statistics, Social, and Biomedical Sciences*. Cambridge University Press.
3. Lauritzen, S. L. (1996). *Graphical Models*. Clarendon Press.
4. Angrist, J. D. and Pischke, J. S. (2008) *Mostly Harmless Econometrics: An Empiricist's Companion*. Princeton University Press.

Additional support

Lecture notes will be provided. Three examples sheets will be provided and three associated examples classes will be given. There will also be a revision class in the Easter Term.

Modern Statistical Methods (M24)

Sergio Bacallado

The remarkable development of computing power and other technology now allows scientists and businesses to routinely collect datasets of immense size and complexity. Most classical statistical methods were designed for situations with many observations and a few, carefully chosen variables. However, we now often gather data where we have huge numbers of variables, in an attempt to capture as much information as we can about anything which might conceivably have an influence on the phenomenon of interest. This dramatic increase in the number variables makes modern datasets strikingly different, as well-established traditional methods perform either very poorly, or often do not work at all.

Developing methods that are able to extract meaningful information from these large and challenging datasets has recently been an area of intense research in statistics, machine learning and computer science. In this course, we will study some of the methods that have been developed to study such datasets. We aim to cover some of the following topics.

- Kernel machines: Ridge regression, the kernel trick, reproducing kernel Hilbert spaces, kernel ridge regression, the support vector machine, the hashing trick.
- Penalised regression: Model selection, the Lasso, variants of the Lasso.
- Conditional independence and high-dimensional inference: Conditional independence graphs, the graphical Lasso, conditional independence testing, the debiased Lasso.
- Multiple testing: the closed testing procedure and the Benjamini–Hochberg procedure.

Prerequisites

Basic knowledge of statistics, probability, linear algebra and real analysis. Some background in optimisation would be helpful but is not essential.

Literature

1. T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning*. 2nd edition. Springer, 2001.
2. P. Bühlmann, S. van de Geer, *Statistics for High-Dimensional Data*. Springer, 2011.

3. T. Hastie, R. Tibshirani and M. Wainwright, *Statistical learning with sparsity: the lasso and generalizations*. CRC Press, 2015.
4. C. Giraud, *Introduction to High-Dimensional Statistics*. CRC Press, 2014.
5. M. Wainwright, *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge University Press, 2019.

Additional support

Four examples sheets will be provided and four associated examples classes will be given. There will be a one-hour revision class in the Easter Term.

Statistical Learning in Practice (L24)

Mr. E. Young

Statistical learning aims to build statistical models for learning relationships and structures from data, mainly with the goal of predicting future outcomes. It blends classical statistics with techniques from modern machine learning with the aim of providing concrete algorithms, for example for regression or classification tasks. This course consists of 12 lectures and 12 practical classes. We will get to know some of the most successful and widely used statistical methodologies in modern applications, with a focus on practical aspects and concrete problems. The practical classes will cover an introduction to the programming language R, exploratory data analysis and the implementation of the statistical methods discussed in the lectures. We aim to cover a selection of the following topics:

- Generalised linear models for regression and classification
- Model selection and regularisation
- Mixed effects models
- Time series
- Principal component analysis
- Linear discriminant analysis and support vector machines
- Nearest neighbours classifiers and random forests
- Introduction to deep learning

Prerequisites

Elementary probability theory, maximum likelihood estimation, hypothesis tests and confidence intervals, linear models. Previous experience with R is not essential and can be obtained during the course (only a good working knowledge of R will be necessary, as provided, for instance, in the lab sessions of the first reference below).

Literature

1. G. James, D. Witten, T. Hastie and R. Tibshirani. *An Introduction to Statistical Learning (with Applications in R)*. Springer, 2013. Also available at <https://www.statlearning.com>.
2. T. Hastie, R. Tibshirani and J. Friedman. *The Elements of Statistical Learning*. Springer, Second Edition.
3. A.J. Dobson and A. Barnett. *An Introduction to Generalized Linear Models*. Third edition. Chapman & Hall/CRC, 2008.
4. P.J. Brockwell and R.A. Davis. *Introduction to time series and forecasting*. Springer, 1996.

Additional support

This course includes practical classes, in which statistical methods are introduced in a practical context and students carry out analysis of datasets using R. In the practical classes, the students have the opportunity to discuss statistical questions with the lecturers. Four example sheets will be provided and there will be four associated example classes (the last one probably to be held in Easter term). There will be a revision class in the Easter Term.

Statistics in Medicine (3 units)

Statistics in Medical Practice (M12)

Lecturers from the MRC Biostatistics Unit

This part of the course includes three modules covering a range of statistical methods and their application in three areas of biostatistics. It is firmly focused on how methods are used in application, rather than rigorous derivations of mathematical results.

Note that this part is examined together with Analysis of Survival Data (Lent), as a single 24-lecture course.

A. Causal Inference in Practice [4 Lectures] (S. Burgess)

It is well known that “correlation is not causation”. But how then do you assess causal claims? Is it possible to show that X is a cause of Y? What does it even mean to say that X is a cause of Y? In this module, we introduce definitions of causal concepts, starting with the work of Rubin, Pearl, and Robins, and discuss practical approaches for assessing causal claims from observational data.

B. Multi-State Models in Health [4 Lectures] (C. Jackson, D. De Angelis, P. Birrell)

(i) Continuous-time multi-state models, applied to modelling the incidence and progression of non-communicable diseases. Defining quantities of applied interest from Markov models. Constructing likelihoods for parametric models given various forms of individual-level data. Interpreting and comparing estimates from fitted models.

(ii) Multi-state modelling to estimate incidence of infectious diseases from population-level data. Backcalculation methods for the estimation of incidence of disease with long incubation periods. Dynamic modelling of infectious disease transmission.

C. Design and Analysis of Clinical Trials [4 Lectures] (M. Law, D. Couturier, S. Villar, D. Robertson)

Introductory concepts in clinical trial design and analysis, including sample size estimation. Early phase dose-finding clinical trials using the continual reassessment method and Bayesian parametric models. Types of randomisation procedures. Optimal response-adaptive procedures. Adaptive and multi-stage designs (including group-sequential designs). Treatment effect estimation following a group-sequential trial.

Statistics in Medicine (3 units)

Analysis of Survival Data (L12)

P. Treasure

This part of the course includes three modules covering the fundamentals of time-to-event analysis with applications to cancer survival.

Note that this part is examined together with Statistics in Medical Practice (Michaelmas) as a single 24-lecture course.

D. Time-to-Event Analysis [4 Lectures]

‘Survival analysis’ is generalised to *time-to-event* analysis. The implications of event times which are unknown or in the future (*censored* data)

are discussed. Time-to-event distributions are introduced and their parametric (maximum likelihood) and non-parametric (*Kaplan-Meier*) characterisations are described. Methods for comparing two time-to-event distributions (as in a clinical trial of an active treatment versus a placebo) are derived (*log-rank* test).

E. Modelling Hazard [4 Lectures]

The *hazard* function (instantaneous event rate as a function of time) is defined. It is shown how the hazard function can naturally be used to model the effect of explanatory variables (such as age, gender, treatment, blood pressure, tumour location and size...) on the time-to-event distribution (*proportional hazards* modelling). Model checking procedures are introduced with an emphasis on excess event (*Martingale*) plots.

F. Population Cancer Survival Analysis [4 Lectures]

Analysis of survival data from real-world cancer studies is complicated by patients also being at risk from other causes of death. Methods of dealing with more than one cause of death are presented for the cases (i) the cause of death is known (*competing risk* analysis) and (ii) the cause of death is unknown (*net survival*). The conceptual difficulties inherent in the notion of a cancer survival distribution relevant to a particular calendar year (e.g. 2019) are addressed: *period* survival analysis.

Statistics in Medicine (3 units)

Additional Information

Prerequisites

Undergraduate-level statistics and probability: including analysis and interpretation of data, maximum likelihood estimation, hypothesis testing, basic stochastic processes.

Literature

There are no course books, but relevant medical papers may be made available before some of the lectures for prior reading. Some literature to complement the course material is listed below.

1. Armitage P, Berry G, Matthews JNS, *Statistical Methods in Medical Research*. Wiley-Blackwell, 2001. [A good introductory companion to the whole course]

2. Burgess S, Thompson SG, *Mendelian Randomization: Methods for Using Genetic Variants in Causal Estimation* Chapman and Hall, 2015 [Module A]
3. van den Hout, A, *Multi-State Survival Models for Interval-Censored Data*. Chapman and Hall, 2016 [Module B]
4. Keeling, MJ, & Rohani, P *Modeling Infectious Diseases in Humans and Animals*. Princeton University Press, 2008 [Module B]
5. Senn, S. *Statistical Issues in Drug Development*. Wiley, 2021. [Module C]
6. Jennison C, Turnbull B, *Group Sequential Methods with Applications to Clinical Trials*. Chapman and Hall, 2000. [Module C]
7. Rosenberger, William F., and John M. Lachin. *Randomization in clinical trials: theory and practice*. John Wiley & Sons, 2015. [Module C]
8. Cox DR, Oakes D, *Analysis of Survival Data*. Chapman and Hall, 1984 [Modules D, E, F: the classic text]
9. Collett D, *Modelling Survival Data in Medical Research*. CRC Press, 2023 [Modules D, E, F: modern, applied, supports and extends lectures.]
10. Aalen OO, Borgen Ø, Gjessing HK, *Survival and Event History Analysis: A Process Point of View*. Springer, 2008 [Modules D, E, F: excellent modern approach]

Additional support

[Modules A, B, C] One-hour example classes for each of the three modules, supported by question sheets and solutions, will be given in Michaelmas Term. Additional support can be arranged with individual lecturers.

[Modules D, E, F] A 90-minute example class, supported by question sheets and solutions, will be given in each of the Lent and Easter Terms. A two-hour revision class will be held just before the examination. There will be regular office hours by Zoom.

Topics in Statistical Theory (M16)

Richard J. Samworth

This course will provide an introduction to the theory behind a selection of statistical problems that play a key role in modern statistics. Most undergraduate statistics courses are restricted to the study of parametric models; here we will no longer assume that our distributions belong to finite-dimensional classes and

will instead study fundamental nonparametric problems such the estimation of a distribution function, a density function or a regression function. We will also study minimax lower bounds, which characterise the intrinsic difficulty of a statistical problem, and provide benchmarks against which statistical procedures can be compared.

An outline of the course is as follows:

- An introduction to nonparametric statistics: the basics of empirical process theory, Glivenko–Cantelli theorem, Dvoretzky–Kiefer–Wolfowitz inequality. Concentration inequalities, including Hoeffding, Bennett and Bernstein inequalities.
- Kernel density estimation: Definition, bounds on bias and variance, uniform nonasymptotic bounds on MSE and MISE. Bandwidth selection via least squares cross validation and Lepski’s method, choice of kernel, multivariate density estimation.
- Nonparametric regression: Local polynomial estimation, bounds on weights, bias, variance and MSE. Cubic splines, natural cubic smoothing splines, choice of smoothing parameter.
- Minimax lower bounds: Reduction to testing, f -divergences, Le Cam’s two point lemma, Assouad’s lemma, the data processing inequality, Fano’s lemma; examples.

Pre-requisites

A good background in undergraduate probability theory, elements of linear algebra and real analysis. Measure theory is not necessary but may be helpful; similarly for a preliminary course in mathematical statistics. Though the material in the Modern Statistical Methods course will not be needed here, the two courses complement each other well.

Literature

The lecturer is currently writing a book based on the course, and this should be available (if not published) in time for the course. Some of the material is covered in:

1. S. Boucheron, Lugosi, G. and Massart, P. *Concentration Inequalities: A Nonasymptotic Theory of Independence*. Oxford University Press, 2013.
2. A. Tsybakov, *Introduction to Nonparametric Estimation*. Springer, 2009.
3. M. J. Wainwright. *High-Dimensional Statistics: A Non-Asymptotic Viewpoint*. Cambridge University Press, 2019.

Additional support

Three examples sheets will be provided and associated examples classes will be given. There will be weekly office hours during Michaelmas and a revision class in the Easter Term.