Higher-order uniformity and applications (L12)

Non-Examinable (Graduate Level)

Julia Wolf

Szemerédi's celebrated regularity lemma states that the vertex set of any sufficiently large graph can be decomposed into a bounded number of components such that almost all pairs of components behave roughly like a random bipartite graph. This result has had countless applications across combinatorics and theoretical computer science since its inception in the 1970s.

In this course we will motivate and develop analogous results for 3-uniform hypergraphs, following the approach developed by Gowers in the late 1990s. As we shall see, the formulation of the hypergraph regularity lemma is inevitably more sophisticated, but also yields significant additional power, leading to a proof of the multidimensional Szemerédi theorem, for example.

We will go on to see how this approach can be bootstrapped, using additional ideas of Green and Tao, and Conlon, Fox and Zhao, to prove the Green-Tao theorem, namely that the primes contain arbitrarily long arithmetic progressions.

If time permits, we will connect these ideas to higher-order Fourier analysis, which has its roots in Gowers's work on Szemerédi's theorem and underpinned the original proof of Green and Tao.

Pre-requisites

This course assumes basic familiarity with Szemerédi's regularity lemma for graphs, and later on with the discrete Fourier transform. The Part III course *Probabilistic Combinatorics* offered in the Michaelmas term will be useful.

Literature

1. B.J. Green, *Montréal lecture notes on quadratic Fourier analysis.* Additive Combinatorics (Montréal 2006, ed. Granville et al.), CRM Proceedings vol. 43, AMS (2007), 69-102. Available at `https://arxiv.org/abs/math/0604089`.

2. W.T. Gowers, *Hypergraph regularity and the multidimensional Szemerédi theorem.* Annals of Mathematics, 166 (2007), 897–946. Available at `https://arxiv.org/abs/0710.3032`.

3. J. Fox, D. Conlon, Y. Zhao, *The Green-Tao theorem: an exposition.* EMS Surv. Math. Sci. 1 (2014), 249-282. Available at `https://arxiv.org/abs/1403.2957`.

Additional support

Two examples sheets will be provided and associated office hours will be offered.